Symmetry and asymmetry of water ages in a one-dimensional flow

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Abstract

Hall and Haine [J. Mar. Syst., in press] briefly addressed the problem of estimating the age of irreducible fluid elements or water parcels in a one-dimensional flow with constant velocity and diffusivity. Herein further developments are achieved on this subject. The age of every water parcel is set to zero at the moment it passes through the point $x=0$, where $x$ is an appropriate space coordinate. As time progresses, the age of the water is seen to increase unboundedly upstream of the point $x=0$, and tend to the steady-state value $A_x/\nu_A$ downstream of the point $x=0$, where $\nu$ is the water velocity. By contrast, the age of the water parcels that have touched at least once the point $x=0$ is symmetric with respect to the point $x=0$ and tends to $|x/\nu|$ as time progresses. Asymptotic expansions are derived for large times.

Keywords: Age; Age of seawater; Ageing; Tracer

1. Introduction

Age is a diagnostic used for understanding complex geophysical flows (e.g., England, 1995; Campin et al., 1999; Hirst, 1999; Khatiwala et al., 2001; Deleersnijder et al., 2001a, 2002; Delhez and Deleersnijder, 2002; Delhez et al., 2003; Waugh and Hall, 2002; Waugh et al., in press). However, as age is the solution of a rather elaborate partial differential problem (e.g., Delhez et al., 1999, Holzer and Hall, 2000), it is also appropriate to evaluate the age in simple, idealised flows, so as to gain insight into the functioning of the “age machinery”. This is why studies were achieved of the age of a tracer released by a point source into an infinite domain with constant velocity and diffusivity. It was found that the age is symmetric with respect to the point source (Beckers et al., 2001; Deleersnijder et al., 2001b; Hall and Haine, in press). To date, the best explanation of this rather counterintuitive property is the one given by Hall and Haine (in press), who resorted to an elegant Lagrangian approach.

Hall and Haine (in press) also briefly addressed the problem of calculating the age of irreducible fluid elements in a one-dimensional flow. At the initial time, the age of all fluid elements was assumed to be zero. The age of every fluid element increased at the same rate as time progressed, and was reset to zero every time the fluid element under consideration would pass through the point $x=0$, where $x$ is an appropriate space
coordinate. The mean age of fluid elements contained in an arbitrarily small water sample taken at location $x$ was seen to be unequal to the age in a water sample taken at the same time at location $-x$. Every water sample consists of fluid elements which have not yet touched the point $x=0$, and those which have passed at least once through $x=0$. The latter make up a distinct water mass, which is different from the water as such. Hall and Haine (in press) mentioned that the age of this particular water mass, unlike the water age, is symmetric with respect to point $x=0$.

Hall and Haine’s (in press) concept of “irreducible fluid element” is equivalent to that of “water parcel” in the Constituent-oriented Age Theory (CAT) of Delhez et al. (1999) and Deleersnijder et al. (2001a). CAT is well suited to investigate the age of tracers (e.g., Delhez and Deleersnijder, 2002; Delhez et al., 2003) as well as that of water masses (Hirst, 1999; Deleersnijder et al., 2002). The objective of the present study is to investigate in some detail the behaviour of the two water ages considered by Hall and Haine (in press). First, in Section 2, the Eulerian equations of the problem are set. Then, their solutions are established, including asymptotic expansions for large times (Section 3).

2. Model equations

Consider a one-dimensional water flow with constant velocity $u (> 0)$ and diffusivity $\kappa (> 0)$ in the domain $-\infty < x < \infty$. At any time $t > 0$ and position, the age $a(t, x)$ of a passive- or inert-tracer is given by (e.g., Delhez et al., 1999; Deleersnijder et al., 2001a)

$$a(t, x) = \frac{\alpha(t, x)}{C(t, x)},$$

where $\alpha$ and $C$ denote the age concentration and the concentration of the tracer under study, respectively. The latter satisfy the following equations

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} = \kappa \frac{\partial^2 C}{\partial x^2},$$

$$\frac{\partial \alpha}{\partial t} + u \frac{\partial \alpha}{\partial x} = C + \kappa \frac{\partial^2 \alpha}{\partial x^2}.\tag{3}$$

The concentration is a dimensionless mass fraction, which is generally much smaller than unity, except the concentration of the water which is almost equal to 1 (Fig. 1).

It is convenient to introduce dimensionless variables (Table 1). As in Deleersnijder et al. (2001b), the time and space scales are taken to be $4\kappa/u^2$ and $4\kappa/u$, respectively. After the introduction of the dimensionless variables, it is readily seen that relation (1) is left unchanged, while Eqs. (2) Eqs. (3) transform to

$$\frac{\partial C}{\partial t} + \frac{\partial C}{\partial x} = \frac{1}{4} \frac{\partial^2 C}{\partial x^2},$$

$$\frac{\partial \alpha}{\partial t} + \frac{\partial \alpha}{\partial x} = C + \frac{1}{4} \frac{\partial^2 \alpha}{\partial x^2}.\tag{5}$$

The concentration of the water being equal to 1, the age of the water $a_w(t,x)$ of such “marked” water parcels in water sample taken at time $t$ and location $x$ obeys the equation

$$\frac{\partial a_w}{\partial t} + \frac{\partial a_w}{\partial x} = 1 + \frac{1}{4} \frac{\partial^2 a_w}{\partial x^2},$$

which is equivalent to Hall and Haine’s (in press) equation for the ideal age. On the other hand, the water parcels that have touched at least once the point $x=0$ can also be considered. The set of these water parcels bears similarities with the “surface water” studied in Deleersnijder et al. (2001a, 2002). The fraction $C_m(t, x)$ of such marked water parcels in water sample taken at time $t$ and location $x$ obeys the equation

$$\frac{\partial C_m}{\partial t} + \frac{\partial C_m}{\partial x} = \frac{1}{4} \frac{\partial^2 C_m}{\partial x^2}.\tag{7}$$

The age concentration $\alpha_m(t,x)$ of the marked particles, which will be regarded as a “water mass” created at $x=0$, satisfies

$$\frac{\partial \alpha_m}{\partial t} + \frac{\partial \alpha_m}{\partial x} = C_m + \frac{1}{4} \frac{\partial^2 \alpha_m}{\partial x^2}.\tag{8}$$

Then, the mean age of the water mass under study is $a_m(t, x) = \alpha_m(t, x)/C_m(t, x)$. No variable is allowed constant diffusivity $\kappa (> 0)$

constant velocity $u (> 0)$

$x=0$

all ages $= 0$

Fig. 1. Schematic representation of the characteristics of the flow in which water ages are estimated.
to grow exponentially in the limit $|x| \to \infty$. All variables are zero at $t=0$. The water age and the age concentration of the water mass are prescribed to be zero at $x=0$. For the concentration $C_m$ to actually represent the fraction of the water parcels that have touched at least once the point $x=0$, it is necessary that the boundary condition $C_m(t, x=0)$ be applied.

By manipulating Eqs. (6)–(8), it may be seen that the age of the water and that of the water mass satisfy

$$a_w = a_m + (t - a_m)(1 - C_m). \quad (9)$$

This relation is similar to that established in Deleersnijder et al. (2002) for the age of the water and the age of the surface water in the World Ocean. Lagrangian arguments similar to those of Deleersnijder et al. (2002) may also be invoked to obtain Eq. (9). At any time and location, the age of the water mass that has already touched $x=0$, and its concentration are no larger than $t$ and 1, respectively. Therefore, Eq. (9) implies that the age of the water is larger than or equal to that of the water mass under study.

### 3. Solution and discussion

The temporal Laplace transform of governing Eqs. (6)–(8) is evaluated, leading to ordinary differential equations involving $x$-derivatives only. Next, taking into account the initial and boundary conditions, the ordinary differential equations are easily solved. Finally, using a standard table of Laplace transforms, the solutions to Eqs. (6)–(8) are obtained:

$$a_w = t(1 - e^{2x}I_{3/2}) + e^{2x}I_{1/2}, \quad (10)$$

$$C_m = e^{2x}I_{3/2}, \quad (11)$$

$$a_m = e^{2x}I_{1/2}, \quad (12)$$

with

$$I_\beta(t, x) = \frac{|x|}{\pi^{1/2}} \int_0^t 0^{-1/2} e^{-0-0-x^2/0} d\theta. \quad (13)$$

The age of the water mass under study is

$$a_m = \frac{I_{1/2}}{I_{3/2}} \int_0^t 0^{-1/2} e^{-0-0-x^2/0} d\theta. \quad (14)$$

The solutions above satisfy relation (9). Clearly, the age of the water is not symmetric with respect to point $x=0$, whereas the water mass age is (Fig. 2), i.e.,

$$a_w(t, x) = a_w(t, -x)$$

and

$$a_m(t, x) = a_m(t, -x). \quad (16)$$

For $\beta = 1/2$ and $\beta = 3/2$, the integral (13) behaves as

$$I_{1/2}(t, x) \sim |x| e^{-2|x|} - \frac{|x| e^{-t-x^2/t}}{\pi^{1/2} t^{1/2}} \quad (17)$$

$$I_{3/2}(t, x) \sim e^{-2|x|} - \frac{|x| e^{-t-x^2/t}}{\pi^{1/2} t^{1/2}} \quad (18)$$

if $t \to \infty$ and $|x|/t \to 0$ (Appendix A). Accordingly, the asymptotic behaviour for large times of the age of the water, the concentration of the water mass that has

### Table 2

<table>
<thead>
<tr>
<th>Dominant terms of the asymptotic expansions for $t \to \infty$ and $</th>
<th>x</th>
<th>/t \to 0$ of the age of the water $a_w$, the concentration $C_m$ of the water parcels having touched at least once the point $x=0$, and the mean age of this water mass $a_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_w(t, x)$</td>
<td>$(1 - e^{-4</td>
<td>x</td>
</tr>
<tr>
<td>$C_m(t, x)$</td>
<td>$e^{-4</td>
<td>x</td>
</tr>
<tr>
<td>$a_m(t, x)$</td>
<td>$</td>
<td>x</td>
</tr>
</tbody>
</table>

Variables are in their dimensionless form (see Table 1).
touched at least once the point \( x = 0 \) and the age of this water mass are

\[
a_w(t,x) \sim (1 - e^{2t^{-2} |x|}) t + |x| e^{2t^{-2} |x|},
\]

(19)

\[
C_m(t,x) \sim e^{2t^{-1} |x|} + \frac{|x| e^{-(x-u)^2/t}}{\pi^{1/2} t^{3/2}},
\]

(20)

\[
a_m(t,x) \sim |x| - \frac{|x| e^{-(|x|-t)2/t}}{\pi^{1/2} t^{1/2}}.
\]

(21)

Upstream and downstream of the point \( x = 0 \), these expansions exhibit significant differences (Table 2). The age of the water and that of the water mass tend to \( x \) for \( x > 0 \), which corresponds to \( x/u \) in dimensional variables. However, in the region \( x < 0 \), the age of the water mass tends to \( |x| \)—i.e., \( x/u \) in dimensional variables—while the water age grows unboundedly as \( t \).

Downstream of the point \( x = 0 \), the fraction \( C_m \) of the water parcels that have touched at least once the point \( x = 0 \) tends to 1 as time progresses. Therefore, in the region \( x > 0 \), the age of the water and that of the water mass under study must tend to the same limit, which, in accordance with elementary intuition, is \( x/u \) in dimensional form. Nonetheless, upstream of the point \( x = 0 \), the larger the distance to this point, the larger the fraction of the water parcels that have not yet touched the point \( x = 0 \)—since there is an infinite reservoir of such water parcels upstream of \( x = 0 \), the domain of interest being infinite. This is why the age of the water tends to be equal to the time that has elapsed since the age of all water parcels was set equal to zero. By contrast, the age of the water mass that has touched at least once the point \( x = 0 \) tends to a value that is independent of the instant at which the age of all water parcels was initialised to zero. This points to a certain ill-foundedness of the concept of water age for an unbounded domain (Hall and Haine, in press), a problem which is unlikely to
arise if the volume of the domain of interest is finite (e.g., England, 1995; Deleersnijder et al., 2002).

Obviously, for any physically relevant problem, the domain of interest is not infinite. However, the infinite-domain solutions discussed herein are believed to be a good approximation of those obtained in a finite domain, provided the latter are considered in the vicinity of the point at which the ages are prescribed to be zero and over a period of time sufficiently smaller than the timescales related to advection and diffusion over length scales comparable to the size of the domain of interest. A relevant illustration thereof is the age symmetry depicted in Fig. 1 of Beckers et al. (2001), which is obtained from a numerical model of the North Sea. A future study will focus on the time evolution of various ages in a Munk loop (Munk, 1966; Hall and Holzer, 2003), with an emphasis on the analysis of the sensitivity of the solutions to the length of the loop.

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Appendix A

The integral defined by relation (10) may be regarded as the following sum:

\[ I_\beta(t, x) = I_\beta(\infty, x) - \tilde{I}_\beta(t, x), \tag{A.1} \]

with

\[ I_\beta(\infty, x) = \frac{|x|}{\pi^{1/2}} \int_0^\infty \theta^{-\beta} e^{-\theta - x^2/\theta} d\theta \]  \tag{A.2}

and

\[ \tilde{I}_\beta(\infty, x) = \frac{|x|}{\pi^{1/2}} \int_0^\infty \theta^{-\beta} e^{-\theta - x^2/\theta - 1} d\theta. \tag{A.3} \]

Clearly, in the limit \( t \to \infty \), the integral \( I_\beta(\infty, x) \) is much larger than \( \tilde{I}_\beta(t, x) \).

By using Gradshteyn and Ryzhik (2000), it is readily seen that

\[ I_\beta(\infty, x) = \frac{|x|^{3/2 - \beta}}{\pi^{1/2}} e^{-2|x|}. \tag{A.4} \]

Setting \( \theta = t(1 + \zeta) \), the integral \( \tilde{I}_\beta(t, x) \) may be transformed to

\[ \tilde{I}_\beta(t, x) = \frac{|x|}{\pi^{1/2}} \int_0^\infty (1 + \zeta)^{-\beta} e^{-\frac{x^2}{t^2}} d\zeta \tag{A.5} \]

where

\[ f(\zeta) = \frac{x^2}{t^2} (1 + \zeta). \tag{A.6} \]

In the interval \( 0 \leq \zeta < \infty \), the maximum rate of \( f(\zeta) \) occurs at \( \zeta = 0 \). Therefore, according to Laplace’s method (e.g., Bender and Orszag, 1978), using the asymptotic expansion

\[ f(\zeta) \sim \frac{x^2}{t^2} + \left( 1 - \frac{x^2}{t^2} \right) \zeta, \quad \zeta \to 0, \tag{A.7} \]

the dominant term of the behaviour of \( \tilde{I}_\beta(t, x) \) as \( t \to \infty \) and \( |x|/t \to 0 \) is obtained

\[ \tilde{I}_\beta(t, x) \sim \frac{|x|}{\pi^{1/2}} \int_0^\infty \exp \left[ -t \left( 1 - \frac{x^2}{t^2} \right) \zeta \right] d\zeta, \tag{A.8} \]

which immediately yields

\[ \tilde{I}_\beta(t, x) \sim \frac{|x|}{\pi^{1/2}} t^{-\beta} e^{-t - x^2/\beta}. \tag{A.9} \]

Combining Eqs. (A.1), (A.4) and (A.9) leads to the asymptotic behaviour of \( I_\beta(t, x) \) as \( t \to \infty \) and \( |x|/t \to 0 \):

\[ I_\beta(t, x) \sim \frac{|x|^{3/2 - \beta}}{\pi^{1/2} t^{\beta}} e^{-2|x|}. \tag{A.10} \]
References


