SMALL SCALES ERRATIC MOTIONS IN THE DEEP OCEAN

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1. INTRODUCTION -

In the ocean, under the surface layer, the flow is quieter and some authors do not hesitate to describe the fluid motion in these regions as a laminar flow displaying occasional disorganized patches answerable to turbulence (Woods, 1969).

In a non-homogeneous medium like the ocean, however, the word "laminar" cannot possibly refer to the sort of peaceful motion one can produce in a pipe or a laboratory channel. Between subsurface layers of varying density, undulating swells, called internal waves, form and multiply by non-linear interactions. These interactions, on the one hand, and the variety of their sources on the other hand, produce an intricated collection of motions of various scales which is best described by a statistical analysis.

The same approach applies then to waves and to turbulence and, indeed, if one defines "turbulence" as a field of chaotic vorticity (Saffman, 1968), the rotational random waves can be incorporated in the definition and the ocean can be regarded as completely turbulent.

The first objective of the study of ocean flow is then the determination of the energy spectrum function which associates to each scale of motion (indicated by its wave number vector $\underline{K}$) a density of energy $F(\underline{K})$ and the definition of transport coefficients describing the average diffusion by erratic stirring and mixing of momentum, heat, salt, radioactivity, pollution etc...

This study has been first approached from two extremes and methods of linear wave analysis (Fofonoff,1969) have been tried simultaneously with phenomenological descriptions of ordinary homogeneous turbulence.

An artificial distinction has thus been introduced, inducing several authors to consider two quite separate types of motions in the deep ocean and Townsend (1958) writes that, dispite "the gradual transition from one flow to another","the two flows are so distinct that no common description is likely to be valid".

It is difficult to accept this opinion, however, in view of the recent works in the domain which, proceeding from the two extremes, have successfully narrowed the difference and support the idea of a unified description.

Standard perturbation techniques, starting from the linear wave model, allowed Hasselmann (1962,1963,1966,1967,1968) and other authors to develop a "weak interaction" theory of ocean waves, analogous to weak plasma turbulence (Kadomtsev,1965), interacting phonon ensembles (Peierls,1955) and to the recent description of Clear Air turbulence proposed by Bretherton (1969). This theory approaches the controversed but enthralling theory propounded by Kraichnan (1959) for homogeneous turbulence. Examining the problem from the other side, Webster (1969) has shown that, even in a range of wave-numbers where one would expect wave models to be very accurate, experimental data were not in disagreement with the famous Kolmogorov law for strong turbulence.

These results, which contribute to the construction of a unified description, agree well with the actual trend, in the theory of non-homogeneous turbulence, to represent a fully developed turbulent shear flow as a superposition of waves (Landahl, 1967, Reynolds, Nihoul,1969, 1970).

The purpose, here, is to present the philosophy of some important features of the existing models and the elements of a general description, in the simplest context possible.

2. FOURIER ANALYSIS -

Ocean hydrodynamics is currently described by the Boussinesq equations. This is an approximation which consists in assuming that the fluid is incompressible and the density constant but taking into account the density variation in the gravity force by the introduction of a buoyancy force.
Let the Brunt-Väisälä frequency N and the buoyancy velocity V_4 be introduced by

$$(1),(2) \qquad N^2 = -\frac{g}{\rho_0}\frac{d\bar{\rho}}{dX_3} \quad ; \quad V_4 = \frac{-g\rho'}{\rho_0 N}$$

where the X_3-axis is vertical upwards, g is the acceleration of gravity, ρ_0 the reference constant density, $\bar{\rho}$ the average deviation from ρ_0 and ρ' the fluctuation around $\bar{\rho}$.

Neglecting the Coriolis effects and assuming the Brunt-Väisälä frequency constant, for simplicity, the Boussinesq equations may be written :

$$(3) \qquad \nabla \cdot \underline{V} = 0$$

$$(4) \qquad \frac{\partial V_1}{\partial t} + \underline{V}\cdot\nabla V_1 + \frac{\partial q}{\partial X_1} - \nu\nabla^2 V_1 = 0$$

$$(5) \qquad \frac{\partial V_2}{\partial t} + \underline{V}\cdot\nabla V_2 + \frac{\partial q}{\partial X_2} - \nu\nabla^2 V_2 = 0$$

$$(6) \qquad \frac{\partial V_3}{\partial t} + \underline{V}\cdot\nabla V_3 + \frac{\partial q}{\partial X_3} - NV_4 - \nu\nabla^2 V_3 = 0$$

$$(7) \qquad \frac{\partial V_4}{\partial t} + V\cdot\nabla V_4 + NV_3 = 0$$

where q denotes the pressure counted from the reference state and divided by ρ_0; V_1, V_2, V_3 are the components of the velocity vector and $\underline{\nabla}$ is the vector-operator $\left(\frac{\partial}{\partial X_1}, \frac{\partial}{\partial X_2}, \frac{\partial}{\partial X_3}\right)$. The Coriolis forces have been neglected to lighten the writing. Since the corresponding terms are linear, there would be no difficulty in incorporating them in the subsequent analysis. The approximation merely excludes the very small frequencies, and disregards eventual complicated interactions between the oscillations considered here and other low frequency mechanisms.

Let

(8) $$V_1(\underline{X},t) = \int W_1(\underline{K},\omega)\, e^{i(\underline{K}.\underline{X}-\omega t)} d\underline{K}\, d\omega$$

(9) $$V_2(\underline{X},t) = \int W_2(\underline{K},\omega)\, e^{i(\underline{K}.\underline{X}-\omega t)} d\underline{K}\, d\omega$$

(10) $$V_3(\underline{X},t) = \int W_3(\underline{K},\omega)\, e^{i(\underline{K}.\underline{X}-\omega t)} d\underline{K}\, d\omega$$

(11) $$V_4(\underline{X},t) = i\int W_4(\underline{K},\omega)\, e^{i(\underline{K}.\underline{X}-\omega t)} d\underline{K}\, d\omega \quad .$$

Four dimensional Fourier transforms of eq(3) to (7) yield, after eliminating the pressure :

(12) $$A_{\alpha\beta} W_\beta = \int C_{\alpha\beta} K_\gamma W_\beta(\underline{K}-\underline{K}',\omega-\omega') W_\gamma(\underline{K}',\omega') d\underline{K}'\, d\omega'$$

where the greek subscripts can take values from 1 to 4. (Note $K_4=0$) and where a sum is understood whenever a subscript is repeated.

The matrix elements $C_{\alpha\beta}$ and $A_{\alpha\beta}$ are given by

(13) $$C_{\alpha\beta} = (\delta_{\alpha\beta} - \frac{K_\alpha K_\beta}{K^2})$$

$$(14)\qquad A = \begin{pmatrix} \omega+i\nu K^2 & 0 & 0 & -N\dfrac{K_1K_3}{K^2} \\ 0 & \omega+i\nu K^2 & 0 & -N\dfrac{K_2K_3}{K^2} \\ 0 & 0 & \omega+i\nu K^2 & N(1-\dfrac{K_3^2}{K^2}) \\ 0 & 0 & N & \omega \end{pmatrix}$$

Denoting ensemble averages by angular brackets, the energy spectrum tensor $\underline{\underline{H}}$ is defined by

$$(15)\qquad H_{\alpha\beta}(\underline{K},\omega)\ \delta(\underline{K}-\underline{\mu})\ \delta(\omega-\sigma)=\langle W_\alpha(\underline{K},\omega)W^*_\beta(\underline{\mu},\sigma)\rangle\ .$$

Energy spectrum functions, measuring the energy distribution in wave number space, are then introduced by :

$$(16)\qquad F_h(\underline{K},\omega) = H_{11}+H_{22}$$

$$(17)\qquad F_c(\underline{K},\omega) = H_{11}+H_{22}+H_{33}$$

$$(18)\qquad F_T(\underline{K},\omega) = H_{11}+H_{22}+H_{33}+H_{44}\ .$$

Four-dimensional Fourier transforms are appropriate to describe fluid motions which are homogeneous in space and time.

This cannot be the case here. On the one hand, no attention has been paid to possible sources of energy (they would appear in the right hand side of eq (4) to (7)) and the dynamics of the random field in consideration may only result in a general decay. On the other hand, the assumption of space homogeneity obviously disregards boundary effects at the surface or the bottom of the ocean and is related to the hypothesis of constant Brunt-Väisälä forces have been neglected to lighten the writing. Since the corresponding terms are linear, there would be no difficulty in incorporating them in the subsequent analysis. The approximation merely excludes the very small frequencies, and disregards eventual complicated interactions between the oscillations considered here and other low frequency mechanisms.

Let

$$V_1(\underline{X},t) = \int W_1(\underline{K},\omega)\; e^{i(\underline{K}.\underline{X}-\omega t)} d\underline{K}\; d\omega \tag{8}$$

$$V_2(\underline{X},t) = \int W_2(\underline{K},\omega)\; e^{i(\underline{K}.\underline{X}-\omega t)} d\underline{K}\; d\omega \tag{9}$$

$$V_3(\underline{X},t) = \int W_3(\underline{K},\omega)\; e^{i(\underline{K}.\underline{X}-\omega t)} d\underline{K}\; d\omega \tag{10}$$

$$V_4(\underline{X},t) = i\int W_4(\underline{K},\omega)\; e^{i(\underline{K}.\underline{X}-\omega t)} d\underline{K}\; d\omega \;. \tag{11}$$

Four dimensional Fourier transforms of eq(3) to (7) yield, after eliminating the pressure :

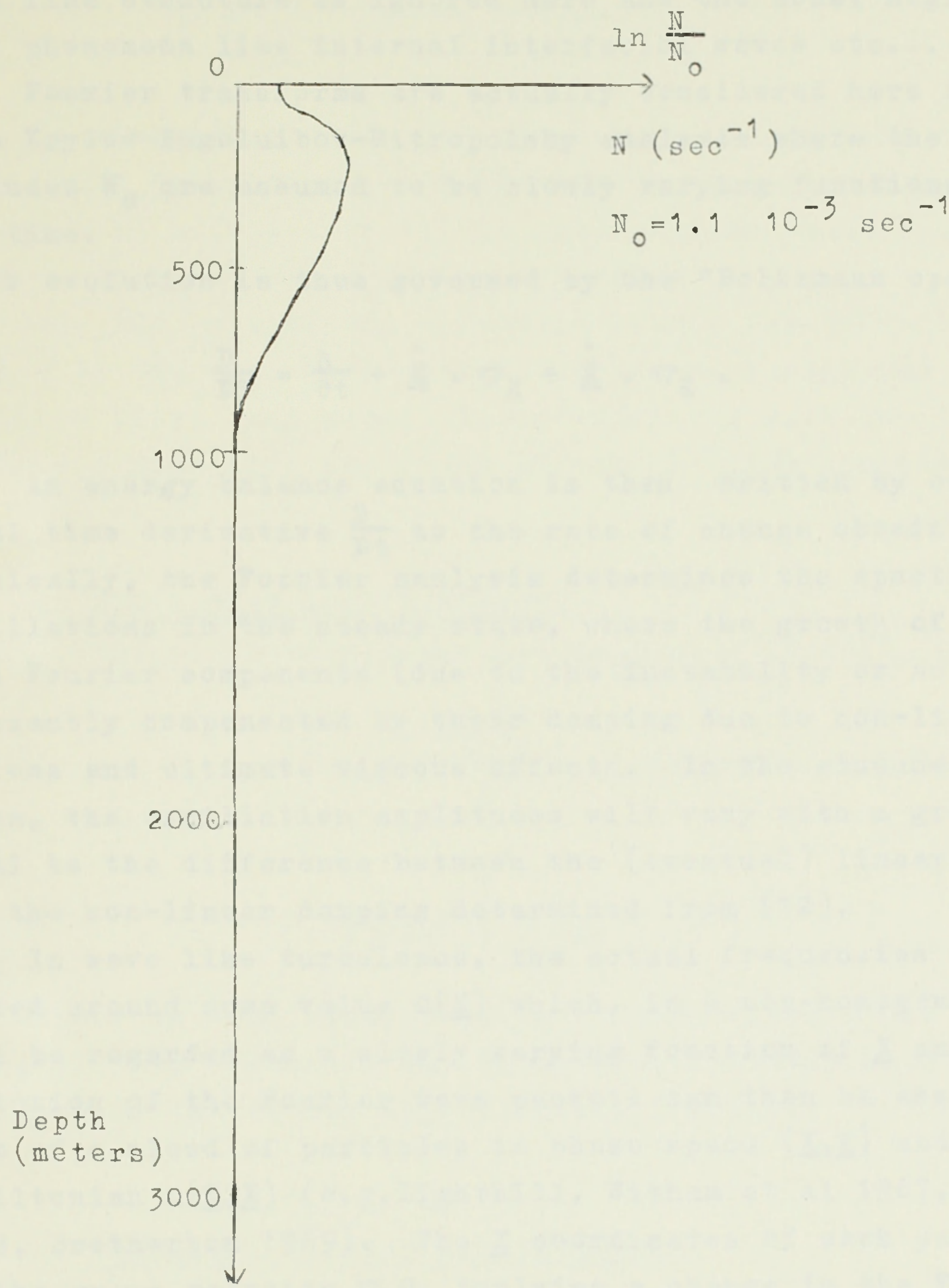
$$A_{\alpha\beta} W_{\beta} = \int C_{\alpha\beta} K_{\gamma} W_{\beta}(\underline{K}-\underline{K}', \omega-\omega') W_{\gamma}(\underline{K}', \omega') d\underline{K}'\; d\omega' \tag{12}$$

where the greek subscripts can take values from 1 to 4. (Note K_4=0) and where a sum is understood whenever a subscript is repeated.

The matrix elements $C_{\alpha\beta}$ and $A_{\alpha\beta}$ are given by

$$C_{\alpha\beta} = (\delta_{\alpha\beta} - \frac{K_\alpha K_\beta}{K^2}) \tag{13}$$

frequency. If N were not constant, there would be an extra term in the right hand side of (7) proportional to $\frac{d \ln N}{dX_3}$. Fig (1) shows a typical variation of ln N with depth.



Clearly, the Brunt-Väisälä frequency is fairly constant in sufficiently deep ocean but varies in the thermocline. In addition, however, to the slow variations shown on fig.1, sharp changes in the Brunt-Väisälä profile have also been observed within narrow sheets. This fine structure is ignored here and the model neglects special phenomena like internal interfacial waves etc...

Fourier transforms are actually considered here in the scope of a Krylov-Bogoluibov-Mitropolsky analysis where the Fourier amplitudes W_α are assumed to be slowly varying functions of space and time.

Their evolution is thus governed by the "Boltzmann operator"

$$\frac{D}{DT} = \frac{\partial}{\partial t} + \dot{\underline{X}} \cdot \nabla_X + \dot{\underline{K}} \cdot \nabla_K \; . \tag{19}$$

An energy balance equation is then written by equating the total time derivative $\frac{D}{Dt}$ to the rate of change obtained from (12). Physically, the Fourier analysis determines the spectrum of the oscillations in the steady state, where the growth of the different Fourier components (due to the instability or acting sources) is exactly compensated by their damping due to non-linear interactions and ultimate viscous effects. In the absence of equilibrium, the oscillation amplitudes will vary with a growth rate equal to the difference between the (eventual) linear growth rate and the non-linear damping determined from (12).

In wave like turbulence, the actual frequencies are concentrated around some value $\Omega(\underline{K})$ which, in a non-homogeneous medium, must be regarded as a slowly varying fonction of $\underline{X}$ and t. The evolution of the Fourier wave packets can then be assimilated to that of a cloud of particles in phase space $(\underline{X},\underline{K})$ under a Hamiltonian $\Omega(\underline{K},\underline{X})$ (e.g. Lighthill, Witham et al 1967, Hasselmann 1968, Bretherton 1969). The $\underline{X}$ coordinates of each packet change at the group velocity $\nabla_K \Omega$, implying a change in the Fourier amplitudes at a fixed point if their gradients in this direction are not zero. The "momenta" $(\underline{K})$ vary at the rate $\nabla_X \Omega$. This can cause a flux through the wave number spectrum. The interaction coefficients must be calculated locally.

Thus

$$\frac{D}{DT} \equiv \frac{\partial}{\partial t} + \nabla_K \Omega . \nabla_X + \nabla_X \Omega . \nabla_K \quad . \tag{20}$$

In stronger turbulence, at smaller scale, this would be a very crude description, but presumably slow variations of the medium will affect very little such motions whose scale is much smaller than its scale of inhomogeneity.
In fact, the part of the medium is played for these turbulent eddies by the wake-like weaker turbulence background and it will be shown below that adiabatic interactions with the background produce a much more important effect of inhomogeneity.

3. THE LINEAR APPROXIMATION.

If the non-linear terms are neglected in (12), one finds :

$$\omega^2 = N^2 \frac{K_1^2 + K_2^2}{K^2} \qquad \text{(dispersion relation)} \tag{21}$$

and

$$H_{11} + H_{22} + H_{33} = H_{44} \; . \tag{22}$$

There is thus equipartition between kinetic and potential energy. This criterion may be used to investigate the limit of validity of the linear approximation (Fofonoff 1969).

4. THE WEAK COUPLING APPROXIMATION.

If the non-linear interactions are small but not negligible, it seems quite natural to seek a solution of the form $\underline{W} = {}^{0}\underline{W} + {}^{1}\underline{W}$

where $\overset{1}{\underline{W}}$ is a small perturbation which results from the non-linear beat interactions and is determined by eq.(12) where $\underline{W}$ in the right hand side is replaced by $\overset{o}{\underline{W}}$.

To take a global account of the action of higher order non-linear terms which are here neglected and which may produce a damping of the individual Fourier component, a term $D_{\alpha\beta}W_\beta$ is introduced in the left hand side of (12), the tensor $\underline{\underline{D}}$ being sofar unknown.

This artificial closure is very important as it incorporates the net non-linear contribution to irreversibility and permits the definition of a rate of change due to non-linear interactions and suitable for (22).

Thus

$$(23) \qquad [A_{\alpha\beta}+D_{\alpha\beta}]\,{}^1W_\beta = \int C_{\alpha\beta}\,{}^oW_\beta(\underline{K}-\underline{K}',\omega-\omega')\,{}^oW_j(\underline{K}',\omega')K_j\,d\underline{K}'d\omega \ .$$

Solving (12) for W_β (after adding $D_{\alpha\beta}W_\beta$ to both sides) and introducing the matrices L and N by

$$(24) \qquad L_{\alpha\beta} = B_{\alpha\gamma}D_{\gamma\beta}$$

$$(25) \qquad N_{\alpha\beta} = B_{\alpha\gamma}C_{\gamma\beta}$$

with

$$(26) \qquad B_{\alpha\beta}(A_{\beta\gamma}+D_{\beta\gamma}) = \delta_{\alpha\gamma}$$

one gets, from (12) :

$$(27) \qquad \langle W_\alpha(\underline{K},\omega)W^*_\beta(\underline{\mu},\sigma)\rangle = L_{\alpha\gamma}\langle W_\gamma(\underline{K},\omega)W^*_\beta(\underline{\mu},\sigma)\rangle$$

$$+\int N_{\alpha\gamma}(\underline{K},\omega)\ \langle W_\gamma(\underline{K}-\underline{K}',\omega-\omega')W^*_\beta(\underline{\mu},\sigma)W_\gamma(\underline{K}',\omega')\rangle K_\gamma\,d\underline{K}'d\omega' .$$

In zero approximation, the oscillations are assumed uncorrelated with a Gaussian distribution (the random phase approximation). The integral in the right hand side of (27) in thus zero at this approximation. The next approximation is obtained by including successively in each of the three factors the first order correction $\overset{1}{\underline{W}}$ which leads to double integrals (over $\underline{K}'$ and $\underline{\mu}'$, ω' and σ', say) of fourth order correlations. These are split into sums of products of second order correlations which are expressed in terms of $H_{\alpha\beta}$ by (15). The integrations over $\underline{\mu}'$ and σ' may then be performed, taking the δ-functions into account. The result includes a linear part and an integral over $\underline{K}'$ and $\underline{\sigma}'$ of all <u>direct</u> non-linear interactions. If $D_{\alpha\beta}$ is chosen as to cancel the linear part (assuming that the global influence of the non linear higher interactions on the energy rate of change is the same at the zeroth and first order), the final result is of the form

(28) $$D_{qs} = -J_{q\alpha\beta} \int K'_{\beta\gamma s} \, H_{\alpha\gamma}(\underline{K}-\underline{K}', \omega-\omega') \, d\underline{K}' \, d\omega'$$

where

(29) $$J_{q\alpha\beta} = C_{q\alpha}(\underline{K}, \omega) K_\beta + C_{q\beta}(\underline{K}, \omega) \, K_\alpha$$

(30) $$K'_{\beta\gamma s} = N_{\beta\gamma}(\underline{K}', \omega') \, K'_s + N_{\beta s} \, (\underline{K}', \omega') K_\gamma$$

and

(31) $$H_{qs} = \int N_{q\alpha} K_\gamma [\overset{*}{N}_{s\beta} K_\delta + \overset{*}{N}_{s\delta} K_\beta] \, H_{\alpha\beta}(\underline{K}-\underline{K}', \omega-\omega') H_{\gamma\delta}(\underline{K}', \omega') d\underline{K}' \, d\omega' \quad .$$

Eq.(28) and (31) constitute a set of coupled integral equations which can be regarded as the generalization of Kraichnan's équations.

In the absence of stratification ($\nu_4 = 0$), assuming homogeneity and isotropy, on can write

(32) $$H_{\alpha\beta}(\underline{K},\omega) = G(K,\omega)\,(\delta_{\alpha\beta} - \frac{K_\alpha K_\beta}{K^2})$$

(33) $$B_{\alpha\beta} = (\omega + i\nu K^2 + \eta)^{-1}\,\delta_{\alpha\beta}.$$

Substituting in (31), one gets :

(34) $$\frac{1}{2} F_c(\underline{K},\omega) = \frac{1}{2}[H_{11}+H_{22}+H_{33}] = G(K,\omega) =$$

$$= |\omega + i\nu K^2 + \eta|^{-2} \int K^2\, S(\underline{K},\underline{K}')\, G(\underline{K}',\omega')\, G(\underline{K}-\underline{K}',\omega-\omega')\, d\underline{K}' d\omega'$$

where

(35) $$S(\underline{K},\underline{K}') = \frac{1}{2}\,[1-2\,\frac{(\underline{K}\cdot\underline{K}')^2(\underline{K}\cdot\underline{K}'')^2}{K^4 K'^2 K''^2} + \frac{(\underline{K}\cdot\underline{K}')(\underline{K}\cdot\underline{K}'')(\underline{K}'\cdot\underline{K}'')}{K^2 K'^2 K''^2}]$$

with $\underline{K}'' = \underline{K} - K'$.

Eq (34) and (35) are identical to Kraichnan's equations. An important characteristic of eqs. (31) or (34) is that they retained only <u>direct</u> interactions in the energy exchanges; the interaction between the Fourier component $\underline{K},\omega$ and the component $\underline{K}'$, ω' appearing as a <u>resonant</u> imput of energy to $\underline{K},\omega$ from $\underline{K}''=\underline{K}-\underline{K}'$, $\omega''=\omega-\omega'$.

This result is obtained here as a consequence of the weak coupling approximation and indeed, turbulence being a mixing process which degrades information, one should expect that the <u>indirect</u> interaction of three modes through the turbulent motion as a whole should not convey phase information among them in the limit where the motion consists of the excitation of an infinitely large number of weakly dependent degrees of freedom.

5. THE LIMITED WEAK COUPLING APPROXIMATION.

A consequence of the weak coupling hypothesis is that an individual mode $\underline{K},\omega$ gains or loses energy only by triangular interactions with modes $\underline{K}',\omega'$; $\underline{K}''\omega''$ such that $\underline{K} = \underline{K}' + K''$, $\omega = \omega' + \omega''$.

Since the number of these modes is large and the amplitude of the oscillations of an individual mode is determined by the integral of its triangular interactions, it is not unreasonable to think that, even in stronger turbulence where the global non-linear interaction is O (1), the interaction of the Fourier component $\underline{K},\omega$ with each separate component $\underline{K}',\omega'$ is comparatively small.

It is thus tempting to extend the weak-coupling approximation to strong turbulence. One can object however that this approximation implicity allows only resonant interactions between Fourier components and that such a model does not seem to be appropriate for interactions between modes of very different wavelengths.

One visualizes, indeed, the non-linear effects as fostering the viscous decay and producing at every wave number an increased damping expressed by the imaginary part of D (or η). Let this damping be grossly described by an "eddy viscosity term" $\hat{\nu}K^2$ (which incorporates the ordinary viscosity). At moderate numbers, this effect (and also the contribution from the real part of D) may be regarded as small. Then, one should expect the energy to be concentrated on the frequency axis around the eigenfrequency (21) given by

$$\det |A| = 0 \qquad (36)$$

and a wave like description to be valid.

In the region of wave numbers O(K), the "waves" will have a lifetime $\tau \sim \hat{\nu}^{-1} K^{-2}$ and although one given mode may initially be localised, it will spread out in general as time passes to fill a region of space whose characteristic size L will be of the order of the distance through which the "wave" propagates during its lifetime i.e.

$$(37) \qquad l \sim \frac{N}{\hat{\nu} K^3} .$$

Thus the state of turbulent motion must be regarded as a system of many wave packets. As long as the ratio (37) is large, these packets exist for a very long time and are almost unlocalised in space so that a wave-like description is appropriate. As the ratio increases, the packets tend to concentrate and appear more as turbulent eddies.

This suggests two apparently different descriptions of the random velocity field; the two descriptions overlapping in this region of wave-numbers where l, estimated from the turbulence concepts (i.e. $l = K^{-1}$) is equal to (37).

If, as a practical experimental situation, the observations made by Woods (1969) off Malta are used, the following estimations can be made (M.K.S units)

Eddy viscosity : $\hat{\nu} \sim 10^{-4}$

Brunt-Väisälä frequency : $N \sim 10^{-3}$

$$l^2 \sim \frac{\hat{\nu}}{N} \sim 10^{-1} \; ; \; l \sim 0.3 .$$

The size of the largest eddy of the typically turbulent motion should thus be of the order of 30cm. This figure is in excellent agreement with Woods's observations (Woods 1969).

This suggests that typical turbulence is found within patches (of size ~ 30cm, in the exemple treated); these patches moving in the non-homogeneous background of the larger scales wave-like motions. One should expect then that, as in (20), wave packets move about in wave number space, leading to a strong correlation between nearby Fourier components which now describe essentially one and the same wave packet.

In this interpretation, the interaction between the mode $\underline{K},\omega$ and a "distant" mode $\underline{K}',\omega'$ cannot be considered as the resonant input to $\underline{K},\omega$ from the nearby component $\underline{K}'',\omega''$ because $\underline{W}$ and $\underline{W}''$ describe the same wave packet and cannot be considered independently of one another.

It is thus necessary to distinguish between two types of interactions and to bring in this distinction it is convenient to break up the region of integration with respect to $\underline{K}'$, ω' in the non-linear terms into three parts : the principal region where K and K', ω and ω' are comparable, the "long wave" region where the prime quantities are much smaller and the short wave region where they are much larger.

The weak coupling analysis can be applied to region (1) the contribution from region (3) is presumably small as little energy is contained in wave numbers $K' \gg K$, $K'' \sim K' \gg K$. The contribution from region (2) can be taken into account by expanding in this region $\underline{W}(\underline{K}-\underline{K}')$ in Taylor series around K.

Limiting the treatment to the first approximation, one writes :

$$(38) \qquad \int_{(2)} C_{\alpha\beta} K_\gamma W_\beta(\underline{K}-\underline{K}',\omega-\omega')W_\gamma(\underline{K}',\omega')d\underline{K}'d\omega' \sim \omega_W \; W_\alpha(\underline{K},\omega)$$

where

$$(39) \qquad \omega_W = K_\gamma \int_{(2)} W_\gamma(\underline{K}',\omega')d\underline{K}' \, d\omega'$$

ω_W is a random quantity given by the sum of a large number of individually random and weakly correlated amplitudes and its distribution may be assumed Gaussian.

The contribution from region (2) may then be incorporated in the matrix A by changing the frequency ω to the relative frequency

$$(40) \qquad \tilde{\omega} = \omega - \omega_{\overline{W}} \; .$$

The final result is that one recovers eq.(31) with the difference that the range of integration in the non-linear term is restricted to cover the region (1) and that the energy spectrum tensor is now calculated in a system of coordinates moving with the long wave pulsations. The true spectral functions H_{qs} are obtained by averaging over $\omega_{\overline{W}}$ i.e.

$$(41) \qquad \underline{\underline{H}}(K,\omega) = \frac{1}{\pi\omega_o} \int H(\underline{K},\omega-\omega_{\overline{W}})\, e^{-\frac{\omega_{\overline{W}}^2}{\omega_o^2}}\, d\omega_{\overline{W}}$$

where

$$(42) \qquad \omega_o^2 = 2\, K_\gamma K_\delta \int_{(2)} H_{\gamma\delta}(\underline{K}', \omega')\, d\underline{K}'\, d\omega' \; .$$

The application by Kraichnan of eq.(34) to ordinary homogeneous isotropic turbulence lead to a $K^{-3/2}$ spectral law instead of the well-known Kolmogorov $K^{-5/3}$ law. The present analysis suggests that this discrepancy might be due to Kraichnan overestimating the part played by the large-scale fluctuations, which is in fact no more than the convection of higher modes which are deformed adiabatically in the process.
Indeed, by setting the lower limit of the integration over $\underline{K}'$, ω' in (34) equal to $\xi \underline{K}$; $\xi\omega$ where ξ is a small constant number ($\xi \sim 1/3$), Kadomtsev (1965) has obtained the Kolmogorov spectrum.

6. PHENOMENOLOGICAL APPROACH.

At present, one has no rigorous method of performing the separation between resonant and adiabatic non-linear interactions and of reducing the integral equations. In considering strong turbulence the limited weak coupling approximation certainly opens new ways prolonging Kraichnan's ideas but, in the meantime, phenomenological approaches like Kolmogorov's theory are still required to interpret the observations.

According to Kolmogorov, over a wide range of scales (if the Reynolds number is large enough) the viscosity plays a negligible role and, in the absence of a direct input or output of energy, a quasi equilibrium is established. The turbulence is there fairly homogeneous and there is a constant energy flux ε through the spectrum. The value of ε determines the local properties of the turbulence. This hypothesis is equivalent to the natural assumption that the energy transfer between modes is of a resonance character, in which the energy of a mode K can be transferred only to modes with nearly the same scale. Thus a portion of energy handed down from a given scale to another must pass through the entire range of intermediate scales of motion. One may assert therefore that for each K the value of ε is determined only by the fluctuation level at this scale, i.e by the value of the spectral function at the corresponding wave number. This means that ε must be expressible in terms of K and, say E(K) where E(K) is the average of $F_c(\underline{K})$ over a sphere of radius K. The only dimensionnally correct combination is then

$$\varepsilon \sim [K^3 \; E(K)]^{1/2} \; K \; E(K) \tag{43}$$

i.e.

$$E(K) \sim \varepsilon^{2/3} \; K^{-5/3} \; . \tag{44}$$

It is interesting to note that Kolmogorov's reasoning applies equally well to three-dimensional and two dimensional turbulence although the interpretation is different. In three-dimensional turbulence the cascade of energy is directed from the large scale eddies where presumably energy is being provided (or has been stored) by external stirring devices, towards the small scale eddies where it is dissipated by viscosity.

As pointed out by Kraichnan (1967), in two dimensional turbulence, the $K^{-5/3}$ range entails backwards energy cascade from higher to lower wave numbers and can only be expected in eddies whose scale is larger than the scale of the energy reservoir.

Quasi-two-dimensional flow is not unrealistic in the ocean in view of the stratification.

In many cases, experimental data are obtained for the horizontal specific energy measured at a given point as a function of time. A Fourier analysis in time gives the distribution in frequency, often interpreted in terms of wave number by invoking Taylor's hypothesis.

According to Taylor's hypothesis, if there is a mean velocity field, large in comparison with the components of turbulent velocity fluctuations, which advects the turbulent cells past the point of measurements, spacial scales are observed as corresponding time scales, i.e. there is some linear relationship between ω and K. Now, from eq.(12), it is readily seen that $\omega[W_1 W_1^* + W_2 W_2^*]$ is the sum of two terms; the first of which is simply the right hand side of (12) multiplied by W_β^* with the summation on β and γ limited to 1 and 2 and the second of which is proportional to K_3 and contains terms in W_3 and W_4. We may thus regard the first term as representing a cascade interaction between components of the horizontal energy of different scales while the second term describes interactions with the vertical and buoyancy fluctuations.

In a stably stratified medium, one may speculate that these interactions and the interactions between W_3 and W_4 result in a net transfer of energy from the horizontal kinetic energy to the potential energy. In this case, the second interaction term expresses an inhibition of the horizontal fluctuations. Since this term is proportional to K_3, it affects predominantly the turbulent eddies whose wave number vector has an important K_3 component. It would be without influence on eddies with an horizontal wave number vector if the non-linear cascade did not redistribute energy over all directions as much as it transfers it from scale to scale.

In weak turbulence however, in the wave like region, this redistribution will presumably not work fast enough and horizontal energy will tend to be concentrated in horizontal wave number vectors while the strength of the inhibition term diminishes.

If this interpretation is correct, the agreement found by Webster (1969) between observational data at Site D and the $K^{-5/3}$ law might indicate, in this range of wave number where buoyancy effects are important, the existence of two-dimensional turbulence with a net energy transfer from the fluctuations at frequency of 1 cycle per hour(*) towards inertial frequencies.

Some evidence in support of the possibility of such a backward transfer in stratified fluids are given by Fjørtoft (1953).

7. REFERENCES.

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(*) It is probably adventurous to imagine sources of energy in this range of frequency although it corresponds roughly to the Brunt-Väisälä frequency and the possibility of instabilities degenerating in turbulence but also losing energy to gravity waves has been mentionned by several authors (Woods 1969).

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