SHEAR EFFECT DISPERSION IN A SHALLOW TIDAL SEA

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INTRODUCTION

The hydrodynamics of shallow continental seas like the North Sea is dominated by long waves, tides and storm surges, with current velocities of the order of 1 m/s. The currents generate strong three-dimensional turbulence and vertical mixing, resulting, in general, in a fairly homogeneous distribution of temperature, salinity and concentrations of marine constituents over the water column.

Vertical gradients of concentrations may exist in localized areas where vertical mixing is partly (and temporarily) inhibited by stratification or during short periods of time - a few hours following an off-shore dumping, for instance - before vertical mixing is completed. However such cases are very limited in space and time and, in most problems, it is sufficient to study, in a first approach, the horizontal distribution of depth-averaged concentrations.

If c denotes the concentration of a given constituant, the three-dimensional "dispersion" equation, describing the evolution of c in space and time, can be written (e.g. Nihoul, 1975)

$$\frac{\partial c}{\partial t} + \underset{\sim}{\nabla}.(c\underset{\sim}{v}) = Q + I - \underset{\sim}{\nabla}.(\underset{\sim}{\sigma}c) + D \qquad (1)$$

In eq. (1),

i) $\underset{\sim}{\nabla}.(c\underset{\sim}{v})$ represents advection and can be separated in two parts corresponding respectively to the horizontal transport $\underset{\sim}{\nabla}.(c\underset{\sim}{u})$ and to the vertical transport $\frac{\partial}{\partial x_3}(cv_3)$; $\underset{\sim}{u} = v_1\, \underset{\sim}{e}_1 + v_2\, \underset{\sim}{e}_2$ denoting the horizontal current velocity.

ii) Q represents the rate of production (or destruction) of the constituent by volume sources (or sinks).

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(*In most practical applications, inputs and outputs are located at the boundaries - in which case, they appear in the boundary conditions and not in Q -, or are localized quasi-instantaneous releases which may be conveniently taken into account in the initial conditions. In the following, one shall assume that this is the case and one shall set* $Q = 0$).

iii) I represents the rate of production (or destruction) of the constituent by (chemical, ecological,...) interactions inside the marine system and I is, in general, a function of coupled variables c' , c'',...

(*A marine constituent is said to be passive when its evolution is not affected by such interactions. In the following, to simplify the formulation, one shall restrict attention to passive constituents and set* $I = 0$. *The generalization of the theory to a system of interacting constituents presents no fundamental difficulty (Nihoul and Adam, 1977)*).

iv) $-\nabla.(\sigma c)$ represents "migration". (sedimentation, horizontal migration of fish,..., e.g. Nihoul, 1975).

(*Migration, at least in the most frequent case of sedimentation, can easily be taken into account (e.g. Nihoul and Adam, 1977). However, to avoid overloading the analysis, one shall assume, in the following, that the constituent is simply transported by the fluid and that the migration velocity* σ *is zero*).

v) D represents turbulent diffusion and can be separated into a vertical turbulent diffusion and a horizontal turbulent diffusion.

(*The horizontal turbulent diffusion is negligible as compared to the horizontal advection. The horizontal dispersion which is observed in the sea is mainly the result of the horizontal transport of the constituent by irregular and variable currents constituting a form of "pseudo horizontal turbulence" extending to much larger scales than the "proper" three-dimensional turbulence (e.g. Nihoul, 1975). In that case,* D *can be simply written*

$$D = \frac{\partial}{\partial x_3} \left(\mu \frac{\partial c}{\partial x_3}\right) \quad (2)$$

where μ *is the vertical turbulent diffusivity*).

In the scope of the hypotheses made above, eq.(1) can be written, in the simpler form

$$\frac{\partial c}{\partial t} + \nabla.(c\mathbf{u}) + \frac{\partial}{\partial x_3}(cv_3) = \frac{\partial}{\partial x_3}\left(\mu \frac{\partial c}{\partial x_3}\right) \quad (3)$$

The velocity field $\underset{\sim}{v} = \underset{\sim}{u} + v_3 \underset{\sim}{e}_3$ is given by the Boussinesq equations and in particular, one has

$$\underset{\sim}{\nabla} . \underset{\sim}{u} + \frac{\partial v_3}{\partial x_3} = 0 \qquad (4)$$

DEPTH-AVERAGED DISPERSION EQUATION

Let

$$\bar{c} = H^{-1} \int_{-h}^{\zeta} c \, dx_3 \quad ; \quad \hat{c} = c - \bar{c} \qquad (5);(6)$$

$$\bar{\underset{\sim}{u}} = H^{-1} \int_{-h}^{\zeta} \underset{\sim}{u} \, dx_3 \quad ; \quad \hat{\underset{\sim}{u}} = \underset{\sim}{u} - \bar{\underset{\sim}{u}} \qquad (7);(8)$$

with

$$\int_{-h}^{\zeta} \hat{c} \, dx_3 = 0 \quad ; \quad \int_{-h}^{\zeta} \hat{\underset{\sim}{u}} \, dx_3 = 0 \qquad (9);(10)$$

and

$$H = h + \zeta \qquad (11)$$

where h is the depth and ζ the surface elevation.

One has

$$\frac{\partial \zeta}{\partial t} + \underset{\sim}{u} . \underset{\sim}{\nabla} \zeta = v_3 \quad \text{at} \quad x_3 = \zeta \qquad (12)$$

$$\frac{\partial h}{\partial t} + \underset{\sim}{u} . \underset{\sim}{\nabla} h = - v_3 \quad \text{at} \quad x_3 = - h \qquad (13)$$

Integrating eqs.(1) and (4) over depth, inverting the order of integration with respect to x_3 and of derivation with respect to t, x_1 or x_2 and using eqs.(12) and (13) to eliminate the corrections due to the variable limits of integration, one obtains (e.g. Nihoul, 1975)

$$\frac{\partial}{\partial t} (H\bar{c}) + \underset{\sim}{\nabla} . (H \, \bar{c} \, \bar{\underset{\sim}{u}}) + \underset{\sim}{\nabla} . \int_{-h}^{\zeta} \hat{c} \, \hat{\underset{\sim}{u}} \, dx_3 = 0 \qquad (14)$$

$$\frac{\partial H}{\partial t} + \underset{\sim}{\nabla} . (H\bar{\underset{\sim}{u}}) = 0 \ . \qquad (15)$$

In the right-hand side of eq.(14), one should have the difference between the fluxes of the constituent at the free surface and at the bottom. The hypothesis is made here that there is no exchange between the water column and the atmosphere and between the water column and the bottom sediments.

In this case, combining eqs.(14) and (15), one gets

$$\frac{\partial \bar{c}}{\partial t} + \bar{\underset{\sim}{u}}.\underset{\sim}{\nabla}\bar{c} = \Sigma \tag{16}$$

where

$$\Sigma = H^{-1}\underset{\sim}{\nabla}.\int_{-h}^{\zeta} (-\,\hat{c}\hat{\underset{\sim}{u}})\,dx_3 \tag{17}$$

Σ contains the mean product of the deviations $\hat{c}$ and $\hat{\underset{\sim}{u}}$ around the mean values $\bar{c}$ and $\bar{\underset{\sim}{u}}$. The observations reveal that this term is responsible for a horizontal dispersion analogous to the turbulent dispersion but many times more efficient. This effect is called the "shear effect" because it is associated with the vertical gradient of the horizontal velocity $\underset{\sim}{u}$ (e.g. Bowden, 1965 ; Nihoul, 1975).

PARAMETERIZATION OF THE SHEAR EFFECT

Substracting eq.(16) from eq.(1), one obtains

$$\frac{\partial \hat{c}}{\partial t} + \bar{\underset{\sim}{u}}.\underset{\sim}{\nabla}\hat{c} + \hat{\underset{\sim}{u}}.\underset{\sim}{\nabla}\hat{c} + \Sigma + v_3\,\frac{\partial \hat{c}}{\partial x_3} + \hat{\underset{\sim}{u}}.\underset{\sim}{\nabla}\bar{c} = \frac{\partial}{\partial x_3}\left(\mu\,\frac{\partial \hat{c}}{\partial x_3}\right) \tag{18}$$

Because of the strong vertical mixing, one expects the deviation $\hat{c}$ to be much smaller than the mean value $\bar{c}$. This is not true for the velocity deviation $\hat{\underset{\sim}{u}}$ which may be comparable to $\bar{\underset{\sim}{u}}$; the velocity increasing from zero at the bottom to its maximum value at the surface. One may thus assume that the first four terms in the left-hand side of eq.(18) are negligible as compared to the sixth one $\hat{\underset{\sim}{u}}.\underset{\sim}{\nabla}\bar{c}$. The fifth term, representing vertical advection, is undoubtedly even smaller than the four neglected terms and eq.(18) reduces to

$$\hat{\underset{\sim}{u}}.\underset{\sim}{\nabla}\bar{c} = \frac{\partial}{\partial x_3}\left(\mu\,\frac{\partial \hat{c}}{\partial x_3}\right) \tag{19}$$

The physical meaning of this equation is clear : weak vertical inhomogeneities are constantly created by inhomogeneous convective transport and they adapt to that transport in such a way that the effects of advection and turbulent diffusion are in equilibrium for them.

Integrating eq.(19) with the condition that the flux is zero at the free surface, one obtains

$$H\underset{\sim}{f}.\underset{\sim}{\nabla}\bar{c} = \mu \frac{\partial \hat{c}}{\partial x_3} \tag{20}$$

where

$$\underset{\sim}{f} = H^{-1}\int_{\zeta}^{x_3} \underset{\sim}{\hat{u}}\, dx_3 \tag{21}$$

Integrating by parts and taking into account that $\underset{\sim}{f} = 0$ at $x_3 = \zeta$ and $x_3 = -h$ (cfr eq.10), one gets

$$\Sigma = H^{-1}\underset{\sim}{\nabla}.(H\, \underset{\approx}{R}\, .\underset{\sim}{\nabla}\bar{c}) \tag{22}$$

where R is the shear effect diffusivity tensor, i.e. :

$$\underset{\approx}{R} = H \int_{-h}^{\zeta} \frac{\underset{\sim}{f}\ \underset{\sim}{f}}{\mu}\, dx_3 \tag{23}$$

To determine $\underset{\approx}{R}$, one must know the turbulent eddy diffusivity μ and the function $\underset{\sim}{f}$, i.e. the velocity deviation $\underset{\sim}{\hat{u}}$.

VERTICAL PROFILE OF THE HORIZONTAL VELOCITY

The evolution equation for the horizontal velocity vector $\underset{\sim}{u}$ can be written, after eliminating the pressure (e.g. Nihoul, 1975)

$$\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{\nabla}.(\underset{\sim}{u}\ \underset{\sim}{u}) + f\underset{\sim}{e}_3 \Lambda\ \underset{\sim}{u} + \frac{\partial}{\partial x_3}(v_3\underset{\sim}{u}) = -\underset{\sim}{\nabla}\left(\frac{p_a}{\rho} + g\zeta\right) + \frac{\partial}{\partial x_3}\left(\nu \frac{\partial \underset{\sim}{u}}{\partial x_3}\right) \tag{24}$$

where f is equal to twice the vertical component of the earth's rotation vector, p_a is the atmospheric pressure, g the acceleration of gravity and ν the vertical turbulent viscosity.

In eq.(24), one has neglected the effect of the horizontal component of the earth's rotation vector (multiplied by $v_3 << u$) and the horizontal turbulent diffusion (because horizontal length scales are always much larger than the depth).

The observations indicate that, in shallow tidal seas, the turbulent viscosity ν can be written as the product of a function of t, x_1 and x_2 and a function of the reduced variable $\xi = H^{-1}(x_3 + h)$ (e.g. Bowden, 1965).

Let

$$\nu = H^2\sigma(t,x_1,x_2)\lambda(\xi) \tag{25}$$

where σ and λ are appropriate functions.

The asymptotic form of ν for small values of ξ is well-known from boundary layer theory :

$$\nu = k\ u_*(x_3 + h) = k\ u_* H\xi \tag{26}$$

where k is the Von Karman constant and u_* the friction velocity given by

$$u_*^2 = \| \underset{\sim}{\tau}_b \| \quad ; \quad \underset{\sim}{\tau}_b = \left\{ \nu \frac{\partial \underset{\sim}{u}}{\partial x_3} \right\}_{x_3=-h} \qquad (27);(28)$$

Hence

$$\sigma H = k u_* \tag{29}$$

and

$$\lambda(\xi) \sim \xi \quad \text{for} \quad \xi << 1 . \tag{30}$$

In a well-mixed shallow sea, where the Richardson number is small and the turbulence fully developed, it is reasonable (e.g. Nihoul, 1975) to take

$$\mu \sim \nu . \tag{31}$$

This hypothesis will be reexamined later.

It is convenient to change variables to (t, x_1, x_2, ξ) in eq.(24) In the final result (Nihoul, 1977), the non-linear terms combine with additional contributions from the time derivative to give three terms, related respectively to the gradients of velocity, depth and surface elevation. These terms are found negligible almost everywhere in the North Sea (Nihoul and Runfola, 1979). Thus although depth-integrated two-dimensional hydrodynamic models of the North Sea may not discard the non-linear terms[1], if one excludes localized singular regions like the vicinity of tidal emphydromic points, the "local" vertical distribution of velocity may be described, with a very good approximation, by a linear model.

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1. It can be shown that these terms are essential in determining the residual circulation (Nihoul and Ronday, 1976b).

Then, the governing equation for the velocity deviation $\hat{u}$ can be written

$$\frac{\partial \hat{\underset{\sim}{u}}}{\partial t} + f \underset{\sim}{e}_3 \wedge \hat{\underset{\sim}{u}} = \sigma \left\{ \frac{\partial}{\partial \xi} \left(\lambda \frac{\partial \hat{\underset{\sim}{u}}}{\partial \xi} \right) - \frac{\underset{\sim}{\tau}_s - \underset{\sim}{\tau}_b}{\sigma H} \right\} \tag{32}$$

where

$$\underset{\sim}{\tau}_s = \left(\nu \frac{\partial \underset{\sim}{u}}{\partial x_3} \right)_{x_3 = \zeta} \tag{33}$$

is the wind stress (normalized with water density).

It is possible to find an analytical solution of eq.(32) giving $\hat{\underset{\sim}{u}}$ in terms of $\underset{\sim}{\tau}_s$, $\underset{\sim}{\tau}_b$ and their derivatives with respect to time ; the coefficients depending on the functions $s(\xi)$, $b(\xi)$ and $f_n(\xi)$ $(n=1,2,..)$ defined by (Nihoul, 1977)

$$s(\xi) = \int_0^{\xi} \frac{\eta}{\lambda(\eta)} \, d\eta \tag{34}$$

$$b(\xi) = \int_{\xi_0}^{\xi} \frac{1 - \eta}{\lambda(\eta)} \, d\eta \tag{35}$$

$$\frac{d}{d\xi} \left(\lambda \frac{df_\eta}{d\xi} \right) = - \alpha_\eta f_\eta \tag{36}$$

with

$$\int_0^1 f_\eta^2(\xi) \, d\xi = 1 \tag{37}$$

$$\lambda \frac{df_\eta}{d\xi} = 0 \quad \text{at} \quad \xi = 0 \quad \text{and} \quad \xi = 1 \, . \tag{38}$$

One should note here that, in the definition of $b(\xi)$, the lower limit of integration is not set equal to zero but to $\xi_0 = \frac{z_0}{H} << 1$ where z_0 is the "rugosity length". z_0 can be interpreted as the distance above the bottom where the velocity is conventionally set equal to zero, ignoring the intricated flow situation which occurs near the irregular sea floor and willing to parameterize its effect on the turbulent boundary layer as simply as possible. In the North Sea, the value of z_0, which varies according to the nature of the bottom, is of the order of 10^{-3}m ($\ln \xi_0 \sim - 10$) (e.g. Nihoul and Ronday, 1976a).

Although $\xi_o << 1$, it cannot be systematically put equal to zero because the linear variation of the vertical eddy viscosity near the bottom leads to a logarithmic velocity profile which is singular at $\xi = 0$. However, in the present description, the difficulty exists only for the function b and the eigenfunction $f(\xi)$ may be determined on the interval $[0,\xi]$.

In a shallow tidal sea like the North Sea, it is readily seen, comparing the orders of magnitude of the different terms, that one obtains a very good approximation with only the first two terms in the series expansion of $\underset{\sim}{\hat{u}}$, i.e. (Nihoul, 1977)

$$\underset{\sim}{\hat{u}} = \underset{\sim}{v}_s\left(s(\xi) - \bar{s}\right) + \underset{\sim}{v}_b\left(b(\xi) - \bar{b}\right) - \left(\frac{s_1}{\alpha_1\sigma}\dot{\underset{\sim}{v}}_s + \frac{b_1}{\alpha_1\sigma}\dot{\underset{\sim}{v}}_b\right) f_1(\xi) \qquad (39)$$

where $\bar{s}$ and $\bar{b}$ are the depth-averages of s and b, s_1 and b_1 two numerical coefficients ($s_1 = \int_0^1 s\, f_1\, d\xi$; $b_1 = \int_0^1 b\, f_1\, d\xi$) ; α_1 the eigenvalue corresponding to $f_1(\xi)$ and where

$$\underset{\sim}{v}_s = \frac{\underset{\sim}{\tau}_s}{\sigma H} \quad ; \quad \underset{\sim}{v}_b = \frac{\underset{\sim}{\tau}_b}{\sigma H} \qquad (40);(41)$$

A dot denotes here a total derivative with respect to time

$$\left(\dot{\underset{\sim}{v}}_s = \frac{d\underset{\sim}{v}_s}{dt} = \frac{\partial \underset{\sim}{v}_s}{\partial t} + f\, \underset{\sim}{e}_3 \wedge \underset{\sim}{v}_s \text{ and similarly for } \underset{\sim}{v}_b\right).$$

Knowing the function $\lambda(\xi)$, one can determine $\underset{\sim}{\hat{u}}$ by eq.(39), $\underset{\sim}{\mathfrak{f}}$ by eq.(21) and $\underset{\approx}{R}$ by eq.(23).

PARAMETERIZATION OF THE BOTTOM STRESS

The functions $\underset{\sim}{\hat{u}}$, $\underset{\sim}{\mathfrak{f}}$ and $\underset{\approx}{R}$ depend on the vectors $\underset{\sim}{v}_s$, $\underset{\sim}{v}_b$ and their derivatives. These can be determined by eqs.(27), (29), (40) and (41) from $\underset{\sim}{\tau}_s$ and $\underset{\sim}{\tau}_b$.

The surface stress $\underset{\sim}{\tau}_s$ can be calculated from atmospheric data, the bottom stress $\underset{\sim}{\tau}_b$ is not given and must be determined by the no-slip condition at the bottom, i.e.

$$\underset{\sim}{\hat{u}} = -\bar{\underset{\sim}{u}} \quad \text{at} \quad \xi = \xi_o \qquad (42)$$

Eq.(42) provides a differential equation for τ_b in terms of τ_s and $\bar{u}$.

In shallow tidal seas like the North Sea, the terms including $\dot{v}_s$ and $\dot{v}_b$ are generally negligible and can only play a part during a relatively short time, at tide reversal (Nihoul and Runfola, 1979). The dominant term is, in fact, the term containing the bottom stress. The effect of the wind stress appears as a first order correction and the "memory" effect involving the derivatives $\dot{v}_s$ and $\dot{v}_b$ as a second order correction. One thus has

$$\bar{u} \sim \bar{b}\, v_b \qquad \text{(zeroth order)} \tag{43}$$

$$\bar{u} \sim \bar{b}\, v_b + \bar{s}\, v_s \qquad \text{(first order)} \tag{44}$$

$$\bar{u} \sim \bar{b}\, v_b + \bar{s}\, v_s + (s_1 \dot{v}_s + b_1 \dot{v}_b)\, \frac{f_{1,0}}{\alpha_1 \sigma} \qquad \text{(second order)} \tag{45}$$

Eq.(43) yields the well-known semi-empirical quadratic bottom friction law. Indeed, combining eqs.(27), (29) and (43), one finds

$$\| \tau_b \| \sim \frac{\sigma H}{\bar{b}} \| \bar{u} \| \sim \left(\frac{\sigma H}{k}\right)^2 \Rightarrow \sigma H \sim \frac{k^2}{\bar{b}} \| \bar{u} \| \tag{46}$$

i.e.

$$\tau_b = \frac{k^2}{\bar{b}^2} \| \bar{u} \| \bar{u} \tag{47}$$

$\frac{k^2}{\bar{b}^2}$ is the so-called "drag coefficient".

At the first order, one gets another classical formula (e.g. Groen and Groves, 1966 ; Nihoul, 1975) :

$$\tau_b = \frac{k^2}{\bar{b}^2} \| \bar{u} \| \bar{u} - \frac{\bar{s}}{\bar{b}}\, \tau_s \tag{48}$$

The second order parameterization is better understood if the last term in the right-hand side of eq.(45) is eliminated using eqs.(32), (34), (35), (36) and (39). One has, indeed

$$\frac{\partial \bar{u}}{\partial t} + f\, e_3 \wedge \bar{u} = - \sigma \left\{ \frac{\partial}{\partial \xi} \left(\lambda \frac{\partial \hat{u}}{\partial \xi} \right) \right\}_{x_3 = -h} + \frac{\tau_s - \tau_b}{H}$$

$$\sim - (s_1 \dot{v}_s + b_1 \dot{v}_b)\, f_{1,0} \sim - \alpha_1 \sigma\, (\bar{u} - \bar{s}\, v_s - \bar{b}\, v_b)$$

i.e., using (46) to estimate σH ,

$$\underset{\sim}{\tau}_b \sim \frac{k^2}{\bar{b}^2} \|\bar{\underset{\sim}{u}}\| \bar{\underset{\sim}{u}} - \frac{\bar{s}}{\bar{b}} \underset{\sim}{\tau}_s + \frac{H}{\alpha_1 \bar{b}} \left(\frac{\partial \bar{\underset{\sim}{u}}}{\partial t} + f \underset{\sim}{e}_3 \wedge \bar{\underset{\sim}{u}} \right)$$

With a typical drag coefficient of the order of $2\ 10^{-3}$ (e.g. Nihoul and Ronday, 1976), one finds, using characteristic values for the North Sea,

$$\frac{k^2}{\bar{b}^2} \|\bar{\underset{\sim}{u}}\| \bar{\underset{\sim}{u}} \sim O(2\ 10^{-3} \bar{u}^2) ,$$

$$\frac{H}{\alpha_1 \bar{b}} \frac{\partial \bar{\underset{\sim}{u}}}{\partial t} \sim \frac{H}{\alpha_1 \bar{b}} f \underset{\sim}{e}_3 \wedge \bar{\underset{\sim}{u}} \sim O(10^{-4} \bar{u}) .$$

Thus, the "acceleration" terms containing $\frac{\partial \bar{\underset{\sim}{u}}}{\partial t}$ and $f \underset{\sim}{e}_3 \wedge \bar{\underset{\sim}{u}}$ (i.e. the terms arising from the time derivatives of $\underset{\sim}{v}_s$ and $\underset{\sim}{v}_b$) are not expected to play an important role except perhaps during relatively short periods of weak currents (when tides reverse, for instance). The total effect of the acceleration terms depends really on the local conditions. Obviously, (fig. 1) if the current velocity vector rotates clockwise during a tidal period, the two terms tend to oppose each other and their global contribution may remain always very small. On the other hand, if the current velocity vector rotates counter-clockwise (as it is often the case in the Southern Bight) the two terms reinforce each other and have a definite - although limited - effect on the velocity profile and the related relationship between the bottom stress and the mean velocity (fig. 2, fig. 3).

Most hydrodynamic models of shallow seas restrict attention to the determination of the depth-mean velocity $\bar{\underset{\sim}{u}}$. The two-dimentional time dependent evolution equation for $\bar{\underset{\sim}{u}}$ is obtained from eq.(24) by integration over depth. It includes the surface elevation ζ and the bottom stress $\underset{\sim}{\tau}_b$ and thus constitutes with eq.(15) and eq.(43) (or 44) a closed system for the depth-averaged circulation (e.g. Nihoul, 1975).

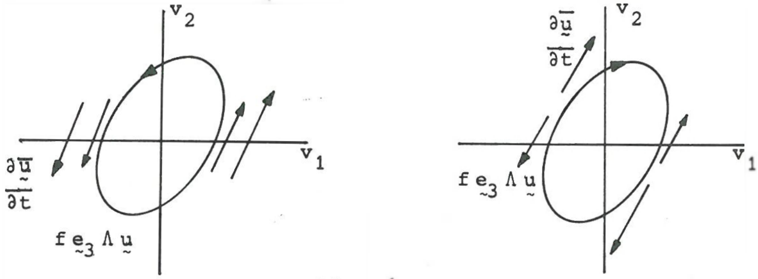

Fig. 1.

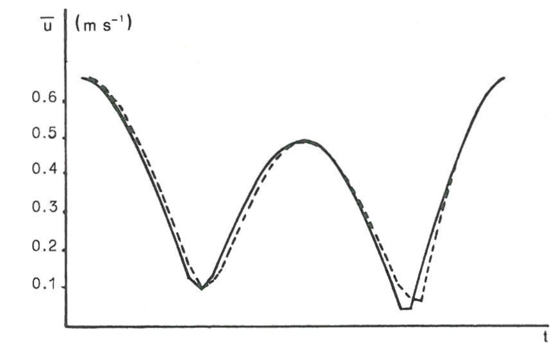

Fig. 2. Comparison between the magnitude of the mean velocity $\bar{u}$ computed at the test point 52°30'N, 3°50'E in the North Sea by a depth-averaged two-dimensional model using an algebraic parameterization of τ_b (full line) and by the three-dimensional model subject to the condition of zero velocity at the bottom (dash line) (Nihoul and Runfola, 1979).

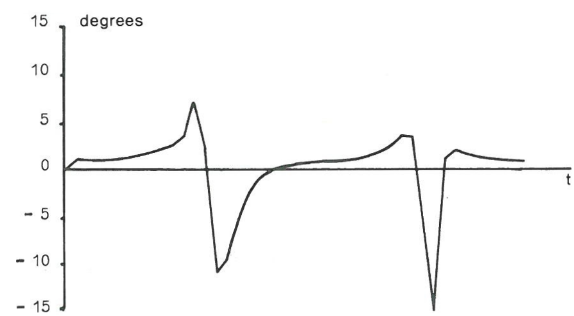

Fig. 3. Difference between the directions of the mean velocity vectors computed as in Fig. 2. (Nihoul and Runfola, 1979).

The determination of the vertical velocity profile can be carried simultaneously using eq.(39).

One can go a step further and devise a three-dimensional model based on the depth-averaged equation, the local depth-dependent equation for the velocity deviation $\hat{u}$ and the refined parameterization of the bottom stress given by eq.(49) ; the acceleration corrections being taken into account when required in the numerical calculation (Nihoul and Runfola, 1979).

The model gives H, $\bar{u}$, τ_b , $\hat{u}$, $\hat{r}$ and R .

Substituting in eq.(16) one obtains explicitly the dispersion equation for the mean concentration $\bar{c}$.

COEFFICIENTS OF THE SHEAR EFFECT DISPERSION

Shear effect dispersion is described by eq.(16) where Σ is given, in terms of the functions $\underset{\sim}{f}$ and μ, by eqs.(22) and (23). The functions $\underset{\sim}{f}$ and μ can be determined by eqs.(21), (25), (31) and (39) provided the function λ is known. Thus the parameterization of the shear effect - as well as the determination of the vertical profile of velocity - reduces to the choice of a single scalar function.

The function

$$\lambda = \xi(1 - 0.5\ \xi) \tag{50}$$

appears to cover a wide range of situations in the North Sea and other shallow tidal seas (e.g. Bowden, 1965 ; Nihoul, 1977).

In this case, the eigenfunctions and the eigenvalues of eqs.(36), (37) and (38) are given by

$$f_n = (4n + 1)^{1/2}\ P_{2n}(\xi - 1) \tag{51}$$

$$\alpha_n = n(2n + 1) \tag{52}$$

where P_{2n} is the Legendre polynomial of even order 2n.

Integrating eq.(39), one obtains, in this case,

$$\underset{\sim}{f} = \underset{\sim}{v}_s\ S(\xi) + \underset{\sim}{v}_b\ B(\xi) + \frac{\dot{\underset{\sim}{v}}_s + 2\dot{\underset{\sim}{v}}_b}{\sigma}\ F(\xi) \tag{53}$$

with

$$S(\xi) = 4\ \ln 2(\xi - 1) + 2(2 - \xi)\ln(2 - \xi) \tag{54}$$

$$B(\xi) = -\ 2\ \ln 2(\xi - 1) + \xi\ \ln \xi - (2 - \xi)\ln(2 - \xi) \tag{55}$$

$$F(\xi) = \frac{5}{36}\ (\xi^3 - 3\ \xi^2 + 2\ \xi) \tag{56}$$

The shear effect diffusivity tensor can then be written, using eqs.(25) and (31)

$$R = \int_{\xi_o}^{1} \frac{\hat{\underset{\sim}{r}}\,\hat{\underset{\sim}{r}}}{\sigma\lambda}\, d\xi$$

$$= \frac{\gamma_{ss}}{\sigma} \underset{\sim}{v}_s \underset{\sim}{v}_s + \frac{\gamma_{sb}}{\sigma} (\underset{\sim}{v}_s \underset{\sim}{v}_b + \underset{\sim}{v}_b \underset{\sim}{v}_s) + \frac{\gamma_{bb}}{\sigma} \underset{\sim}{v}_b \underset{\sim}{v}_b$$
$$+ \frac{\gamma_{sf}}{\sigma^2} (\underset{\sim}{v}_s \dot{\underset{\sim}{v}}_s + 2\, \underset{\sim}{v}_s \dot{\underset{\sim}{v}}_b + \dot{\underset{\sim}{v}}_s \underset{\sim}{v}_s + 2\, \dot{\underset{\sim}{v}}_b \underset{\sim}{v}_s) \qquad (57)$$
$$+ \frac{\gamma_{bf}}{\sigma^2} (\underset{\sim}{v}_b \dot{\underset{\sim}{v}}_s + 2\, \underset{\sim}{v}_b \dot{\underset{\sim}{v}}_b + \dot{\underset{\sim}{v}}_s \underset{\sim}{v}_b + 2\, \dot{\underset{\sim}{v}}_b \underset{\sim}{v}_b)$$
$$+ \frac{\gamma_{ff}}{\sigma^3} (\dot{\underset{\sim}{v}}_s + 2\, \dot{\underset{\sim}{v}}_b)(\dot{\underset{\sim}{v}}_s + 2\dot{\underset{\sim}{v}}_b)$$

with

$$\gamma_{ss} = \int_{\xi_o}^{1} \frac{S^2}{\lambda}\, d\xi \sim 0.048 \qquad \gamma_{sf} = \int_{\xi_o}^{1} \frac{SF}{\lambda}\, d\xi \sim -\,0.015 \qquad (58),(59)$$

$$\gamma_{sb} = \int_{\xi_o}^{1} \frac{SB}{\lambda}\, d\xi \sim 0.090 \qquad \gamma_{bf} = \int_{\xi_o}^{1} \frac{BF}{\lambda}\, d\xi \sim -\,0.031 \qquad (60),(61)$$

$$\gamma_{bb} = \int_{\xi_o}^{1} \frac{B^2}{\lambda}\, d\xi \sim 0.196 \qquad \gamma_{ff} = \int_{\xi_o}^{1} \frac{F^2}{\lambda}\, d\xi \sim 0.005 \qquad (62),(63)$$

APPLICATION TO THE SOUTHERN BIGHT OF THE NORTH SEA

In the Southern Bight of the North Sea, the depth is small and the bottom stress $\underset{\sim}{\tau}_b$, maintained by bottom friction of tidal currents, wind induced currents and residual currents is always fairly important. One can estimate that, in general, the characteristic time σ^{-1} is one order of magnitude larger than the characteristic time of variation of $\underset{\sim}{v}_s$ and $\underset{\sim}{v}_b$ (Nihoul and Ronday, 1976 ; Nihoul, 1977).

The terms of eq.(57) which contain the derivatives $\dot{\underset{\sim}{v}}_s$ and $\dot{\underset{\sim}{v}}_b$ - the coefficients of which are already smaller than the others - may then be neglected.

The shear effect diffusivity tensor reduces then to

$$\underset{\approx}{R} = \frac{H}{\|\underset{\sim}{v}_b\|} \left(\beta_1 \underset{\sim}{v}_b \underset{\sim}{v}_b + \beta_2 (\underset{\sim}{v}_s \underset{\sim}{v}_b + \underset{\sim}{v}_b \underset{\sim}{v}_s) + \beta_3 \underset{\sim}{v}_s \underset{\sim}{v}_s\right) \qquad (64)$$

with

$$\beta_1 \sim 1.2 \quad ; \quad \beta_2 \sim 0.6 \quad ; \quad \beta_3 \sim 0.3 \qquad (65)(66)(67)$$

In weak wind conditions ($v_b \le 10^{-2}\ \bar{u}$), the first term in the bracket is largely dominant and, using eq.(43), one obtains, with a good approximation

$$R = \alpha \frac{H}{\bar{u}} \bar{u}\,\bar{u} \qquad (68)$$

$$\Sigma = H^{-1} \nabla . \left(\alpha \frac{H^2}{\bar{u}} \bar{u}(\bar{u}.\nabla\bar{c})\right) \qquad (69)$$

with

$$\alpha \sim 0.14 \qquad (70)$$

However, in weak wind conditions, the approximation which consists in neglecting the derivatives $\dot{v}_b$ and $\dot{v}_s$ is less justified and, furthermore, one may question the validity of eq.(50). If the wind is too weak to maintain turbulence in the sub-surface layer, one may expect, in some cases, a turbulent diffusivity which, instead of growing continuously from the bottom to the surface, instead passes through a maximum at some intermediate depth to decrease afterwards to a smaller surface value. This type of behaviour is described by the family of curves

$$\lambda = \xi(1 - \delta\xi) \qquad (71)$$

Eq.(50) corresponds to the case $\delta = 0.5$. Values of δ from 0.5 to 1 correspond to lower intensity turbulence in the surface layer and the limiting value $\delta = 1$ would correspond to the case of an ice cover and the existence, below the surface, of a logarithmic boundary layer analogous to the bottom boundary layer.

In the Southern Bight of the North Sea, it is reasonable to assume that δ does not differ significantly from 0.5 and, in any case, never reaches extreme values close to 1. Nevertheless, to estimate the maximum error one can make on α, it is interesting to compute the coefficients β_1, β_2 and β_3 for some very different values of δ.

One finds

δ	0.5	0.7	0.9
β_1	1.2	1.5	2
β_2	0.6	0.8	1.3
β_3	0.3	0.5	1
α	0.14	0.17	0.23

The increase of the coefficients β_1, β_2, β_3 and α with δ is obviously associated with more important variations of u over depth, i.e. with larger values of $\hat{u}$.

One should note also that the existence of a vertical stratification, even a weak one, reduces the turbulent diffusivity ($\mu = \eta\nu$ with $\eta < 1$) and contributes similarly to increase the value of α (e.g. Bowden, 1965).

In the Southern Bight of the North Sea, eventual modifications of the magnitude ($\eta < 1$) or of the form ($\delta > 0.5$) of the turbulent diffusivity are not likely to be very important and eq.(68) can presumably be used with $\alpha = 0.14$ or some slightly higher value obtained by calibration of the model with the observations.

VERTICAL CONCENTRATION PROFILE

When $\bar{c}$ has been calculated, it is possible to compute the deviation $\hat{c}$ by eq.(20). Changing variable to ξ and using eqs.(25) and (31), one gets

$$\frac{\partial \hat{c}}{\partial \xi} = \frac{\hat{\underset{\sim}{r}}}{\lambda} \cdot \frac{\underset{\sim}{\nabla} \bar{c}}{\sigma} \qquad (72)$$

with, from eq.(9),

$$\int_0^1 \hat{c} \; d\xi = 0 \ . \qquad (73)$$

Restricting attention to the dominant terms, one finds

$$\hat{c}(\xi) = \left(H(\xi) \; \underset{\sim}{v}_s + G(\xi) \; \underset{\sim}{v}_b\right) \cdot \frac{\underset{\sim}{\nabla} \bar{c}}{\sigma}$$

where

$$H(\xi) = 4 \; \{P(\xi) + \ln \; (2-\xi) \; \ln \frac{\xi}{4} + \ln 2 \; (4 \ln 2 - 1 - \ln \xi) - 2\} \qquad (75)$$

$$G(\xi) = - 2 \; \{2 \; L_2(\tfrac{1}{2}) + \ln \frac{2-\xi}{2} \ln \frac{\xi}{2} + \ln (2) \; \ln (2) + 2\} \qquad (76)$$

and

$$P(\xi) = L_2(\tfrac{\xi}{2}) + 2 \; L_2(\tfrac{1}{2}) - \bar{L}_2 \qquad (77)$$

$$L_2(x) = \text{Dilo} \; (1-x) = \sum_{\nu=1}^{\infty} (-1)^{\nu} \; \frac{(x-1)^{\nu}}{\nu^2} \qquad (78)$$

The functions H and G are shown in fig. 4. They are both negative near the surface and positive near the bottom. This is what one should expect from a physical point of view. Higher velocities near the surface carry water masses farther. If this transport is directed towards increasing mean concentrations the corresponding inflow of lower concentration fluid decreases the local concentration below the mean value $\bar{c}$. If the transport in the upper layer is directed towards decreasing mean concentrations, the corresponding inflow of high concentration fluid increases the local concentration above the mean value $\bar{c}$. The opposite situation occurs near the bottom. This is illustrated in fig. 5 showing the concentration profiles at two points situated downstream and downwind and respectively upstream and upwind on the same isoconcentration curve following a dumping.

The combination of eqs.(16) and (42) with a three-dimensional hydrodynamic model provides a three-dimensional dispersion model for the calculation of the evolution with time and the spatial - horizontal and vertical - distribution of any passive buoyant marine constituant.

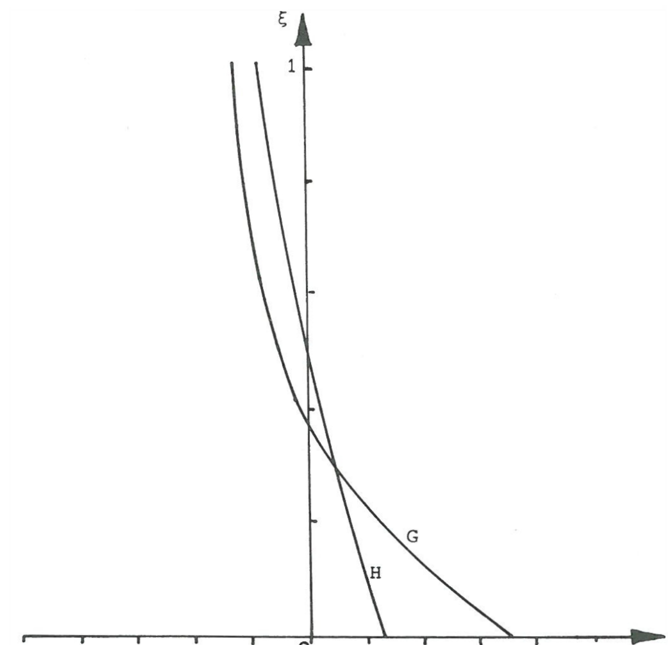

Fig. 4.

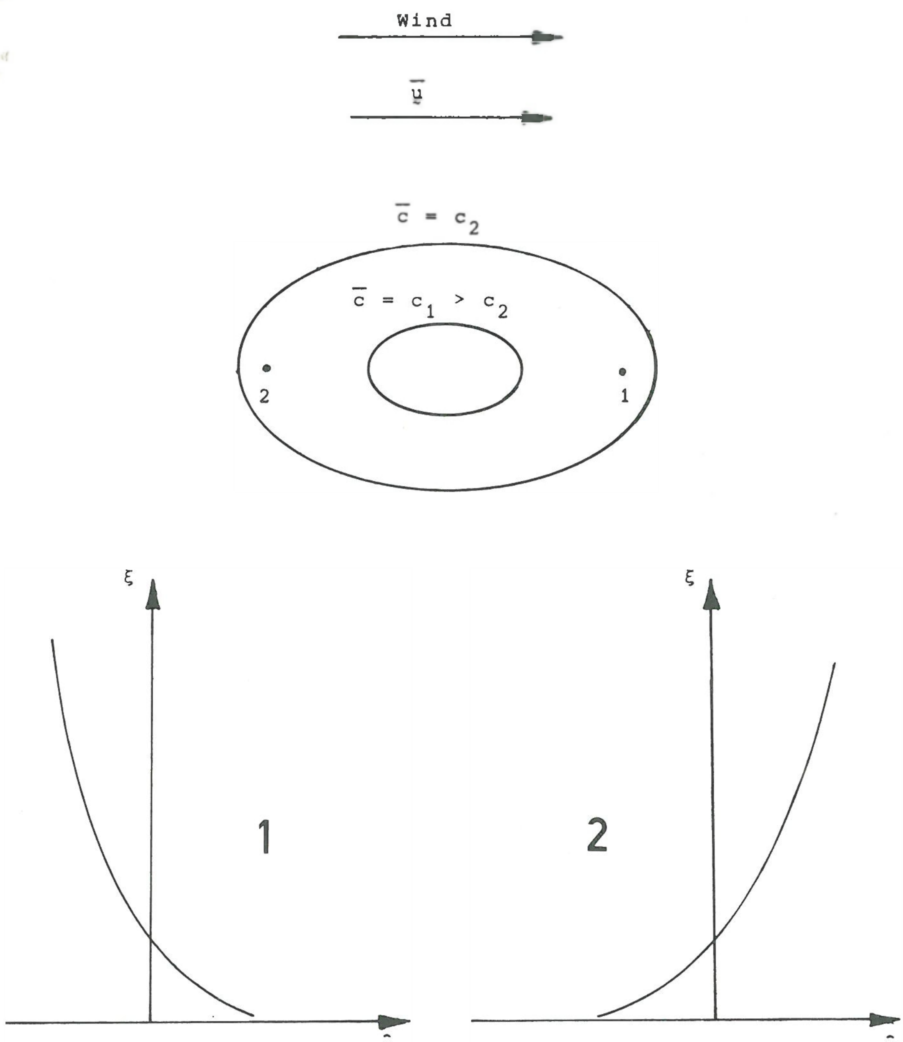

Fig. 5.