GENERATION OF TRANSIENT NEARLY INERTIAL INTERNAL WAVES BY THE INTERACTION BETWEEN INTERNAL WAVES AND A GEOSTROPHIC SHEAR CURRENT

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1. INTRODUCTION.

There is evidence that the nonlinear coupling between modes is weak in the internal wave frequency range of the ocean-current spectrum (Fofonoff, 1969, Hasselmann, 1968). Hence, other mechanisms by which internal gravity waves exchange energy with their surroundings or with other types of motion might be mainly responsible, in some cases, of the observed spectrum.

Such energy exchange occurs when internal waves are in the presence of a mean shear flow (Bretherton, 1966, Phillips, 1966). An early attempt to evaluate the influence of this mechanism on the shape of the frequency spectra was made by Phillips (1966) for a simple non-rotating model. His result must nevertheless be interpreted with care, as they are restricted to a narrow band of frequencies and obscured by an algebraic error.

The influence of a steady shear on the internal wave spectra in a rotating medium has been recently investigated by Frankignoul (1970), on the basis of Phillips'work. The mean shear current was supposed to be in ageostrophic balance and the wave propagation directions isotropically distributed in the mean with respect to the horizontal. The comparison of the equilibrium spectral shapes with some relevant oceanic data indicated a good agreement.

In most deep sea situations, it is known that, if a steady

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shear flow is to be assumed, the hypothesis of geostrophic balance is more relevant. In the present paper, attention will be focused on the effect of a horizontal density gradient on the time-behavior and the equilibrium spectral shapes of the internal waves. An attempt will be made to evaluate the influence of the dissipations.

2. DERIVATION OF THE WAVE EQUATION.

It is assumed that the fluid is incompressible and non-dissipative. The horizontal component of the earth rotation is neglected and the mean shear flow is supposed to be steady, unidirectional and depending only upon the depth. To the Boussinesq approximation, the linearised equations of motion reads

$$\frac{du}{dt} - fv + wU' = -\frac{\partial \pi}{\partial x} \qquad (1)$$

$$\frac{dv}{dt} + fu = -\frac{\partial \pi}{\partial y} \qquad (2)$$

$$\frac{dw}{dt} - b = -\frac{\partial \pi}{\partial z} \qquad (3)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (4)$$

$$\frac{db}{dt} - v\, M^2 + w\, N^2 = 0 \qquad (5)$$

where $(U(z),0,0)$ is the mean velocity, (u,v,w) the fluctuating velocity, b the fluctuating part of the buoyancy, π the non-hydrostatic pressure per unit mass, f the Coriolis parameter. $N^2 = -\frac{g}{\rho_0}\frac{\partial \bar{\rho}}{\partial z}$ and $M^2 = \frac{g}{\rho_0}\frac{\partial \bar{\rho}}{\partial y}$ are the Brunt-Väisälä frequency and its horizontal analogue, g is the gravitational acceleration, $\bar{\rho}$ the mean non-hydrostatic density; $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + U\frac{\partial}{\partial x}$.

Geostrophic and hydrostatic equilibria lead to

$$fU' = M^2 = -sN^2 \qquad (6)$$

where s is the vertical slope of an isopycnal in the mean density field. For simplicity, it is assumed that

$$U'(z) \equiv U' \equiv \text{constant}$$
$$N^2(z) \equiv N^2 \equiv \text{constant} . \qquad (7)$$

For all these assumptions to be approximately valid, very small and very large scale motions must be disregarded. A typical region where the model applies roughly is the region below the main thermocline, far from boundaries.

By elimination, one gets an equation for w

$$\frac{d^3}{dt^3}\nabla^2 w+\frac{d}{dt}\left(f^2\frac{\partial^2}{\partial z^2}+2M^2\frac{\partial^2}{\partial y\partial z}+N^2\nabla_n^2\right)w-2M^2U'\frac{\partial^2 w}{\partial x\partial y}-2M^2 f\frac{\partial^2 w}{\partial x\partial z}=0. \qquad (8)$$

Instead of using a normal mode expansion, we assume a wave decomposition of the form

$$w = W(t)\, e^{i(kx+ly\ -\ kU'zt)} \qquad (9)$$

k and l being constant, on the basis of the stretching action of the shear, as found by examination of the ray equations (Jones, 1969, Mooers, 1970). The time-origin is chosen when the wave-number is horizontal so that negative time is allowed. In order to set-up a model which is amenable to analysis, W is assumed to be depth-independent. The medium is supposed to be infinitely deep and there is no boundary conditions.

As only wave-number components in the direction of the mean shear undergo its influence, the form (9) is of no use if $k \equiv 0$: there is no energy exchange between the shear flow and the waves (at least to the Boussinesq approximation, see Healey and Leblond, 1969), and the present model does not apply. If the wave propagates obliquely, its wave number is rotated and increases as soon as it becomes horizontal. This was first noted by Phillips (1966).

Using the form (9), equation (8) reduces to

$$K^{-2} \frac{\partial^3}{\partial T^3}[(1+T^2)W]+ \frac{\partial}{\partial T}[(1+2s \sin\Phi\, T+F^2T^2)W] + 2(F^2T+s \sin\Phi)W = 0 \tag{10}$$

where Φ specifies the angle between the horizontal wave number vector and the steady current direction, and where

$$T = U' \cos\Phi\, t$$
$$K^2 = \frac{N^2}{U'^2 \cos^2\Phi} \tag{11}$$

3. ASYMPTOTIC SOLUTION FOR LARGE K^2.

As in the previous works based on the wave decomposition (9), an asymptotic solution can be found when K^2 is large. This corresponds usually to a very large dynamical stability of the mean flow. To the first order in K^{-1}, one has

$$W \sim G(1+T^2)^{-\frac{3}{4}} (1+2\, s \sin\Phi\, T+F^2T^2)^{\frac{1}{4}}\, e^{\pm i \frac{K}{N}\int n\, dT} \tag{12}$$

where G is a constant and n the intrinsic frequency defined by

$$n = N\left(\frac{1+2\, s \sin\Phi\;\; T+F^2T^2}{1 + T^2}\right)^{\frac{1}{2}} . \tag{13}$$

The frequency n changes continuously with time and tends to the inertial frequency as t increases, so that energy is transferred towards low frequencies for $T > 0$. The frequency band in the geostrophic case is larger than in the cases where there is no mean flow or when the mean flow is in ageostrophic balance. Taking for simplicity F^2 and s^2 much smaller than unity, as is verified in most physical situations, the frequency limits are given by

$$f\left(1 - \frac{s^2 \sin^2 \Phi}{F^2}\right)^{\frac{1}{2}} < n < N \left(1+s^2\sin^2\Phi\right)^{\frac{1}{2}} . \qquad (14)$$

As discussed by Mooers (1970) in the case where $\Phi = 90°$, only the low frequency limit can be effectively affected.

The vertical component of the group velocity Cg, given by

$$\frac{\partial n}{\partial t} = - k\, U'\, Cg ,$$

allows for an estimation of the vertical distance d over which the energy at any point runs from a time t_1 to a time t_2, by integrating (15) between t_1 and t_2. Typical value of d is d_Φ corresponding to $t_1 \equiv 0$, $t_2 \equiv \infty$. For conditions found at Site D (39°20'N,70°W) at a depth of 2000 meters, one has $d_o \sim O(2\lambda)$, λ being the horizontal wave-length. For not too large scale, the vertical displacement of the wave energy is small; this could partly justify the omission of boundary conditions. The dispersion relation is

$$(n^2-f^2)m^2 = (N^2-n^2)(k^2+l^2) + 2lm\, f\, U' \qquad (16)$$

where m is the vertical component of the wave number.

There is an energy exchange between the wave and the mean flow. The mean Reynolds stress produced by the wave is given by

$$-\rho_o \overline{uv} \sim -\frac{1}{2} \rho_o G^2 \cos \Phi \, T(1+T^2)^{-\frac{3}{2}}(1+2s \sin \Phi \, T+F^2T^2)^{\frac{1}{2}} \quad (17)$$

an overbar indicating the mean value. For $T > 0$, the Reynolds stress is negative and energy is extracted from the wave motion and transferred to the mean current. The rate of energy transfer is indicated by the behavior of the horizontal and vertical kinetic energy density of the wave

$$E_h \sim \frac{1}{4} \rho_o G^2T^2(1+T^2)^{-\frac{3}{2}} (1+2s \sin \Phi \, T+F^2T^2)^{1/2} \quad (18)$$

$$E_v \sim \frac{1}{4} \rho_o G^2(1+T^2)^{-\frac{3}{2}} (1+2s \sin \Phi \, T+F^2T^2)^{1/2} , \quad (19)$$

After a time long enough, E_h tends to a constant value whereas E_v vanishes, and the intrinsic frequency becomes the inertial frequency. Hence, the effect of the shear is to transform the internal wave motions into inertial oscillations, provided nothing interrups the process (see below).

If there is an energy transfer between waves and mean flow, one cannot rigourously assume the steadiness of the mean current. However, if the isotropy of the wave propagation directions is assumed, all wave energy contributions to the mean current balance. Then, the assumption of a steady shear, though being invalid locally, can be justified.

It must be noted that the transfer of the wave energy to the mean flow is identical to the phenomenon of critical layer absorption, as showed Booker and Bretherton (1967). No critical level occurs here as it corresponds to a fixed frequency (normal mode decomposition), while, in Phillips'model, the frequency is varying and the absorption of vertical momentum continuous.

4. SHEAR INSTABILITY AND DISSIPATIVE EFFECTS.

As an effect of the mean shear, the vertical scale of the wave motion is reduced, increasing the shear produced by the wave. This shear tends to become infinite as n tends to f, so that the Richardson number unavoidably becomes smaller than 1/4, the critical value for shear instability (necessary but not sufficient condition).

To calculate the frequency below which the motion might become unstable, we define a minimum Richardson number by adding the shear produced by the wave to the overall mean shear, which can be neglected as $K^2 >> 1$. Calling a the wave amplitude at t = 0 and λ the horizontal wave-length, a critical curve can be drawn in a $(n, \frac{a}{\lambda})$ diagram (fig.1). The larger is the wave amplitude, the smaller is the frequency below which instability may occur. Let us mention that the appearance of a dynamical instability does not imply that the wave itself is destroyed.

So far, dissipative effects have been neglected as is usually done in a wave theory. However, the evolution towards a vanishingly small scale in the vertical direction accelerate the viscous damping. This should prevent the wave from the occurrence of shear instability, in some cases.

Detailed investigation of the combined effects of viscosity, heat and salt diffusion leads to very tedious algebra, but bounds of their influence can be obtained more easily by using

$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + U \frac{\partial}{\partial x} - \nu\nabla^2$ in the basic equations instead of the former value. According to the choice of ν-kinematic viscosity or thermal diffusivity (the salinity being neglected)- one gets upper or lower bounds of the dissipative effects. The analysis is rigourous for unitary Prandlt number.

Calculation leads to an equivalent asymptotic solution; it can be shown that solution (12) is still valid, except that there is an additionnal factor

$$e^{-K\eta} \int_{T_o}^{T} (1+T^2)\, dT$$

with (20)

$$\eta = \frac{4\pi^2 \nu}{\lambda^2 N^2} \,.$$

T_o corresponds either to the generation of the wave or to the time of entrance of a wave in such a region of shear, and must be introduced because the dissipations are irreversible.

It can be shown that, provided the ratio of the wave amplitude to the wave length is not large, there will be no dynamical instability. For illustration of the damping effect, one can calculate that the amplitude of a wave (λ=200 m) generated at Site D (2000 m) is reduced by a factor D such that $1.05 < D < 1.45$, after the lapse of time necessary for the frequency to decrease from N to 0.15 N.

The statement of section 3 must be modified to take into account the phenomenom described in this section. In most cases, the wave frequency will become rather close to f, but will never reach it. Two mechanisms - shift towards lower frequencies (when n becomes close to f), damping of the horizontal oscillations - will strongly contribute to give a transient character to a given low frequency Fourier component. Hence, the shear will act to transform the internal waves into nearly inertial transient oscillations, if the inertial angle of incidence of the wave is appropriate.

The action of the shear on internal waves could then be an important mechanism for the appearance of inertial oscillations in deep sea, and the observed transient character of those waves is easily explained in our model.

5. EQUILIBRIUM SPECTRAL SHAPES.

The main interest of the unusual wave decomposition (9) is that it allows to derive equilibrium spectral shapes from simple arguments.

In his paper, Phillips (1966) assumed there was a steady source of internal waves which has been acting long enough for a steady spectrum to develop. Then, the frequency spectra can be derived simply because, as each wave is rotated, its amplitude, velocity and frequency are modified. A fundamental problem is to find a suitable energy source in deep sea, far from boundaries; very little is known on this subject.

Phillips'results are based on the Doppler-shifted wave frequency and their use for time-dependent spectra at a fixed point must be interpreted with care. In fact, provided the source is fixed and steady, there will be no temporal variation in the frequency spectra at a given place, but only a direct dependance to the source frequency spectrum. Also, all wave energy transferred to the mean flow will accumulate and U cannot be supposed to depend only upon the depth, except to a rough approximation.

An assumption of more statistical character was made by Frankignoul (1970) and will be used here. We assume there is a large supply of many waves coming in the region of shear, in an undetermined manner but isotropically distributed with respect to the horizontal. This does not answer to the question of the source of the waves, but is not too restrictive. Because of the isotropy of the wave propagation directions, all Doppler-Shift effects balance in the mean and the frequency n can be used for measurements at a fixed point, provided they are long enough. Also, it is easy to see that the frequency spectra will be similar to the spectra obtained by assuming the existence of a steady source (by considering a pair of waves, symetric about the horizontal plane). Using Phillips'method (1966) to derive the spectral shapes, it is found that the horizontal and vertical kinetic energy density spectra E_h and E_v , and the potential energy spectrum $P_{\varphi\varphi}$ are respectively of the form

$$E_h \propto T^2(1+2s\sin^2\Phi T+F^2T^2)(s\sin\Phi T^2+(1-F^2)T-s\sin\Phi)^{-1} \quad (21)$$

$$E_v \propto (1+2s\sin^2\Phi T+F^2T^2)(s\sin\Phi T^2+(1-F^2)T-s\sin\Phi)^{-1} \quad (22)$$

$$P_{\varphi\varphi} \propto (1+T^2)(s\sin\Phi T^2+(1-F^2)T-s\sin\Phi)^{-1} . \quad (23)$$

To express the spectra in function of the frequency, formula (13) must be used. A simple analytical form is found when $\Phi \equiv 0$:

$$E_h \propto n^2(N^2-n^2)^{\frac{1}{2}} (n^2-f^2)^{-\frac{3}{2}} \quad (24)$$

$$E_v \propto n^2(N^2-n^2)^{-\frac{1}{2}} (n^2-f^2)^{-\frac{1}{2}} \quad (25)$$

$$P_{\varphi\varphi} \propto (N^2-n^2)^{-\frac{1}{2}} (n^2-f^2)^{-\frac{1}{2}} . \quad (26)$$

The hypothesis of geostrophic balance only affects the low frequency part of E_v and $P_{\varphi\varphi}$ - they show a peak at the inertial frequency - , as compared with results obtained in the ageostrophic case, otherwise very similar.

To compare (24) - (26) with observations, it must be noted that both ends of the curves cannot be representative of the ocean-current spectra. Indeed, at very low frequency, the waves we have studied are subject to shear instability, whereas at very high frequencies, the model itself is invalid, since the micro-structure has been neglected and only a mean N has been used. As, in the real oceans, other motions are also present at very low and very high frequency, a good agreement cannot be expected near the limits of the internal wave frequency range.

The model can be tested in a region, far from boundaries, where the mean Brunt-Väisälä frequency is roughly constant - this is often observed below the main thermocline - and where there is a mean shear which is roughly steady and unidirectional, at least in periods long enough for the transfer process to occur. The value of the shear affects only the time scale of the frequency shift of the waves, and contributions from different periods of time where the model is valid will add in the same way, no matter what is the direction and magnitude of the mean shear.

Measurements made at Site D (39°20'N,70°W), below the main thermocline, seem to be suitable for a comparison with the computed spectral shapes (Frankignoul, 1970). Fig. 2 shows a comparison between the calculated horizontal kinetic energy spectrum and observations made at a depth of 2000 meter, in a region where the model applied best. The agreement is good, as in the ageostrophic case. There is no other really relevant observations which can be used to check the validity of the model. In (Frankignoul, 1970), comparison has been made with vertical kinetic energy and temperature fluctuations spectra recorded at the lower edge of the thermocline (where N is not constant) but the accuracy of the measurements is weak and the model rather inappropriate so that no conclusion can be drawn.

The effects of the dissipation on the equilibrium spectra cannot be investigated in detail but it can be seen intuitively that they will not seriously affect the value (24)-(26) in considering two waves coming in the region of shear, with propagation directions symetric about the horizontal plane. The frequency of one wave is shifted towards higher frequencies, while the frequency of the other one is shifted towards lower frequencies. As the dissipation depends only upon the time elapsed, the low and high frequency regions of the spectra will be affected in a similar way and the spectral shape will not be influenced by the dissipations.

6. CONCLUSIONS.

The effect of a constant steady and unidirectional shear on internal waves is to modify their frequency, wave-number and amplitude, and finally to transform them into inertial oscillations, after all vertical kinetic energy has been transferred to the mean current. Consideration of dissipative effects introduces a limited life-time of the wave and gives a transient character to the low-frequency waves, in agreement with the observations of inertial oscillations.

The good agreement between spectra of the horizontal kinetic energy computed from the model and some relevant observations bears out the interest of studying the effect of the shear on internal waves.

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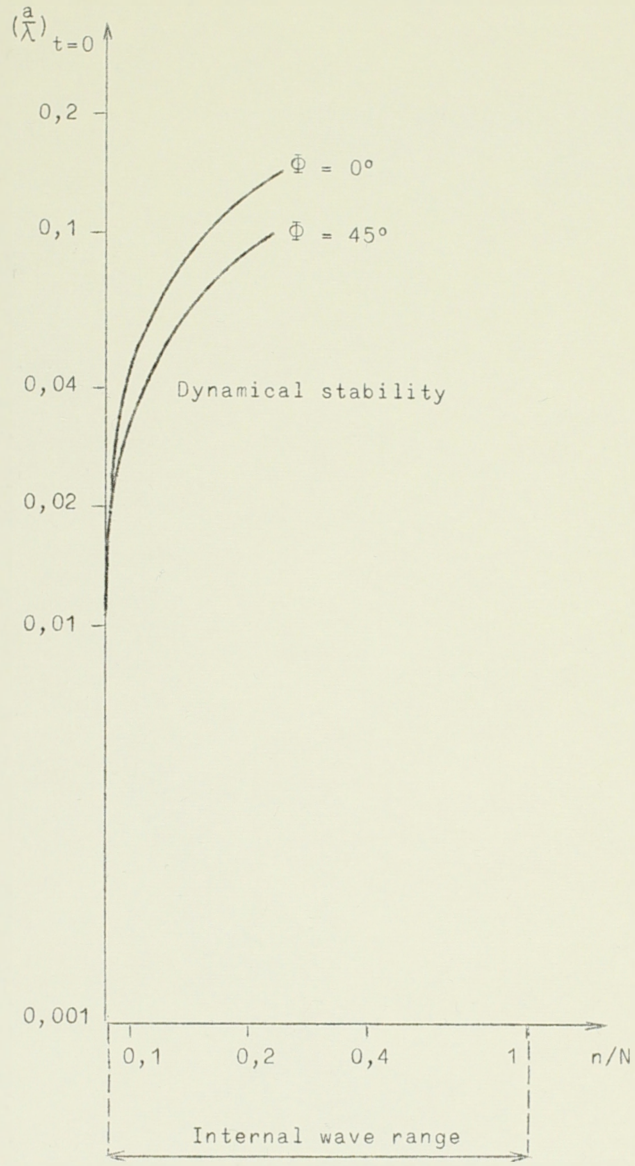
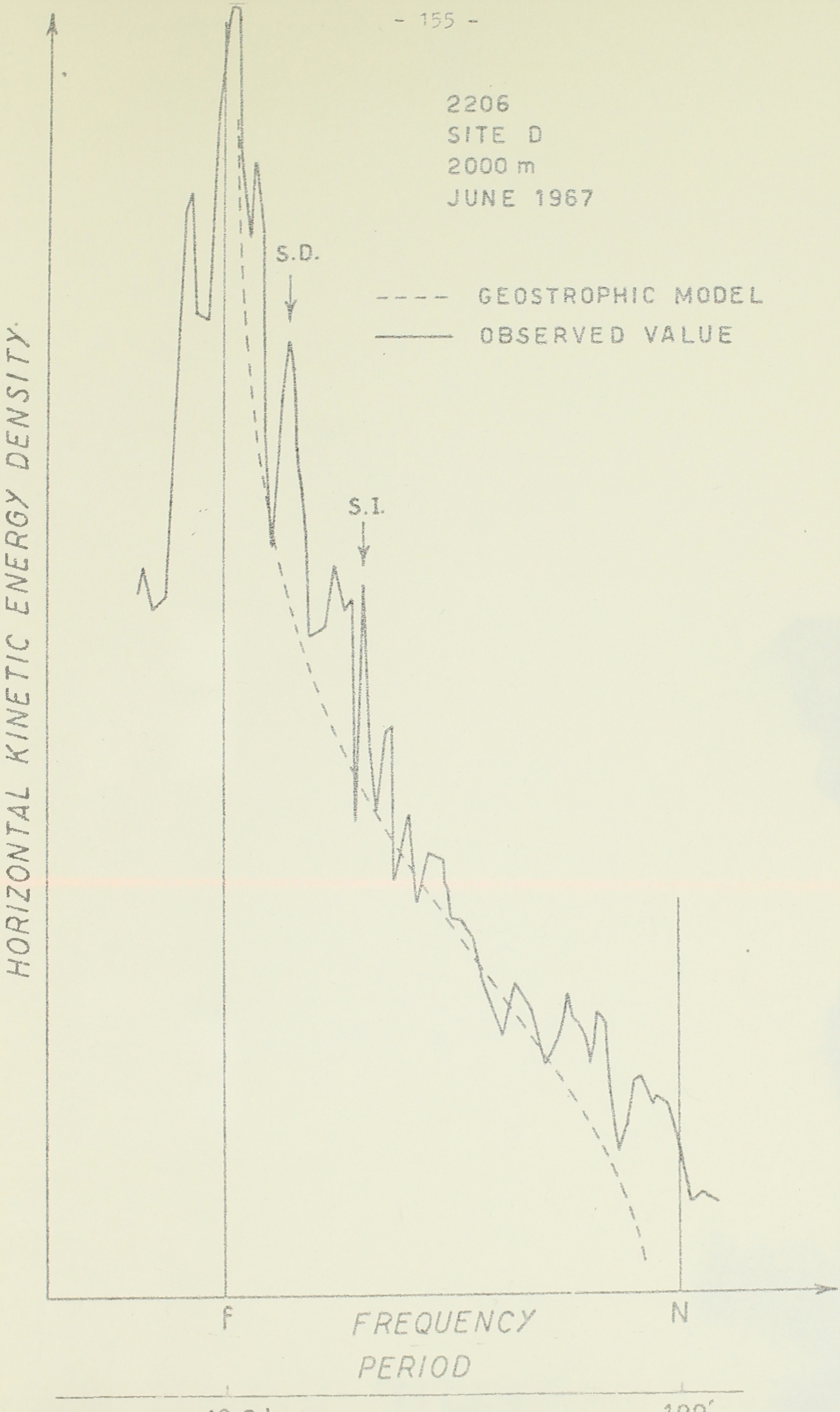
LEGENDS FOR FIGURES.

Figure 1. Curves relating the non-dimensional frequency under which instability may occur to the ratio of the wave amplitude to the wave length, at t = 0. Numerical values correspond to site D, 2000 m. Dissipations are neglected.

Figure 2. A comparison of the horizontal kinetic energy density spectrum observed at Site D, 39°20'N,70°W; (2206, 2000m, June 1967) with the calculated spectrum. Arbitrary magnitude of the computed curve has been chosen to fit the observations.


$(\frac{a}{\lambda})_{t=0}$
0,2
0,1
0,04
0,02
0,01
0,001
Φ = 0°
Φ = 45°
Dynamical stability
0,1
0,2
0,4
1
n/N
Internal wave range


2206
SITE D
2000 m
JUNE 1967
S.D.
S.I.
GEOSTROPHIC MODEL
OBSERVED VALUE
HORIZONTAL KINETIC ENERGY DENSITY
F
N
FREQUENCY
PERIOD
18,9 hours
100'