35th PIANC World Congress, 29 April – 23 May 2024, Cape Town, South Africa
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Exact and Approximated Solutions to the Critical Ship Speeds in Canals

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Abstract: A sailing ship displaces water and this amount of water needs to return along the hull. When the environment is confined, as in a channel or canal, the water is squeezed in the gap between the ship and the canal boundaries, increasing the return flow. The available space for the water to return is expressed as the blockage ratio m, which is the ratio of the cross sectional area of the ship and the cross section of the canal.

With increasing ship speed the necessary return flow will no longer be met and the water will start to accumulate in front of the ship. At this point the flow condition starts to change, which is commonly referred to as the (first) critical speed. In the 1950s, critical speeds as a function of the blockage ratio were solved graphically, but afterwards elegant goniometric formulations of these critical speeds as a function of the blockage were established. Nevertheless, the appropriate proofs could not be traced back in literature. Because of the importance of the speed ranges, and their relationship with squat and resistance of ships navigating in canals, the present paper provides not only a proof of these goniometric relationships, but also introduces approximated solutions.

Keywords: speed, canal, critical, mathematics, proof, confined

1. Introduction

A displacement vessel, sailing at a (forward) speed needs to, as the name suggests, displace water. This displaced water results in a buoyancy force preventing the ship to sink. As a consequence, with the ship moving, there is a flow of water in the opposite direction of the ship known as the return flow. This return flow is omnipresent but is of higher importance in shallow and confined water. When there is a restriction due to the close environment of the ship, the speed of this return flow may increase to maintain the conservation of mass at all times.

At some combinations of speed and confinement, however, the water can no longer keep up with this need for speed and starts to pile up in front of the vessel. This speed is known as the *first critical speed* and represents the boundary between **subcritical** and **transcritical** flow, the latter being characterized by an unsteady flow regime. With further increasing ship speed (unrealistic for common displacement ships), a new steady situation occurs again, denoted by a **supercritical** flow, which occurs at the *second critical speed*.

These speed regimes for a ship in a narrow channel, compared to her breadth, were first discussed by Schijf [5] from the perspective of the protection of bank revetments. The latter are affected by the speed of the water flow, which in turn is the result of the return flow created by the sailing displacement ship passing by. This is not only dependent on the ship's speed and geometry, but also on the space available for the return flow. Moreover, this space is further limited due to the squat of the ship.

The theory elaborated by Schijf(1) [5] is based on the combination of Bernoulli's theorem with the principle of continuity. It supposes that the vessel sails at a constant speed V in a waterway with cross sectional area Ω at a constant water depth h (at rest) and width W. Further assumptions by Schijf were:

- the squat is proportional to the water level drop (basically the hydrostatics of the ship remain, but as the water level drops, the ship will endure a vertical motion known as running sinkage or squat);
- the return flow is uniformly distributed along the entire cross section excluding the area taken by the vessel, if present;
- viscous and turbulence effects are neglected.

When the vessel sails with a speed over ground V on a straight course along the centreline of such a rectangular fairway, a return flow will be initiated at each longitudinal position, so that the velocity of the water at the entire section is increased by δV. δV being the return flow in the opposite direction of the velocity of the ship. Because of the increased water speed along this section (combination of V and δV), the pressure will decrease (Bernoulli principle), causing a decrease of the water level over a vertical distance ζ proportional to the pressure drop (Figure 1).

1 Schijf used other symbols, that are presently no longer in use. In order to facilitate comparison the following is valid: $F=\Omega$; $f=A_{\mathrm{M}}$; $B=W$; $z=\zeta$; $v=V$; $u=\delta V$; $v+u=V_1$; $s=V^2/(2g)$; $\frac{f}{F}=m$. Section 5 explains the meaning of the present symbols.

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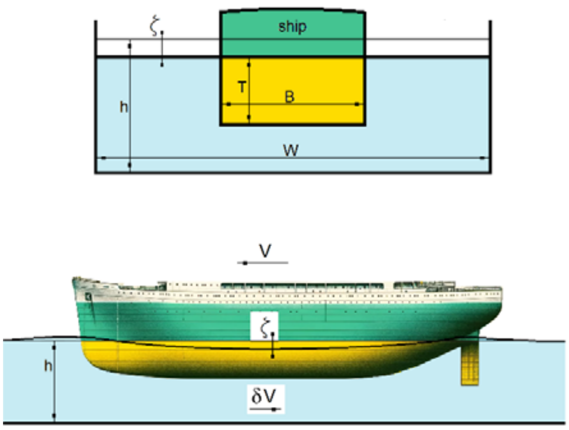

Figure 1 Overview of channel parameters (subcritical speed).

The conservation of mass at the midship can be written as

$$\Omega V = V_1(\Omega - A_M - W\zeta) \qquad (1)$$

with

$$V_1 = V + \delta V \qquad (2)$$

Equation (1) assumes that the free surface sinks over the full width of the canal section. Figure 1 shows a level drop ζ, while the section of the vessel at this position sinks over the same distance. This decrease of the water depth ζ can be calculated with Bernoulli's Law:

$$\zeta = \frac{1}{2g}\left({V_1}^2 - V^2\right) = \frac{V^2}{2g}\left(\left(\frac{V_1}{V}\right)^2 - 1\right) \qquad (3)$$

The level drop ζ can be eliminated from the continuity equation (1):

$$\zeta = \frac{\Omega}{W} - \frac{A_M}{W} - \frac{\Omega}{W}\frac{V}{V_1} = h\left(1 - \frac{A_M}{\Omega} - \frac{V}{V_1}\right)$$

$$= h\left(1 - m - \frac{1}{\frac{V_1}{V}}\right) \qquad (4)$$

The ratio $\frac{A_M}{\Omega}$ is known as the blockage ratio m, equation (3) can now be written as:

$$\frac{1}{2g}\left({V_1}^2 - V^2\right) = h\left(1 - m - \frac{1}{\frac{V_1}{V}}\right) \qquad (5)$$

or

$$\frac{1}{2}\frac{V^2}{gh}\left(\left(\frac{V_1}{V}\right)^2 - 1\right) + \frac{V}{V_1} + m - 1 = 0 \qquad (6)$$

Substituting the water depth dependent Froude number Fr_h and multiplying by $\frac{V_1}{V}$ in equation (6):

$$\frac{1}{2}{Fr_h}^2\left(\frac{V_1}{V}\right)^3 - \left(\frac{1}{2}{Fr_h}^2 + 1 - m\right)\frac{V_1}{V} + 1 = 0 \qquad (7)$$

Equation (7) has always three solutions for V_1:

- One solution is always real and negative, and has no (physical) meaning.
- When the blockage factor m is less than a critical blockage factor m_{crit}, both remaining solutions are real and positive; for larger values of m these solutions are complex conjugated numbers which means that no steady solution exists.

The two physically realistic solutions correspond to the earlier introduced critical flow speeds. The smallest solution $(V < V_1)$ corresponds to the first critical speed and is accompanied by a water level drop ζ, whereas the largest solution $(V > V_1)$ requires a water level rise ζ instead and corresponds to the second critical speed.

The equation for the critical blockage factor was plotted as an envelope by Schijf (see Fig. 2), including a correct mathematical equation for that envelope, but without further explanation on how this was to be obtained. In the notations used in this paper, the critical blockage is:

$$m_{crit} = 1 - \frac{3}{2}\,{Fr_h}^{\frac{2}{3}} + \frac{1}{2}Fr_h^2 \qquad (8)$$

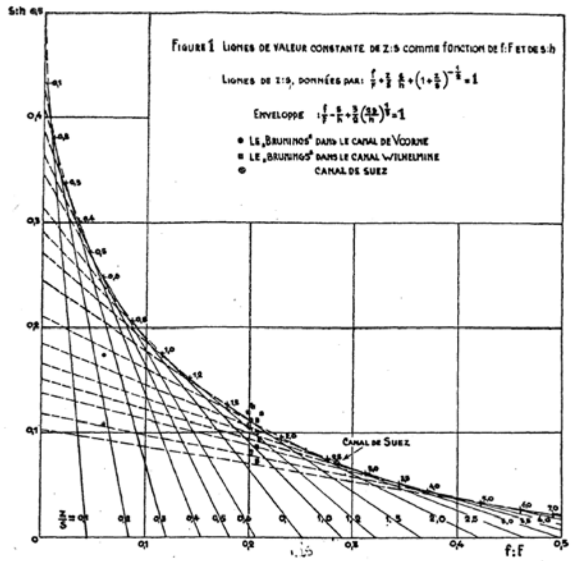

Figure 2 Figure 1 from [5]: lines of constant $\frac{z}{s} = \zeta/(\frac{V^2}{2g})$ as a function of $\frac{f}{F} = m$ and $\frac{s}{h} = \frac{1}{2}Fr_h^2$. The shown envelope is $\frac{f}{F} - \frac{s}{h} + \frac{3}{2}\left(\frac{2s}{h}\right)^{\frac{1}{3}} = 1 = \mathrm{m} - \frac{1}{2}{Fr_h}^2 + \frac{3}{2}{Fr_h}^{\frac{2}{3}}$.

With this critical blockage two critical Froude depth numbers can be associated, which can be expressed as:

$$Fr_{h,crit1} = \left(2\sin\left[\frac{\arcsin(1-m)}{3}\right]\right)^{3/2} \qquad (9)$$

$$Fr_{h,crit2} = \left(2\sin\left[\frac{\pi - \arcsin(1-m)}{3}\right]\right)^{3/2} \qquad (10)$$

corresponding to the two real and positive solutions. The first expression gives the boundary between subcritical and transcritical speed, whereas the second one between transcritical and supercritical speed.

Although these expressions are always referred to as discovered by Schijf [5], there is no mention of these in [5]. Nor is there an explanation on how to derive (8). The goal of this paper is to prove that the above equations are the correct expressions. Moreover, approximated expressions are included as well.

2. Mathematical elaboration

2.1 Historical overview

In order to trace back the derivations of the above formulations, an USACE manual [6] states that "*The value of the Schijf limiting Froude number can be calculated using the following explicit formula (Huval 1980, Balanin et al. 1977, Zernov et al. 1970).*" [3,1,7]. The oldest reference by Zernov [7] could not be found by the authors. The second oldest reference, Balanin et al. [1] (where Zernov appears as co-author) gives a solution of the Bernoulli – continuity combination, including friction losses.

The maximal speed in a canal, which they call *stalling speed*, is equal to, neglecting the friction losses introduced and using the symbols of the present paper:

$$V_{crit1} = \sqrt{gh}\sqrt{8\cos^3\left(\frac{\pi+\arccos(1-m)}{3}\right)} \qquad (11)$$

Mind that also Balanin et al. [1] did not indicate how this expression is obtained. This expression can also be written in a similar format as previously:

$$Fr_{h,crit1} = \left(2\cos\left[\frac{\pi+\arccos(1-m)}{3}\right]\right)^{3/2} \qquad (12)$$

Although the expression (12) looks different from (9) it will be shown that it leads to the same result.

A report from Delft Hydraulics Laboratory (nowadays known as Deltares) [2] also discusses the paper of Schijf [5]. They explicitly mention that the solution was derived by Maus [4] and Balanin et al. [1]. As already stated Balanin et al. [1] provided a solution, however, without derivation. Maus [4] though provides more information. Zernov [7] is not mentioned.

According to Maus [4], equation (7) can be solved based on the solutions for a so-called depressed cubic equation:

$$x^3 + px + q = 0 \qquad (13)$$

To have three real roots, the discriminant of this equation has to be positive:

$$\Delta = -(4p^3 + 27q^2) \geq 0 \qquad (14)$$

The combination (7) and (14) yields:

$$p = -\frac{\frac{1}{2}Fr_h{}^2 + 1 - m}{\frac{1}{2}Fr_h{}^2} = -1 - 2Fr_h{}^{-2}(1-m) \qquad (15)$$

$$q = \frac{1}{\frac{1}{2}Fr_h{}^2} = 2Fr_h{}^{-2} \qquad (16)$$

The step by step solution is not shown by Maus [4], but is straightforward:

$$\begin{aligned}
\Delta = \left(-4\left(1+2Fr_h{}^{-2}(1-m)\right)^3 + 27\left(2Fr_h{}^{-2}\right)^2\right) &\leq 0 \\
-4\left(1+2Fr_h{}^{-2}(1-m)\right)^3 &\leq -4\cdot 27\left(Fr_h{}^{-2}\right)^2 \\
\left(1+2Fr_h{}^{-2}(1-m)\right)^3 &\geq 27Fr_h{}^{-4} \\
1+2Fr_h{}^{-2}(1-m) &\geq 3Fr_h{}^{-4/3} \\
2Fr_h{}^{-2}(1-m) &\geq 3Fr_h{}^{-\frac{4}{3}} - 1 \\
1-m &\geq \frac{3Fr_h{}^{-\frac{4}{3}} - 1}{2Fr_h{}^{-2}} \\
-m &\geq \frac{3}{2}Fr_h^{\frac{2}{3}} - \frac{1}{2}Fr_h^2 - 1 \\
m &\leq 1 - \frac{3}{2}Fr_h^{\frac{2}{3}} + \frac{1}{2}Fr_h^2 \\
m_{\mathrm{crit}} &= 1 - \frac{3}{2}Fr_h^{\frac{2}{3}} + \frac{1}{2}Fr_h^2
\end{aligned}$$

which proves the envelope equation of Schijf [5].

Maus states further that it is convenient to use complex numbers to show that the critical speeds are (rewritten with the symbols of the present paper):

$$Fr_{h,crit1} = 2\left(2\cos\left[\frac{4\pi+\arccos(1-m)}{3}\right]\right)^{3/2} \qquad (17)$$

$$Fr_{h,crit2} = 2\left(2\cos\left[\frac{\arccos(1-m)}{3}\right]\right)^{3/2} \qquad (18)$$

However, again no further information was provided.

In summary, the original theory was set forth by Schijf [5], yet he only gave an expression for the critical blockage (without proof). A first appearance of the above expressions was given by Maus [4], including an incomplete proof. Although Zernov [7] is an older reference, the authors could not trace it back and cannot provide credits for it.

2.2 Proof of the correct solutions

Goniometric expressions are used to find a goniometric solution of the critical speed. The authors assume that the readers are comfortable with goniometric expressions and that it is straightforward to show the validity of a goniometric relationship like:

$$\sin(3\alpha) = 3\sin\alpha - 4\sin^3\alpha \tag{19}$$

The choice of the above example is not arbitrary as it seems a good candidate to find the critical speed. Assuming that α is defined so that:

$$\sin\alpha = \frac{Fr_h^{\frac{2}{3}}}{2} \Leftrightarrow \alpha = \arcsin\frac{Fr_h^{\frac{2}{3}}}{2} \tag{20}$$

Mind that, because of the periodic nature of harmonic functions, this is only one possible solution. Then,

$$\sin(3\alpha) = 3\frac{Fr_h^{\frac{2}{3}}}{2} - \frac{1}{2}Fr_h^2 \tag{21}$$

and based on (8):

$$m_{crit} = 1 - \sin(3\alpha) \tag{22}$$

or

$$m_{crit} = 1 - \sin\left(3\arcsin\frac{Fr_h^{\frac{2}{3}}}{2}\right) \tag{23}$$

This expression is equivalent to equation (8), but makes it easier to derive the expression for the lowest critical Froude depth number:

$$\sin\left(3\arcsin\frac{Fr_h^{\frac{2}{3}}}{2}\right) = 1 - m$$

$$3\arcsin\frac{Fr_h^{\frac{2}{3}}}{2} = \arcsin(1-m)$$

$$\frac{Fr_h^{\frac{2}{3}}}{2} = \sin\left[\frac{\arcsin(1-m)}{3}\right]$$

$$Fr_{h,\mathrm{crit1}} = \left(2\sin\left[\frac{\arcsin(1-m)}{3}\right]\right)^{3/2}$$

The second critical Froude number follows from the observation that besides (20)

$$\sin\alpha = \frac{Fr_h^{\frac{2}{3}}}{2} \Leftrightarrow \alpha = \pi - \arcsin\frac{Fr_h^{\frac{2}{3}}}{2} \tag{24}$$

is also a valid solution.

An alternative to (19) is:

$$\cos(3\alpha) = -3\cos\alpha + 4\cos^3\alpha \tag{25}$$

with following possible solutions:

$$\cos\alpha = \frac{Fr_h^{\frac{2}{3}}}{2} \Leftrightarrow \alpha = \arccos\frac{Fr_h^{\frac{2}{3}}}{2}$$

$$\cos\alpha = \frac{Fr_h^{\frac{2}{3}}}{2} \Leftrightarrow \alpha = \pi + \arccos\left(-\frac{Fr_h^{\frac{2}{3}}}{2}\right)$$

$$\cos(3\alpha) = -3\frac{Fr_h^{\frac{2}{3}}}{2} + \frac{1}{2}Fr_h^2$$

$$m_{\mathrm{crit}} = 1 + \cos(3\alpha)$$

$$m_{crit} = 1 + \cos\left(3\arccos\frac{Fr_h^{\frac{2}{3}}}{2}\right) \tag{26}$$

yielding

$$\cos\left(3\arccos\frac{Fr_h^{\frac{2}{3}}}{2}\right) = m - 1$$

$$3\arccos\frac{Fr_h^{\frac{2}{3}}}{2} = \arccos(m-1)$$

$$\frac{Fr_h^{\frac{2}{3}}}{2} = \cos\left[\frac{\arccos(m-1)}{3}\right]$$

$$Fr_{h,crit2} = \left(2\cos\left[\frac{\arccos(m-1)}{3}\right]\right)^{3/2} \tag{27}$$

Representing the second critical Froude number this time and for the first:

$$Fr_{h,crit1} = \left(2\cos\left[\frac{\pi+\arccos(1-m)}{3}\right]\right)^{3/2} \tag{28}$$

which proves and shows the equivalency of the expression used by Balanin et al. [1]. It is straightforward to see that due to the periodicity of goniometric functions multiple expressions can be found.

3. Approximate formulations of the critical speeds

Although the elegancy of the solution for the critical speeds, the formula by itself does not show a clear relationship between critical speed and blockage. Obviously the critical speed ranges grow apart from each other with increasing blockage. As it is not possible to find an explicit (polynomial) solution to (8) for Fr_h a McLaurin series expansion could be tried, however, the magnitude of the first derivative of $Fr_{h,\mathrm{crit1}}$ with m is driven by the magnitude of the derivative of the $\arcsin$ function:

$$\lim_{m\to 0}\frac{dFr_{h,crit1}}{dm} \propto \lim_{m\to 0}\frac{1}{\sqrt{1-(1-m)^2}} = \infty \tag{29}$$

which explains why the slightest blockage has a significant effect on the first critical Froude number, but also that an expansion of the critical speed with polynomials is not straightforward.

A function that has a similar behaviour is $f = \sqrt{m}$:

$$\lim_{m\to 0}\frac{df}{dm} = \lim_{m\to 0}\frac{1}{\sqrt{m}} = \infty \qquad (30)$$

A square root of m indeed delivers a good trend for small blockages as can be seen in Figure 3. In this figure, the computed values have been normalized, so that the origin corresponds to the open water condition, on the left hand side the subcritical range is shown and on the right hand side the supercritical range.

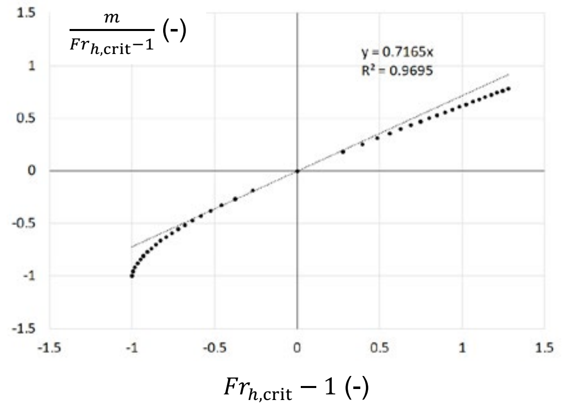

Figure 3 Relationship between blockage and critical Froude number, elaborated based on the goniometric relationships in steps of 0.05.

From Figure 3 it seems that for small blockages ($m < 0.25$), the following relationship can be used:

$$\frac{m}{Fr_{h,crit}-1} \approx 0.7165\left(Fr_{h,crit}-1\right)$$
$$\rightarrow \left(Fr_{h,crit}-1\right)^2 \approx \pm 1.4m \qquad (31)$$

Given the fact that 1.4 approaches $\sqrt{2}$, the following approximation can be established:

$$Fr_{h,crit1,2} \approx 1 \mp \sqrt{\sqrt{2}m} \qquad (32)$$

For larger blockages a correction is needed, that is different for subcritical and supercritical range. In the subcritical range the offset has a higher order with the blockage, compared to the supercritical range. In the former it seems that a parabolic correction is needed, whereas for the latter a linear correction seems sufficient. This suggests:

$$Fr_{h,crit1} \approx 1 - \sqrt{\sqrt{2}m + \gamma m^{2.5}} \qquad (33)$$

$$Fr_{h,crit2} \approx 1 + \sqrt{\sqrt{2}m + \beta m^{1.5}} \qquad (34)$$

The addition of these higher order terms also imply that their influence for small m is negligible. The values for γ and β can be found based on closure considerations, namely:

$$Fr_{h,crit1}(m=1) = 0$$

$$\rightarrow Fr_{h,crit1} = 1 - \sqrt{\sqrt{2}m + \left(1-\sqrt{2}\right)m^{2.5}} \qquad (35)$$

$$Fr_{h,crit2}(m=1) = 3^{3/4}$$
$$\rightarrow = 1 + \sqrt{\sqrt{2}m + \left(\left(3^{\frac{3}{4}}-1\right)^2 - \sqrt{2}\right)m^{1.5}} \qquad (36)$$

Figure 4 shows that the above approximations perform very well over the entire range of blockages.

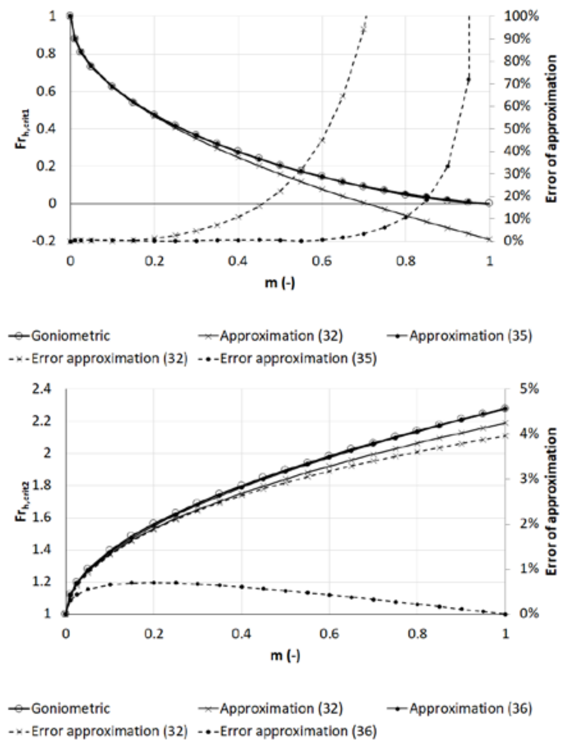

Figure 4 Comparison between goniometric and approximated critical Froude numbers as a function of the blockage. Top for subcritical speeds, bottom for supercritical speeds.

4. Conclusions

The concept of critical flow is commonly used to explain the changed behaviour of ships in channels and well known goniometric expressions exist that relate the first (and second) critical speed to the ship's speed and the blockage of the ship in a given cross section. Over the past decades these expressions were usually attributed to the research performed by Schijf, now 75 years ago. Revisiting his original publication revealed that, although he presented a formula for the critical blockage, he did not show any expression for critical speeds.

For that reason the authors tried to trace back these expressions in history, however, without finding any document – openly available in literature – that gives a full elaboration. Credit for such expressions can certainly be given to Zernov, Balanin or Maus, but the original, full credit cannot be awarded.

The present paper provided a proof for the commonly well-known geometric expressions for the critical speed, without claiming any credit for it, but the work to put the state-of-the-art in this article. It is thus a proven fact that the first critical speed can be written as (while other alternatives exist as well):

$$Fr_{h,\mathrm{crit1}} = \left(2\sin\left[\frac{\arcsin(1-m)}{3}\right]\right)^{3/2}$$

$$Fr_{h,crit1} = \left(2cos\left[\frac{\pi + arccos(1-m)}{3}\right]\right)^{3/2}$$

and the second:

$$Fr_{h,crit2} = \left(2\,sin\left[\frac{\pi - arcsin(1-m)}{3}\right]\right)^{3/2}$$

$$Fr_{h,crit2} = \left(2\,cos\left[\frac{arccos(m-1)}{3}\right]\right)^{3/2}$$

While enjoying the math, the authors also delivered approximate expressions, that in present days of computation power have become less relevant, but still allow a better understanding between critical speed and blockage for human intelligence. Up to realistic blockages, $m < 0.25$, the following approximations are valid:

$$Fr_{h,\mathrm{crit1,2}} \approx 1 \mp \sqrt{\sqrt{2}m}$$

whereas for the full range of blockages:

$$Fr_{h,\mathrm{crit1}} = 1 - \sqrt{\sqrt{2}m + (1-\sqrt{2})m^{2.5}}$$

$$Fr_{h,\mathrm{crit2}} = 1 + \sqrt{\sqrt{2}m + \left(\left(3^{\frac{3}{4}} - 1\right)^2 - \sqrt{2}\right)m^{1.5}}$$

5. Used symbols

A_{M}	midship's cross section	m²
Fr_h	Froude depth number	-
g	gravity acceleration constant	m/s²
h	water depth	m
m	blockage ratio	-
W	channel width	m
V	ship's velocity	m/s
V_1	return velocity	m/s
α	auxiliary angle	rad
ζ	water level drop	m
Ω	channel's cross section	m²

Subscripts

crit critical

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