WAVE DRIVEN INERTIAL OSCILLATIONS

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1. INTRODUCTION.

The spectra of ocean currents usually show a sharp peak at the inertial frequency. Within the peak, the two components of horizontal velocity are found to be highly coherent, in quadrature, and of equal amplitude, as expected for the rotating current vector of an inertial oscillation of large wave length.

Inertial oscillations have been observed in the open ocean and in enclosed basins, at all latitudes and depths. Typically, the amplitudes are ~ 0 (10 cm/sec) (but sometimes considerably larger), the vertical coherence scale of $\sim 0(10\text{ m})$; estimates of the horizontal scale vary from 5 to 100 km. All records show a characteristic intermittency of the oscillations, each distinct burst of 5 to 20 oscillations long. There is evidence that the bursts near the surface are correlated with high local winds.

At greater depths, no clear dependence on surface conditions has been found. Surveys of inertial current observations and various hypotheses of their origin have been given by Webster (1968; this issue).

In a recent paper (Hasselmann, 1970+), a mechanism has been investigated in which the driving forces are attributed to non-linear interactions between high-frequency gravity waves. The apparent "damping" of the inertial oscillations is interpreted as the diffusion due to phase-mixing of a large ensemble of modes with closely neighbouring frequencies. Thus the generation process is regarded as weakly non-linear and the decay process as linear.

+Reference to this paper will be indicated in the following by(I).

The approach is in accordance with the "weak-interaction" interpretation of oceanic turbulence which suggests that a broad range of the ocean-current spectrum can be regarded as a superposition of linear wave motions, rather than strongly non-linear fluctuations. The non-linear Reynolds stresses driving low-frequency currents in the ocean may then be interpreted as interactions between higher-frequency wave fields, rather than turbulent stresses. The total stress can be divided into a mean term, arising from quadratic self interactions of the waves, and a fluctuating term, arising from difference interactions between pairs of waves. Only the mean term has been considered.

The response of the ocean to the mean stress exerted by the waves is closely related to the mass transport of a wave field.

In a non-rotating system, the shear components of the Reynolds stress tensor vanishes, since the horizontal and vertical components of the orbital velocity are exactly in quadrature. The mass transport reduces then to the Stokes current which is the difference between the local Lagrangian and Eulerian currents.

In a rotating system, the Lagrangian current cannot remain constant but rotates with the local inertial frequency (Ursell, 1950). Here, it is shown that the wave-induced shear stresses are non zero, which is equivalent. It is also shown that the mass transport currents are not simply a property of the local wave field but represente the cumulative low-frequency response of the ocean to a variable, wave induced force field.

The solution can be represented formally as a superposition of normal modes. The prominence of the inertial peak in the low-frequency spectrum is due to the degeneracy of the modes at zero wave number (in the f-plane approximation, the frequency ω of all modes converges to f, the horizontal Coriolis component, as the wave number k approaches zero).

If the driving wave field is homogeneous in the horizontal, corresponding to excitation at zero wave number, the current vectors of all modes rotate with the same frequency f and their superposition yields an inertial oscillation which keeps rotating infinitely with its initial vertical distribution. If the scale of the driving field is large but finite, the initial horizontal and vertical distribution is gradually modified by the phase mixing of modes rotating with slightly different frequencies, inducing a dispersion of the oscillation in the horizontal and vertical directions.

The same dispersion-type behaviour is yielded by any effect which removes the degeneracy at zero wave number and introduces mode-dependent frequency shifts : horizontal inhomogeneities of the wave guide, the horizontal component of the Coriolis vector, non-linear interactions with geostrophic flows, planetary variations of the Coriolis parameters.

Instead of the standart normal mode approach (not practicable for all stratifications), the Green function representation is used to expand the solution about the degenerate state, yielding a characteristic operator for each phase-mixing process. The operators can be obtained directly from the equations of motion using eigenvalue formulae from the perturbation theory of linear operators.

Computations have been made for phase-mixing due to wave field inhomogeneities in a continuously stratified model representative of the Baltic. The orders of magnitude of the amplitude response and decay time are in reasonable agreement with observations.

2. EQUATIONS OF MOTION.

It is assumed that the motions are of a horizontal scale small compared with the radius of the earth or the lateral dimension of the ocean. The ocean can thus be described, to the first order, as an incompressible, stratified fluid of infinite horizontal extent. The equations of motion are given in the Boussinesq approximation by

$$\frac{\partial u_1}{\partial t} - fu_2 + \frac{\partial}{\partial x_1} p = - \frac{\partial}{\partial x_j} (u_1 u_j) \tag{1}$$

$$\frac{\partial u_2}{\partial t} + fu_1 + \frac{\partial}{\partial x_2} p = - \frac{\partial}{\partial x_j} (u_2 u_j) \tag{2}$$

$$\frac{\partial u_3}{\partial t} - b + \frac{\partial p}{\partial x_3} = - \frac{\partial}{\partial x_j} (u_3 u_j) \tag{3}$$

$$\frac{\partial b}{\partial t} + N^2 u_3 = - \frac{\partial}{\partial x_j} (b u_j) \tag{4}$$

$$\frac{\partial u_j}{\partial x_j} = 0 \; . \tag{5}$$

where $\underline{u} = (u_1, u_2, u_3)$ is the current velocity, p is the deviation of pressure from equilibrium $/\rho$, (x_1, x_2, x_3) are the Cartesian coordinates, x_1 Eastwards, x_2 Northwards, x_3 upwards, b is the buoyancy field, N^2 the Brunt-Väisälä frequency. At the free surface $x_3 = 0$, the solution must satisfy the dynamical and kinematical boundary conditions

$$\frac{\partial}{\partial t} p - gu_3 = - \frac{\partial}{\partial t}\left[\zeta\left(\frac{\partial}{\partial x_3} p + \frac{N^2 \zeta}{2}\right)\right] - g \frac{\partial}{\partial x_\alpha} (u_\alpha \zeta) \; ; \; x_3 = 0; \; (\alpha = 1,2) \tag{6}$$

ζ being the vertical displacement, g the gravitational acceleration. At the bottom,

$$u_3 = 0 \; , \quad x_3 = - h \; . \tag{7}$$

Gradual variation of f, h and N^2 will be considered as perturbations of the homogeneous state; the inclusion of $\hat{f}$, the horizontal Coriolis component will be also incorporated in the perturbation scheme.

The low-frequency response of the linear system on the left hand side of (1) - (7) to the non-linear forcing terms on the right is of interest. It is assumed that the components in the non-linear terms can be represented to first order as wave solutions which satisfy the linearised equations of motion and which are of "high" frequency $\omega >> f$. The response is determined for "low" frequencies $\omega << N$. In practice, the two frequency ranges are well separated.

The linearised equations (1) - (7) have normal-mode solutions $\underline{\varphi} \sim e^{i(\underline{k}\underline{x}-\omega t)}$, $k = (k_1, k_2, 0)$.

For the low-frequency modes ($\omega << N$), the hydrostatic approximation is used. By elimination, the linearised equation of motion can be written

$$\frac{\partial}{\partial t}\underline{\varphi}_c - i H_c \underline{\varphi}_c = 0 \qquad (8)$$

where $\underline{\varphi}_c = \begin{pmatrix} u_1 \\ u_2 \\ p \end{pmatrix}$ and the linear operator H_c is

$$H_c = \begin{pmatrix} 0 & -if & i\frac{\partial}{\partial x_1} \\ if & 0 & i\frac{\partial}{\partial x_2} \\ iI\frac{\partial}{\partial x_1} & iI\frac{\partial}{\partial x_2} & 0 \end{pmatrix} \qquad (9)$$

with

$$I = \int_{x_3}^{0} N^2 dx_3' \int_{h}^{x_3'} dx_3'' + g\int_{h}^{0} dx_3' \,. \qquad (10)$$

For small wave numbers, the dominant part of H_c is the 2×2 rotation matrix in the top left of the matrix. It is convenient to diagonalise this submatrix by transforming to rotary velocity components

$$\begin{aligned} u_+ &= u_1 + iu_2 \\ u_- &= u_1 - iu_2 \,. \end{aligned} \tag{11}$$

Defining

$$\partial_{\pm} = \frac{\partial}{\partial x_1} \pm i \frac{\partial}{\partial x_2} \,, \quad k_{\pm} = k_1 \pm ik_2 \tag{12}$$

(8) reads

$$\frac{\partial}{\partial t} \varphi - i H \varphi = 0 \tag{13}$$

where $\varphi = \begin{pmatrix} u_+ \\ u_- \\ p \end{pmatrix}$ and

$$H = \begin{pmatrix} -f & 0 & i\partial_+ \\ 0 & f & i\partial_- \\ \frac{iI\partial_-}{2} & \frac{iI\partial_+}{2} & 0 \end{pmatrix} . \tag{14}$$

The operator H has separable eigenfunctions

$$H \varphi^s_{n\underline{k}} + \omega \varphi^s_{n\underline{k}} = 0$$

where

$$\underline{\varphi}^{s}_{n\underline{k}} = \underline{\beta}^{s}\ \psi_n(x_3)\ e^{i(\underline{k}\cdot\underline{x}-\omega t)}\ . \tag{15}$$

Solution of the eigenvalue problem gives the eigenfrequencies

$$\omega^{\pm} = \pm\ (f^2+\lambda_n k^2)^{\frac{1}{2}} \qquad \text{(gravity waves)} \tag{16}$$

$$\omega^{o} = 0 \qquad \text{(geostrophic flow)} \tag{17}$$

and the associated eigenvectors

$$\underline{\beta}^{s} = \begin{pmatrix} (\omega^s+f)k_+ \\ (\omega^s-f)k_- \\ \lambda_n k^2 \end{pmatrix}, \quad s = \pm \quad ; \quad \underline{\beta}^{o} = \begin{pmatrix} -\,{}^{k_+}/_{f} \\ {}^{k_-}/_{f} \\ 1 \end{pmatrix} . \tag{18}$$

The sequence of eigenvalues λ_n (λ_n being a constant) decreases monotonically, the eigenvalue of the barotropic mode (λ_o) standing out several orders of magnitude above the eigenvalue of the internal modes (n = 1,2,...).

To construct the general solution of (13), we shall involve only a partial decomposition of the solution into the three normal mode branches s = $\pm$ 0, leaving the horizontal and vertical mode structure unresolved. To do this, $\underline{\beta}^{s}$ is interpreted in (15) as an operator, independent of the wavenumber and vertical mode index. This is achieved simply by replacing ik_i by $\frac{\partial}{\partial x_i}$ and λ_n by I. Introducing

$$\Omega^{s} = s(f^2-I\nabla^2)^{1/2} = sf\left(1-\frac{I\nabla^2}{2f^2}-\frac{I^2\nabla^4}{8f^4}-\ldots\right) \tag{19}$$

$$\equiv s\ \Omega \quad ; \quad \nabla^2 \equiv \frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}$$

(15) can be written

$$H \; \underline{\beta}^s = -\underline{\beta}^s \; \Omega^s \qquad (20)$$

and defining the orthogonal projection operators $\underline{\tilde{\beta}}_s$, the general solution of (13) may be written

$$\underline{\varphi} = \sum_{s=\pm,0} \underline{\beta}^s \; \varphi^s \quad , \text{ where } \varphi^s = \underline{\tilde{\beta}}^s . \; \underline{\varphi}.$$

The equation for the scalar field φ^s follows by multiplying (13) from the left with $\underline{\tilde{\beta}}_s$

$$\frac{\partial}{\partial t} \varphi^s + i \, s \, \Omega \; \varphi^s = 0 \; . \qquad (21)$$

The equation of motion

$$\frac{\partial}{\partial t} \underline{\varphi} - i \, H \; \underline{\varphi} = \underline{q} \qquad (22)$$

in the presence of a forcing field $\underline{q} = \begin{pmatrix} q_+ \\ q_- \\ q_o \end{pmatrix}$ reduces similarly to

$$\frac{\partial}{\partial t} \varphi^s + i \; s \; \Omega \; \varphi^s = q^s \qquad (s=\pm,0) \qquad (23)$$

where

$$q^s = \underline{\tilde{\beta}}^s \cdot \underline{q} \quad . \qquad (24)$$

3. THE HOMOGENEOUS PROBLEM.

The low frequency response for the forcing field $\underline{q}$ arising from quadratic interactions between high frequency waves is investigated. The correct solution for a rotating system is obtained by allowing for first order Coriolis effects in the high - frequency wave field which is treated as strictly homogeneous, but is allowed to vary slowly with time.

If the frequencies of the wave field are high compared with the inertial period, we can presumably take the mean value of the quadratic terms in (1) - (6), in considering the low - frequency response; this contribution arises from quadratic self interactions of waves with their complex conjugates. A rough estimation of the fluctuating term indicates that its contribution is comparatively negligible for low - frequency response.

Since the wave - field is statistically homogeneous, the mean forcing terms are independent of x_1, x_2 and the resulting velocity field must then be horizontal. Hence, equations (1) and (2) need only to be considered. Provided the wave - field is homogeneous and stationnary, (1) and (2) reduce to :

$$\frac{\partial}{\partial t} \underline{u} + \underline{f} \wedge \underline{u} = - \underline{f} \wedge \underline{u}^{st} \qquad (25)$$

$\underline{u}$ and $\underline{u}^{st}$ being horizontal. The Stokes current $\underline{u}^{st}$ is the difference between the mean Lagrangian and Eulerian currents;
$f = (0,0,f)$

$$\underline{u}^{st} = \underline{u}^{l} - \underline{u}^{e} . \qquad (26)$$

To quadratic order, one has

$$\langle u_i^l(\underline{r})\rangle = \langle u_i^e(\underline{r})\rangle + \langle\frac{\partial}{\partial x_j} u_i(\underline{r})\ \zeta_j(\underline{r})\rangle \qquad (27)$$

where $\underline{\zeta}$ is the displacement of a particle from its position of rest $\underline{r}$; $\underline{u}^e(\underline{x}) = \underline{u}^e(\underline{r})$ is the fluid velocity at the particle position $\underline{x}(\underline{r}) = \underline{r} + \underline{\zeta}(\underline{r})$.

For constant $\underline{u}^{st}$, the general solution of (27), (28) is

$$\underline{u} = \underline{u}^e = - \underline{u}^{st} + \underline{U} \cos ft - (\underline{z}_o \times \underline{U}) \sin ft \qquad (28)$$

where $\underline{z}_o$ is the unit vector upwards and $\underline{U}$ is a constant amplitude dependent on the initial value of u^e. The Lagrangian current, given by (26) is a purely rotary current.

If $\underline{u}^e = 0$ at $t = 0$, the solution corresponds to a step function onset of the high frequency wave field in a previously calm ocean. It is easy to show that the earth's rotation must usually be taken into account in considering the wave induced mass transport in the ocean.

By relaxing the condition of stationary of the wave field, it is found that a free undamped inertial oscillation remains in the fluid indefinitely, after the excitation has died away. The amplitude of the residual oscillation depends strongly on the detailed time history of the excitation, which might help to explain the observed variability of inertial oscillations generated by different storms. The order of magnitude of the response is comparable with observed inertial currents for generation by both surface and internal wave fields.

4. PHASE-MIXING.

In order to explain the observed damping of inertial oscillations, various idealisations of the model are relaxed, yielding a serie of phase-mixing processes which produce damping by vertical and horizontal dispersion.

An obvious idealisation in the present model is the statistical homogeneity of the driving wave field, an assumption which greatly simplified the analysis in the previous section. The system was excited at zero wave number, a point of degeneracy at which all gravity modes have the same frequency $\pm$ f. Consequently, the vertical coordinate entered only as a parameter and there was no need for decomposition into modes.

If the fields are allowed to vary horizontally, the source function excites an ensemble of low - frequency modes of finite wave number, each of which rotates with a slightly different frequency. Velocity components of neighbouring modes which were originally excited in phase therefore gradually loose their phase relation; the field becomes "randomised". In the present problem, the essential dynamical features of the oscillations are governed by the transition from an initially coherent mode ensemble to the asymptotic random state.

The time scale of the phase - mixing process depends on the frequency separation between neighbouring modes. For large spatial scale of the driving wave field, the frequencies of the excited modes are close to f and the time scale T of the phase-mixing is large compared with f. The solution is hence expanded with respect to the parameter $\frac{t}{T}$ which avoids mode decomposition and is applicable for arbitrary stratification.

PHASE-MIXING DUE TO INHOMOGENEOUS FIELDS.

For a weakly inhomogeneous wave field, the source vector of the field equation (22) takes the form

$$\underline{q} = f \begin{pmatrix} -iu_+^{st} \\ i\ u_-^{st} \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{\partial}{\partial x_\alpha} & T_{\alpha+} \\ \frac{\partial}{\partial x_\alpha} & T_{\alpha-} \end{pmatrix} \qquad \alpha = 1,2. \qquad (29)$$

The second vector represents the divergence of the interaction stress tensor, $T_{\alpha\beta} = \langle u_\beta^2 \rangle\ \delta_{\alpha\beta} - \langle u_\alpha u_\beta \rangle$ which vanished before on account to the homogeneity, but is non zero for slowly varying field (Hasselmann,1970). The third component of $\underline{q}$ is irrelevant for the generation of inertial oscillations.

Using (29), equation (23) can be integrated to yield the inertial current response

$$\varphi^s(\underline{x},t) = e^{-is\Omega t} \varphi^s_{t=0} + \int_0^t e^{-is\Omega(t-t')} q^s(\underline{x},t')dt' . \quad (30)$$

A convergent serie is then obtained by expanding (30) with respect to the perturbed frequency operator X_w

$$\Omega = f + X_w = f - \frac{I^2}{2f} - \frac{I^{2\ 4}}{8f^3} - \ldots \quad . \quad (31)$$

For $X_w = 0$, one finds the solution for a homogeneous wave field. The deviation from this solution is governed by the phase-mixing operator $X_w . X_w \ll \Omega \simeq f$ for small inhomogeneities.

In the case $q^s = 0$, (i.e. we considered the evolution of the inertial oscillations after the generation wave field has passed by), equation (30) represents an inertial oscillation with slowly varying amplitude and phase; the rate of change of phase can be interpreted as a frequency shift. The convergence of the expansion of (30) can be analysed in the normal-mode viewpoint. Quantitative estimates can be found in (I), in the simple case N^2 = const.

Since $X_w \ll \Omega$, it is normally sufficient to retain the first term in the expansion (31). This could also have been derived using the standart formulae for eigenvalue perturbation. If the operator H in the eigenvalue equation (13) is of the form

$$H = H_o + H' \quad (32)$$

where $H' \ll H_o$, the (negative) eigenvalue Ω^s can be expanded in a serie

$$\Omega^s = \Omega^s_o + \Omega^s_1 + \Omega^s_2 + \ldots \qquad (33)$$

where Ω^s_o is an eigenvalue of H_o and

$$\Omega^s_1 = - \underline{\tilde{\beta}}^s_o \, H' \, \underline{\beta}^s_o \; ; \quad \Omega^s_2 = \ldots \qquad (34)$$

Here, $\underline{\beta}^s_o$, $\underline{\tilde{\beta}}^s_o$ refer to the eigenvectors and the orthogonal vectors of the unperturbed operator H_o. In our case

$$H_o = \begin{pmatrix} -f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$H' = \begin{pmatrix} 0 & 0 & i\partial_+ \\ 0 & 0 & i\partial_- \\ \dfrac{iI\partial_-}{2} & \dfrac{iI\partial_+}{2} & 0 \end{pmatrix} \qquad (35)$$

and $\Omega^s = sf$. The vectors $\underline{\tilde{\beta}}^s_o = \underline{\beta}^s_o$ are the three unit vectors parallel to the coordinate axes. The eigenvectors $\underline{\beta}^s$ can be similarly expanded but, in the lowest - order phase - mixing approximation, one can set $\underline{\beta}^s = \underline{\beta}^s_o$ and need only the perturbation of the eigenoperator. This approach is used to consider further perturbation which destroy the zero wave number and produce phase-mixing.

PHASE-MIXING DUE TO THE HORIZONTAL CORIOLIS COMPONENT.

The analysis (I) shows that the horizontal Coriolis parameter $\hat{f}$ cannot always be neglected, particularly for weak stratifications and large lateral scales typical of deep-ocean conditions. The inclusion of $\hat{f}$ presents no computational difficulty.

PHASE-MIXING DUE TO WAVE-GUIDE INHOMOGENEITIES.

Lateral variations of the wave-guide parameters f, N^2 and h can be treated as perturbations if the lateral length scales are large compared with the ocean depth. The classical methods of geometrical optics are not applicable, since we are concerned with an ensemble of modes, rather than a single mode. Thus slow wave-guide variations again result in a diffusion- type behaviour. Detailled investigation (I) leads to the conclusion that planetary effects (f variation) can be neglected compared with the lateral variations of the wave-guide or the field.

PHASE-MIXING DUE TO NON LINEAR INTERACTIONS WITH GEOSTROPHIC CURRENTS.

Since the geostrophic currents are time independent in the linear, f-plane approximation, the quadratic interaction with an inertial oscillation yields a perturbation which is linear in the inertial oscillation, with a time-independent coefficient.

The phase-mixing operator for the geostrophic interactions follows from (34)

$$X_g^s = i\left[-U_j \frac{\partial}{\partial x_j} - \frac{(\partial_{-s} U_s)}{2} + \frac{\left(\frac{\partial}{\partial x_3} U_s\right) I_a \partial_{-s}}{2}\right] \quad (s = \pm) \quad (36)$$

where $\underline{U}$ is the heostrophic current and $I_a = \int_{-h}^{x_3} dx_3'$. There is no summation over s.

The contribution to phase-mixing from this process appears at present marginal (I).

5. A NUMERICAL EXAMPLE.

The decay of an inertial oscillation due to lateral inhomogeneities of the initial field distribution has been computed using the phase - mixing expansion (30), (31); the initial distribution was taken as axisymmetric Gaussian in the horizontal coordinates, with an exponential vertical profile corresponding to generation by surface waves. The numerical values were adjusted to agree with observations at the site near Bornholm (Tomczak Jr., 1969).

Detailled discussion and figures will not be reproduced here and can be found in (I). In fact, many of the gross features of inertial oscillations observed in the Baltic were reproduced. These follow largely from a surface - wave source and are independent of the form of the decay process.

The time scale of the computed decay was not inconsistent with measurements, but the details were not everywhere convincing. Inclusion of lateral wave - guide inhomogeneities would probably remedy the shortcomings of the model.

6. CONCLUSIONS.

Estimates of the non linear generation of inertial oscillations by high frequency gravity waves agree in order of magnitude with inertial currents observed both near the surface and in the interior of the ocean. Surface gravity waves can drive inertial oscillations either through horizontal stresses (radiation stresses) or the vertical shear stress induced by the rotation of the earth. For horizontal scales smaller than 100 km, the horizontal stress is more important, whereas the shear stress dominates for larger scales. In the case of internal gravity waves, the horizontal stress is always negligible; estimates of the shear stress based on observed internal wave spectra yield values comparable with shear stresses for surface waves.

The decay of inertial oscillations due to phase-mixing has been investigated for five processes. Lateral inhomogeneities of the inertial oscillation and lateral wave-guide variations yield comparable decay rates in reasonable agreement with observations; the influence of the horizontal component of the Coriolis vector and interactions with geostrophic currents appear to be of marginal significance; planetary effects are negligible.

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