Accurate determination of internal kinematics from numerical wave model results

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Abstract

The internal kinematics for surface waves propagating over a locally constant depth are expressed as convolution integrals. Given the wave kinematics at the still water level (SWL), this provides explicit and exact potential flow expressions for the internal kinematics as convolutions in space with appropriate impulse response functions. These functions are derived in closed form and they are shown to decay exponentially. This effectively reduces the limits of the convolution integral to a horizontal distance of approximately three water depths from the water column of interest. The SWL kinematics must be provided within this region. The source of SWL kinematics may, e.g. be one of the recently developed highly accurate Boussinesq-type formulations. The method is valid for multidirectional, irregular waves of arbitrary nonlinearity at any constant water depth.

Keywords: Wave kinematics; Convolution; Fourier transform; FFT; Boussinesq; Waves

1. Introduction

For many years, a substantial effort has been devoted to the development of models for nonlinear water wave propagation. Due to the prohibitive computational requirements associated with models resolving the whole three-dimensional fluid domain, a class of models reducing the problem to two horizontal dimensions, under certain assumptions, have attracted considerable attention. Boussinesq-type models constitute an important sub-class. For recent reviews, see Kirby (1997, 2003) and Madsen and Schäffer (1999).

Common to all Boussinesq developments is that the solution of the equations does not directly provide the internal wave kinematics. However, the kinematics can be obtained by post-processing the model results, e.g. using the expressions derived in the process of obtaining the governing Boussinesq equations.

A common feature of all Boussinesq models is that, irrespective of the choice of kinematic expression, the range of application is much smaller for the kinematics than for the dependent variables of the Boussinesq model. A detailed discussion of various formulations of kinematics in the framework of Boussinesq theory has recently been given by Madsen and Agnon (2003).

This paper devises an alternative method for obtaining the internal kinematics from the velocity at the still water level (SWL). Recognising that even...
for highly nonlinear waves, the internal kinematics are governed by linear equations, a linear convolution technique is applied. The method requires that the velocity at SWL is available in a neighbourhood of roughly three water depths. The SWL velocity can, e.g., be provided by an appropriate version of the recent Boussinesq formulation by Madsen et al. (2003) (referred to as MBS in the following), see also Madsen et al. (2002). The present convolution technique provides the correct velocity profiles for nonlinear multidirectional irregular waves with no depth limitation other than what might be inherited from the model providing the SWL velocity, i.e., $kh < 40$ when using MBS ($k$ is wave number, $h$ is water depth). The water depth is assumed to be (locally) constant, but mild-slope extensions are expected to be possible.

Accurate evaluation of wave kinematics is a basic prerequisite for an appropriate evaluation of loads on traditional offshore structures, marine pipelines, offshore wind turbine foundations, etc.

1.1. Consistency

For the task of post-processing wave model results to obtain wave kinematics, it is sometimes argued that an associated kinematics model should be applied for consistency. However, inspection shows that there is not a unique correspondence between a given set of Boussinesq equations and an associated relation for the internal kinematics. For example, the mass equation can be obtained from either the kinematic surface condition or from the depth-integrated continuity equation. In the first case, the underlying kinematic field must be developed to an order higher than in the second case to obtain exactly the same resulting mass equation. This means that the kinematic field that appears as the consistent expression for a given set of Boussinesq equations depends on the derivation procedure adopted, and thereby, it is not unique. Thus, we might as well look for an independent model for the kinematics that can match the application range of the Boussinesq model (and not the reduced application range for the associated kinematics). This can be done without considering the origin of the SWL kinematics to be processed.

The issue of consistency is only addressed to comply with possible concerns of the reader. This should not be confused with the purpose of this paper, which is to get accurate wave kinematics.

1.2. Source of SWL kinematics

The present convolution technique for internal kinematics is not linked to any particular source of SWL kinematics. Nevertheless, the present work was triggered by the recent Boussinesq-type developments by MBS, building on Agnon et al., (1999). Their work shows unprecedented accuracy for highly nonlinear waves spanning from solitary waves in shallow water to steep waves in very deep water, $kh \sim 40$. MBS’s model provides very accurate results for both the surface elevation and the horizontal and vertical particle velocities in the region between wave crest and wave trough level. However, the results for the kinematics in the rest of the water column break down when $kh$ exceeds roughly 12. Recently, Madsen and Agnon (2003) extended the range of validity further to about $kh = 16$, still within the framework of Boussinesq type expansions.

1.3. The structure of this paper

Section 2 lists some basic definitions and conventions, Section 3 concerns the formulation of the velocity. In physical space, the infinite series formulation by Madsen and Schäffer (1998) is taken as the starting point. This involves gradient operators of up to infinite order. Applying the Fourier transform, we arrive at the equivalent formulation in wave number space. Invoking the kinematic boundary condition of the solid bottom, the vertical velocity is expressed in terms of the vertical velocity at SWL alone and likewise for the horizontal velocity. Returning to physical space by the inverse Fourier transform, this leads to the convolution approach described in Section 4. By these transactions, the infinite gradient operators from Madsen and Schäffer are replaced by convolution integrals involving certain impulse response functions. These are evaluated in Section 5 for the respective situations of one and two horizontal dimensions. To show the validity of the theory, generic situations containing the basic properties of wave irregularity, nonlinearity and directionality are studied in Section 6. Finally, conclusions are given in Section 7.
2. Definitions and conventions

We adopt a Cartesian coordinate system \((\hat{x}, \hat{y}, \hat{z})\) with the \(\hat{z}\)-axis pointing upwards from the SWL and define the non-dimensional system

\[
(x, y, z) = \frac{1}{h} (\hat{x}, \hat{y}, \hat{z})
\]

where \(h\) is the constant water depth.

Capitalised symbols indicate Fourier-space functions, while functions of physical space are specified using lower case. Among the various definitions of the Fourier transform and its inverse, we choose the following transform pairs

\[
F(\kappa) = \int_{-\infty}^{\infty} f(x) e^{-i\kappa x} dx
\]

and

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\kappa) e^{i\kappa x} d\kappa
\]

With \(x \equiv (x, y)\) and \(\kappa \equiv (\kappa_x, \kappa_y)\), the similar definitions in two horizontal dimensions are

\[
F(\kappa) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) e^{-i\kappa x} dx dy = \int_A f(x) e^{-i\kappa x} dx
\]

and

\[
f(x) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\kappa) e^{-i\kappa x} d\kappa_x d\kappa_y
\]

\[
= \frac{1}{(2\pi)^2} \int_A F(\kappa) e^{-i\kappa x} d\kappa
\]

where \(A\) is the entire \(x\) or \(\kappa\) plane. For later use, we further mention that 2D Fourier transform pairs become zero order Hankel transform pairs in the case of radial symmetry, where we have

\[
F(\kappa) = 2\pi \int_0^{\infty} f(r) J_0(\kappa r) dr
\]

and

\[
f(r) = \frac{1}{2\pi} \int_0^{\infty} F(\kappa) \kappa J_0(\kappa r) d\kappa
\]

with \(\kappa = |\kappa|\). Here \(J_0\) is the Bessel function of first kind and order zero. Apart from the factors \(2\pi\) and \(1/2\pi\), which vanish in some definitions, the Hankel transform equals the inverse Hankel transform, see, e.g. Sneddon (1972).

3. The kinematics formulation

Although the formulation pursued in the present paper finally departs from Boussinesq-type theory, we take an expansion from shallow water as the starting point of our discussion. The horizontal particle velocity vector is denoted as \(\mathbf{u}\), the vertical velocity is \(w\), while \(t\) is time. Following Madsen and Schäffer (1998) in the formulation given by MBS, an exact representation of the velocity field (satisfying conditions of irrotationality and continuity) is given by the series representations

\[
\mathbf{u}(x, z, t) = \cos(z \nabla) \mathbf{u}(x, 0, t) + \sin(z \nabla) w(x, 0, t)
\]

and

\[
w(x, z, t) = \cos(z \nabla) w(x, 0, t) - \sin(z \nabla) \mathbf{u}(x, 0, t)
\]

where the \(\cos\) and \(\sin\) operators are

\[
\cos(z \nabla) \equiv \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{(2n)!} \nabla^{2n},
\]

\[
\sin(z \nabla) \equiv \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \nabla^{2n+1}
\]

These operate on vectors as well as scalars and the following interpretation of the powers of the horizontal gradient operator \(\nabla\) is understood

\[
\nabla^{2n} f = \nabla (\nabla^{2n-2} (\nabla \cdot f)), \quad \nabla^{2n+1} f = \nabla^{2n} (\nabla \cdot f),
\]

\[
\nabla^{2n} f = \nabla^{2n} f, \quad \nabla^{2n+1} f = \nabla (\nabla^{2n} f)
\]

Applying the two-dimensional Fourier transform to Eqs. (8) and (9), both of which are linear in \(\mathbf{u}\) as well as \(w\), yields

\[
U(\kappa, z, t) = \cosh(\kappa z) U(\kappa, 0, t)
\]

\[
-\frac{\kappa}{K} \sinh(\kappa z) W(\kappa, 0, t)
\]
and

\[ W(\kappa, z, t) = \cosh(\kappa z)W(\kappa, 0, t) + i\frac{\kappa}{\kappa} \sinh(\kappa z)U(\kappa, 0, t) \]  \tag{12} \]

Both of these expressions exhibit the undesirable property of the exponential growth for increasing arguments of the hyperbolic functions. For large \( \kappa z \) the right-hand-side of each equation becomes a delicate balance between two very large contributions. Boussinesq-type developments over the past decade have indeed concentrated on manipulating the equivalent physical-space formulation to avoid this problem for operators truncated at finite order. Here we use a different approach, which does not require finite order truncation, and thus, it has no water depth limitation.

Invoking the boundary condition of an impermeable bottom, \( W(\kappa, -1, t) = 0, \) Eq. (12) yields a simple relation between \( W(\kappa, 0, t) \) and \( U(\kappa, 0, t), \) which allows us to eliminate \( W(\kappa, 0, t) \) from Eq. (11) and \( U(\kappa, 0, t) \) from Eq. (12) to get

\[ U(\kappa, z, t) = \frac{\cosh(\kappa z + 1)}{\cosh \kappa} U(\kappa, 0, t) \]  \tag{13} \]

and

\[ W(\kappa, z, t) = \frac{\sinh(\kappa z + 1)}{\sinh \kappa} W(\kappa, 0, t) \]  \tag{14} \]

These expressions are in obvious agreement with Stokes’ linear wave theory. The important thing to note is that they are also valid for waves of arbitrary nonlinearity, since neither of the surface boundary conditions were used in their derivation.

We have now eliminated the problematic balance between the hyperbolic functions as both Eqs. (13) and (14) simply express the decay of the velocities below SWL through functions staying between zero and unity. Our final step is now to transform back to physical space by the inverse Fourier transform. For this purpose, it is convenient to define the transfer functions

\[ R_u(\kappa; z) = \frac{\sinh(\kappa z + 1)}{\sinh \kappa} \]  \tag{15} \]

and

\[ R_w(\kappa; z) = \frac{\sinh(\kappa z + 1)}{\sinh \kappa} \]  \tag{16} \]

by which

\[ U(\kappa, z, t) = R_u(\kappa; z)U(\kappa, 0, t) \]  \tag{17} \]

and

\[ W(\kappa, z, t) = R_w(\kappa; z)W(\kappa, 0, t) \]  \tag{18} \]

Although each of the transfer functions is the same for one and two horizontal dimensions, their associated impulse response functions are not. Thus, we address the respective cases of one and two horizontal dimensions separately.

4. The convolution approach

4.1. The one-dimensional case

Applying the inverse Fourier transform to the one-dimensional versions of Eqs. (13) and (14), the convolution theorem provides us with the convolution integrals

\[ u(x, z, t) = \int_{-\infty}^{\infty} u(x - x', 0, t)r_u(x'; z)dx' \]  \tag{19} \]

and

\[ w(x, z, t) = \int_{-\infty}^{\infty} w(x - x', 0, t)r_w(x'; z)dx' \]  \tag{20} \]

where \( r_u(x; z) \) and \( r_w(x; z) \) are impulse response functions given by the inverse Fourier transform of \( R_u(\kappa; z) \) and \( R_w(\kappa; z) \), respectively. These are evaluated below.

4.2. The two-dimensional case

Applying the 2D inverse Fourier transform to Eqs. (13) and (14), the convolution theorem provides us with the 2D convolution integrals

\[ u(x, y, z, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x - x', y - y', 0, t) \times \tilde{R}_u(x', y'; z)dx'dy' \]  \tag{21} \]
\[ w(x, y, z, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(x - x', y - y', 0, t) \times \tilde{\rho}_w(x', y'; z) dx' dy' \] (22)

Since \( R_u(\kappa; z) \) and \( R_w(\kappa; z) \) are radially symmetric in the \( \kappa \) plane, the inverse 2D Fourier transform reduces to the inverse Hankel transform of order zero, see Eq. (7). Leaving out the subscript \( u \) or \( w \) for generality, we have

\[ \tilde{\rho}(x, y; z) = \rho(\sqrt{x^2 + y^2}; z) \] (23)

where

\[ \rho(r; z) = \frac{1}{2\pi} \int_0^\infty R(\kappa; z) \kappa J_0(\kappa r) d\kappa \] (24)

5. The impulse response functions

5.1. The trivial point at the SWL

At \( z = 0 \), we have

\[ R_u(\kappa; 0) = R_w(\kappa; 0) = 1 \] (25)

corresponding to the trivial statement that the SWL velocities equal themselves. In 1D, the associated impulse response functions are

\[ r_u(x, 0) = r_w(x, 0) = \delta(x) \] (26)

where \( \delta \) is the Dirac delta function. In 2D, we similarly have

\[ \tilde{\rho}_u(x, y, 0) = \tilde{\rho}_w(x, y, 0) = \delta(x, y) = \delta(x)\delta(y) \] (27)

or, in terms of the radial coordinate

\[ \rho_u(r, 0) = \rho_w(r, 0) = \frac{\delta(r)}{\pi r} \] (28)

The latter result may be obtained by acknowledging that its Hankel transform (Eq. (6)) must equal unity, (Eq. (25)).

5.2. The one-dimensional case

The impulse response functions are given as the inverse Fourier transforms (see Eq. (3)) of the transfer functions (16) and (15), respectively, and we obtain

\[ r_u(x; z) = \frac{1}{2} \frac{\sin(\pi z)}{\cos(\pi z) - \cosh(\pi x)}, \quad -1 < z < 0 \] (29)

and

\[ r_w(x; z) = \frac{\sin(\frac{\pi}{2} z) \cosh(\frac{\pi}{2} x)}{\cos(\pi z) - \cosh(\pi x)}, \quad -1 < z < 0 \] (30)

Contour plots of these impulse response functions are shown in Figs. 1 and 2, which clearly indicate the peak near \( (x, z) = (0, 0) \). In practice, the
limits of the convolution integrals can be taken as roughly $\pm 3$ or less corresponding to a horizontal distance of approximately three water depths. This is illustrated in Figs. 3 and 4, which show the two impulse response functions versus $x$ for different values of $z$. The decay is exponential in $x$. The above expressions are valid throughout the water column, except for the trivial point $z=0$, where they vanish instead of providing the correct values as given above.

5.3. The two-dimensional case

Neither $R_u(\kappa; z)$ nor $R_w(\kappa; z)$ allow us to evaluate this integral directly and we turn to series formulations. An obvious idea is to regard $\kappa$ as a parameter and investigate the Fourier series of $R_u$ and $R_w$. In terms of the variable $z+1$, $R_u$ is an even function while $R_w$ is odd. The region between the bottom and the SWL corresponds to the $z+1$-interval $[0,1]$. This makes it a natural choice to construct a periodic function of $z+1$ with period 2 and defined by $R(\kappa; z)$ in the $z+1$-interval $[0,1]$. This leads to

$$R_w(\kappa; z) = \frac{\sinh(\kappa(z+1))}{\sinh \kappa} = -\sum_{n=1}^{\infty} \frac{(-1)^n 2n \pi}{n^2 \pi^2 + \kappa^2} \sin n \pi (z+1), \quad -1 \leq z < 0$$

(31)

and

$$R_u(\kappa; z) = \frac{\cosh(\kappa(z+1))}{\cosh \kappa} = \frac{\tanh \kappa}{\kappa} \left( 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \kappa^2}{n^2 \pi^2 + \kappa^2} \cos n \pi (z+1) \right),$$

$$-1 \leq z < 0$$

(32)

This technique works well for $R_w$, as shown below. An exception is the trivial point $z=0$, where $R_w=1$ and where the series becomes zero, and in practice also its vicinity. However, for $R_u$, the series retain a hyperbolic function of $\kappa$, and thus, it brings us no further in our search for an analytical integration of Eq. (24). This difference may be understood by looking at the poles of $R_u$ and $R_w$ in the complex $\kappa$-plane. For $R_u$, these poles are given by $\kappa=\text{int} \pi$, where $n$ is an integer, and the rational coefficients of the Fourier series in Eq. (31) clearly retain these poles. For $R_u$, the poles are $i\kappa=(2n+1)\pi/2$, but the rational coefficients still exhibit poles at $\kappa=\text{int} \pi$. Clearly, the poles at $i\kappa=(2n+1)\pi/2$ must still be part of the expression and in Eq. (32), they are represented through the overall factor $\tanh \kappa$. To obtain a formulation without the hyperbolic factor, we regard $i\kappa$ as the variable and $z$ as a parameter and expand $R_u$ in partial fractions. To get the poles to
be represented by rational functions, we introduce the expansion

\[ R_u(\kappa; z) = \frac{\cosh \kappa (z + 1)}{\cosh \kappa} = \sum_{n=-\infty}^{\infty} \frac{a_n}{(2n + 1) \frac{\pi}{2} - i \kappa} \quad (33) \]

The coefficients \( a_n \) can be obtained by looking at the limit as \( i \kappa \) approaches the \( n \)th pole. There, the \( n \)th term dominates the series completely, and thus, all other terms can be disregarded. This yields

\[ a_n = (-1)^n \frac{\pi (2n + 1)(z + 1)}{2} \quad (34) \]

Collecting the contributions from positive and negative values of \( 2n + 1 \) gives the final expansion

\[ R_u(\kappa; z) = \frac{\cosh \kappa (z + 1)}{\cosh \kappa} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi (2n + 1) \cos \left( \frac{2n + 1}{2} \right) + \kappa^2}{\kappa^2} \]

\[ -1 \leq z < 0 \]

Although this result was not derived as a Fourier series it certainly resembles one. The argument of the cosine shows a periodicity of 4 and inspection reveals that it is indeed a Fourier series of a periodic function constructed by taking \( R_u \) in the interval \(-1 < z + 1 < 1\) and shifting \(-R_u\) a distance of two either way along the abscissa to complete the period of 4. This is illustrated by Fig. 5, which shows an example of the partial sum of the series in Eq. (35). \( R_u \) is shown for reference. Having identified Eq. (35) as a Fourier series, it is clear that it converges in the open interval \(-2 < z < 0\). As only \(-1 \leq z < 0\) belongs to the water column, this is the range specified in Eq. (35).

The series in Eqs. (31) and (35) have the same kind of very slow convergence. Without the trigonometric factor and the alternating sign, the asymptotic value of the \( n \)th term would be proportional to \( 1/n \) and thus the series would diverge. It is actually possible to derive series with significantly faster convergence. However, this advantage turns out to disappear during the evaluation of the integral in Eq. (24), resulting in the same series for the final formulation of \( \rho (r, z) \).

The expressions (31) and (35) may be further simplified by including the alternating sign in the trigonometric term using the identities

\[ (-1)^n \sin n\pi(z + 1) = \sin n\pi z \quad (36) \]

and

\[ (-1)^n \cos \frac{\pi (2n + 1)(z + 1)}{2} = -\sin \frac{\pi (2n + 1)z}{2} \quad (37) \]

Proceeding with the integration of the impulse response functions, we substitute Eq. (31) with Eq. (36) in Eq. (24). Uniform convergence of the series except at \( z = 0 \) allows for term-by-term integration and using the identity

\[ \int_{0}^{\infty} \frac{x}{x^2 + a^2} J_0(bx) dx = K_0(ab), \quad a, b > 0 \quad (38) \]

where \( K_0 \) is the modified Bessel function of the second kind of order zero, we get

\[ \rho_u(r; z) = -\sum_{n=0}^{\infty} K_0(n\pi r) \sin n\pi z, \quad -1 \leq z < 0 \quad (39) \]

Similarly, Eq. (24) with Eq. (37) in Eq. (35) yields

\[ \rho_u(r; z) = -\sum_{n=0}^{\infty} \frac{2n + 1}{2} K_0 \left( \frac{(2n + 1) \pi r}{2} \right) \]

\[ \times \sin \frac{\pi (2n + 1)z}{2}, \quad -1 \leq z < 0 \quad (40) \]
Both of these formulations are valid throughout the water column except for the trivial point $z=0$, where they vanish instead of providing the correct results, as given previously.

The asymptotic behaviour of the Bessel function (see below) ensures very fast convergence for large $r$. On the other hand, the convergence is extremely slow for small $r$ due to the singularity of $K_0$ for vanishing argument. For $r=0$, the series diverge. This difficulty is illustrated in Figs. 6 and 7 showing examples of the partial sums of the series in Eqs. (39) and (40). The problem is solved by deriving the alternative forms

$$
\rho_w(r; z) = \frac{1}{2\pi} \left( \frac{-z}{(r^2 + z^2)^{3/2}} + \sum_{n=1}^{\infty} \left( \frac{2n - z}{(r^2 + (2n - z)^2)^{3/2}} - \frac{2n + z}{(r^2 + (2n + z)^2)^{3/2}} \right) \right), \quad -1 \leq z < 0
$$

(41)

and

$$
\rho_u(r; z) = \frac{1}{2\pi} \left( \frac{-z}{(r^2 + z^2)^{3/2}} + \sum_{n=1}^{\infty} (-1)^n \left( \frac{2n - z}{(r^2 + (2n - z)^2)^{3/2}} - \frac{2n + z}{(r^2 + (2n + z)^2)^{3/2}} \right) \right), \quad -1 \leq z < 0
$$

(42)

see Appendix A. These series have opposite properties of convergence rate as they converge fast for small $r$ (including the point $r=0$) and slowly for large $r$. Thus, Eqs. (39) and (40) should be used for the far field and Eqs. (41) and (42) for the near field. Figs. 8 and 9 show $\rho_w$ and $\rho_u$ computed from Eqs. (41) and (42). A sufficient number of terms in the series may well be retained, since the impulse response functions need only be computed once. Switching between the near field and far field formulations at $r=0.25$ and retaining 20 terms in all the series, the absolute error on $\rho_w$ and $\rho_u$ stays well below $10^{-4}$.

Figs. 10 and 11 show $\rho_w$ and $\rho_u$ versus $r$ for different values of $z$. Comparing with the impulse response for one horizontal dimension (Figs. 3 and 4), the near-field peak is seen to be wider, while the far-field decay is slightly faster, see below.
5.4. The connection between the 1D and 2D impulse response functions

A plane wave may be described either in one or two horizontal dimensions. Since the kinematics must be independent of this choice, we get the following relation between the 1D and 2D impulse response functions

\[
r(x; z) = \int_{-\infty}^{\infty} \tilde{p}(x, y; z) dy = \int_{-\infty}^{\infty} \rho(\sqrt{x^2 + y^2}; z) dy
\]

(43)

It may be shown that this relation is indeed satisfied by Eqs. (29) and (41) as well as by Eqs. (30) and (42).

5.5. Asymptotic behaviour

In the far field, we have

\[
r_w(x; z) \approx -\sin(\pi z)e^{-\pi x}, \quad x \gg 1
\]

(44)

\[
r_u(x; z) \approx -\sin\left(\frac{\pi}{2} z\right)e^{-\pi x}, \quad x \gg 1
\]

(45)

For the radial impulse response functions, we utilise that

\[
K_0\left(\frac{\pi}{2} r\right) \approx \frac{e^{-\pi r}}{\sqrt{r}}, \quad r \gg 1
\]

(46)

and get

\[
\rho_w(r; z) \approx -\sin(\pi z)e^{-\pi r}, \quad r \gg 1
\]

(47)

\[
\rho_u(r; z) \approx -\sin\left(\frac{\pi}{2} z\right)e^{-\pi r}, \quad r \gg 1
\]

(48)

Apart from exhibiting the exponential decay, these expressions show that all the impulse response functions have a very simple structure of the vertical variation in the far field. Thus, for the horizontal velocity both impulse response functions show a quarter of a sinusoid over the water column with extreme value at the bed and zero at the SWL. For
the vertical velocity, the distribution is half a sinusoid peaking at mid-depth and vanishing both at the bed and at the SWL.

6. Verification of the convolution approach

The convolution integrals provide the internal kinematics in terms of the velocities at SWL. The expressions are explicit and exact. The only assumptions are irrotational flow of an incompressible fluid and locally constant water depth. Thus, the approach is valid for multidirectional, irregular waves of arbitrary nonlinearity at any water depth. To demonstrate this, we look at three generic situations for which exact solutions exist. The first example is bichromatic, unidirectional, linear waves that contain the basic feature of wave irregularity. The second example adds bi-directionality to exhibit the simplest form of multidirectionality. The third example demonstrates the validity for nonlinear waves. A highly nonlinear regular wave is considered and we take the ‘stream function theory’ (Rienecker and Fenton, 1981) as the ‘numerically’ exact solution.

6.1. Bichromatic wave example

Fig. 12 gives an example of a vertical profile of horizontal and vertical velocity under a linear bichromatic wave. Both velocities are normalised by \( \sqrt{gh} \). The two wave numbers are \( kh = 2 \) and \( kh = 5 \) and the two wave amplitudes are both 1% of the water depth. With the two wave crests coinciding at \((x,t) = (0,0)\), the example shows the situation for \((x,t) = (3\pi/8,0)\). The thin, solid curve shows the target as computed from linear wave theory explicitly using the two values of \( kh \) involved. The thick, dashed curve shows the result from the convolution technique, which implicitly retrieves the equivalent information from the horizontal variation of the SWL kinematics. Thus, the convolution formulation (Eq. (19) and (20)) with the impulse response functions (Eqs. (30) and (29)) does not involve \( kh \). The convolution result fits exactly on top of the target solution.

Although this is an example in one horizontal dimension, we should still be able to get the correct result by regarding it as a two-dimensional case and
apply the convolution integrals in Eqs. (21) and (22). We have checked that this yields the same results.

### 6.2. Bi-directional wave example

A simple bi-directional example is made from the above bi-chromatic example by rotating the longer wave $30^\circ$. The two crests coincide at $(x,y,t)=(0,0,0)$ and the point studied is $(x,y,t)=(3\pi/8,0,0)$. Now, the two-dimensional convolution integral (Eq. (21)) is needed for providing the two horizontal velocity components, $u=(u,v)$ and Eq. (22) is needed for calculating $w$. Apart from that, the explanation of the results shown in Fig. 13 is the same as for Fig. 12. Again the convolution technique matches the target completely.

### 6.3. Nonlinear regular-wave example

Before showing the results of the convolution technique, we illustrate some results of the Boussinesq...
model by MBS. Their work provided some of the inspiration for the present convolution method and the Boussinesq results are relevant as a provider of the SWL kinematics needed for the convolution method. Furthermore, the Boussinesq-results for the velocity profile over depth are relevant for comparison with the convolution approach.

Figs. 14 and 15 show surface elevation and the horizontal and vertical surface velocity for a highly nonlinear, regular wave in deep water. The solid, thin curve is the ‘stream function’ solution while the thick dashed curve is the MBS Boussinesq result (their method III with $\sigma = -0.5$). The wave steepness is $H/L = 0.135$ and $kh$ is 12. The results are graphically indistinguishable.

At this very high value of $kh$, the kinematics model of MBS still provides quite accurate results near the region between wave crest and trough. However, further down the water column the solution starts to oscillate. Figs. 16 and 17 illustrate this for the whole range from the bottom to the surface. The inaccuracies increase with $kh$ (not shown).

We now turn to the results of the convolution method. Using the SWL kinematics from the MBS Boussinesq model as input, we can compute the velocity variation below SWL by Eqs. (19) and (20). The results are shown in Figs. 18 and 19, which also include the variation above SWL as it is accurately computed by the Boussinesq method. Again the ‘stream function’ solution is shown as the thin, full curve. The perfect graphical match confirms the feasibility of the convolution approach.

7. Conclusions

A convolution method is developed for the determination of internal wave kinematics in terms of the kinematics at SWL. The convolution integrals involve impulse response functions which are derived analytically by the inverse Fourier transform of simple relations in wave number space.

For the case of one-horizontal dimension, the impulse response functions appear as functions of the horizontal coordinate with the vertical coordinate appearing as a parameter. For the two-dimensional case, the impulse response functions have rotational symmetry around the vertical axis, and thus, they depend on the horizontal radius with the vertical coordinate as a parameter. These radially symmetric functions are expressed as infinite series. To facilitate their evaluation, two forms of these series are given, one with fast far-field convergence and one with fast near-field convergence.

The method involves no approximations other than those appearing from the evaluation of the convolution integrals. Thus, any desired accuracy can be attained. With high quality SWL kinematics as input, the method produces high quality internal kinematics. The method is valid for irregular, multidirectional, non-breaking waves of arbitrary nonlinearity. This is illustrated using three examples containing the basic features of irregularity, multidirectionality and nonlinearity.

The only assumptions behind the method are potential flow and constant depth.

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Appendix A. Alternative formulation of impulse response functions

The radially symmetric impulse response functions as given by Eqs. (39) and (40) are suitable for far-field evaluation, but practically useless in the near field. The alternative formulations (Eqs. (41) and (42)) have the opposite properties. This appendix shows how to obtain the alternative formulations. Note that all formulations are mathematically exact.

A few ingredients are needed in the derivation. The first one is an integral representation of \( K_0 \) with a product argument. From Abramowitz and Stegun (1970; Eq. 9.6.25), we have

\[
K_0(n \pi r) = \int_0^\infty \frac{\cos(n \pi t)}{\sqrt{r^2 + t^2}} dt \tag{49}
\]

Changing the limits and integrating by parts yields

\[
K_0(n \pi r) = \frac{1}{2n \pi} \int_{-\infty}^\infty \frac{t \sin(n \pi t)}{(r^2 + t^2)^{3/2}} dt \tag{50}
\]

which is the result needed below.

Furthermore, we need the following identity

\[
\frac{1}{2} + \sum_{n=1}^{\infty} \cos(n \pi t) = \sum_{p=-\infty}^{\infty} \delta(t - 2p) \tag{51}
\]

The right-hand side of this equation is a periodic sequence of delta functions with period 2 and the left-hand side is the associated Fourier series, which may be derived directly from the classical definition.

Substituting Eq. (50) into (39) and expressing the resulting product of sine functions as a sum of cosines, we interchange integration and summation to get

\[
\rho_w(r; z) = \frac{1}{4\pi} \int_{-\infty}^\infty \frac{t}{(r^2 + t^2)^{3/2}} \sum_{n=1}^{\infty} \left( \cos(n \pi (t - z)) - \cos(n \pi (t + z)) \right) dt \tag{52}
\]

Using Eq. (51) to replace the cosines with delta functions allows for analytical integration giving

\[
\rho_w(r; z) = \frac{1}{4\pi} \sum_{p=-\infty}^{\infty} \left( \frac{2p - z}{(r^2 + (2p - z)^2)^{3/2}} - \frac{2p + z}{(r^2 + (2p + z)^2)^{3/2}} \right) \tag{53}
\]

of which terms may be collected two by two to provide the desired form as given in Eq. (41).

We omit the derivation of Eq. (42) from (40) as it follows a similar path as the above.

The equivalence of the different formulations was checked numerically.

References


