JOURNAL OF GEOPHYSICAL RESEARCH, VOL. 97, NO. C6, PAGES 9755–9757, JUNE 15, 1992

Comment on "A Variational Inverse Method for the Reconstruction of General Circulation Fields in the Northern Bering Sea" by Pierre P. Brasseur

ERIC DELEERSNIJDER

G. Lemaître Institute of Astronomy and Geophysics, Catholic University of Louvain, Louvain-la-Neuve, Belgium

SUMMARY OF BRASSEUR'S APPROACH

In an excellent article, *Brasseur* [1991] (hereinafter PB) describes the application of a variational inverse model to the interpolation of two-dimensional (in a horizontal plane) scalar data, such as temperature and salinity, originating from the region of the Bering Strait. PB addresses the problem of constructing a continuous interpolation function, ϕ, in accordance with the following requirements: ϕ must be as close as possible to the data, should be as smooth as possible, and should verify, if necessary, some relevant mathematical constraints. The interpolation function stems from a minimum principle,

$$\min_{D,\,\phi} J(\phi)$$

where D represents the domain of interest. The minimum of the functional J corresponds to

$$\delta_\phi J(\phi) = 0 \tag{1}$$

δ_ϕ denoting the first-order variation operator with respect to ϕ. The functional J reads,

$$J(\phi) = \int_D S(\phi)\, dD + \int_D C(\phi, \mathbf{x})\, dD \tag{2}$$

where $S(\phi)$ is an appropriate "smoothness operator"; $C(\phi, \mathbf{x})$ encompasses all the constraints applied to ϕ at location $\mathbf{x}$. The numerical solution of (1)–(2) is obtained by means of the finite element method.

RECONSTRUCTION OF VELOCITY FIELDS

Taking advantage of the computer program developed to solve (1)–(2), PB uses the minimum principle stated above to model the depth-averaged climatic velocity field, $\mathbf{u}$, in D. If the sea depth is denoted H, the transport $H\mathbf{u} = \mathbf{U}$ is given by

$$\mathbf{U} = -\nabla \times \psi \mathbf{e}_3 = -\nabla\psi \times \mathbf{e}_3 \tag{3}$$

where ∇, ψ, and $\mathbf{e}_3$ stand for the horizontal gradient operator, the stream function, and the vertical unit vector, respectively. The flow is assumed to be time-independent so that (3) implies that the mass conservation equation, $\nabla \cdot \mathbf{U} = 0$, is identically satisfied.

The stream function is the solution of (1)–(2), where ϕ is replaced by ψ. The "smoothness operator" is defined as

$$S(\psi) = (\nabla\nabla\psi) : (\nabla\nabla\psi) + \frac{\alpha}{L^2} |\nabla\psi|^2 \tag{4}$$

In (4), α is a dimensionless weighting factor and L stands for an appropriate length scale. PB states that in the absence of significant wind stress, the flow in the region of the Bering Strait roughly obeys a simple momentum equation, namely

$$f\mathbf{e}_3 \times \mathbf{U} = -H\nabla q - \frac{c_D}{H}\left(\nu_* + \frac{|\mathbf{U}|}{H}\right)\mathbf{U} \tag{5}$$

where f is the Coriolis factor, q is the reduced pressure, c_D is the bottom drag coefficient, and ν_* is a velocity taking into account the influence of the tides on the long-term average of the bottom stress [*Overland and Roach*, 1987]. The dynamic relation (5) is taken into account via the mathematical constraint C. Hence

$$C(\psi) = \frac{\gamma}{V}\left[\frac{c_D}{H^2}\left(\nu_* + \frac{|\mathbf{U}|}{H}\right)|\nabla\psi|^2 + 2\,\frac{f}{H^2}\,(\nabla H \cdot \mathbf{U})\psi\right] \tag{6}$$

where γ and V are a dimensionless weighting factor and a proper velocity scale, respectively.

First, PB takes the variation of $J(\psi)$ with respect to ψ only, $\mathbf{U}$ being considered a given fixed function. The resulting equation, which is linear in ψ, is then solved. Finally, definition (3) is used to update $\mathbf{U}$. Thus the solution of the minimum principle is obtained after a certain number of iterations.

The boundary Γ of the computational domain is the sum of two sets, Γ_1 and Γ_2, of boundary segments. The set Γ_1 encompasses the impermeable boundary segments, on which ψ is imposed. On Γ_2, the set of open sea boundaries, no condition on ψ is required a priori. Thus with the finite element method, it is not necessary to prescribe the distribution of the normal velocity on inflow boundaries. PB states that this fact is a distinct advantage of the method he is using. It must however be stressed that additional boundary conditions, sometimes termed "natural conditions," ensue from the application of the minimum principle.

PB did not calculate these natural conditions. Deriving them would permit assessment of to what extent the finite element method leads to a better treatment of open boundary conditions.

The objective of this note is to caculate the Euler-Lagrange equations arising from PB's minimum principle so as to obtain the natural boundary conditions.

One must bear in mind that when the finite element method is used, the natural conditions are satisfied only in the form of a weighted average along the boundaries, the weight function depending upon the shape functions utilized.

Paper number 92JC00868.
0148-0227/92/92JC-00868$02.00

EULER-LAGRANGE EQUATIONS AND BOUNDARY CONDITIONS

Combining (2), (4), and (6), one gets

$$J(\psi) = \int_D \left[(\nabla\nabla\psi):(\nabla\nabla\psi) + \frac{\alpha}{L^2} |\nabla\psi|^2 + \frac{\gamma}{V}\frac{c_D}{H^2}\left(\nu_* + \frac{|\mathbf{U}|}{H}\right)|\nabla\psi|^2 + 2\frac{\gamma}{V}\frac{f}{H^2}(\nabla H \cdot \mathbf{U})\psi \right] dD \quad (7)$$

In (7), ":" represents the so-called "double dot product" or "colon product." The latter is a tensor operator defined as follows:

$$\mathbf{A}:\mathbf{B} = \sum_i \sum_j A_{i,j}B_{i,j} \quad (8)$$

where $A_{i,j}$ and $B_{i,j}$ are the components of tensors **A** and **B**, respectively. Introducing (7) into (1), a relation is obtained from which the relevant Euler-Lagrange equations may be derived:

$$\int_D \left\{ (\nabla^2\nabla^2\psi) - \frac{\alpha}{L^2}\nabla^2\psi - \frac{\gamma}{V}\nabla\cdot\left[\frac{c_D}{H^2}\left(\nu_* + \frac{|\mathbf{U}|}{H}\right)\nabla\psi\right] + \frac{\gamma}{V}\frac{f}{H^2}(\nabla H\cdot\mathbf{U})\right\}\delta\psi\, dD + \int_\Gamma \mathbf{n}\cdot(\nabla\nabla\psi)\cdot\nabla(\delta\psi)\, d\Gamma$$
$$+ \int_\Gamma \mathbf{n}\cdot\left[-\nabla(\nabla^2\psi) + \frac{\alpha}{L^2}\nabla\psi + \frac{\gamma}{V}\frac{c_D}{H^2}\left(\nu_* + \frac{|\mathbf{U}|}{H}\right)\nabla\psi\right]\delta\psi\, d\Gamma = 0 \quad (9)$$

In the domain of interest, (9) leads to

$$\frac{f}{H^2}(\nabla H\cdot\mathbf{U}) - \nabla\cdot\left[\frac{c_D}{H^2}\left(\nu_* + \frac{|\mathbf{U}|}{H}\right)\nabla\psi\right] = \frac{V}{\gamma}\left(\frac{\alpha}{L^2}\nabla^2\psi - \nabla^2\nabla^2\psi\right) \quad (10)$$

The left-hand side of the above equation is associated with the constraint (6) or, equivalently, with the curl of the simplified momentum equation (5). The right-hand side stems from the smoothing operator. If the weight of the dynamic constraint increases, i.e., if γ increases, one may expect the smoothing operator to become less important. And, in the limit $\gamma \rightarrow \infty$, Euler-Lagrange equation (10) tends to the curl of (5), in the absence of any singular perturbation problem. The influence of γ on the flow field illustrated by the numerical experiments of PB is in agreement with the present reasoning.

Along the boundary of the computational domain, it is worth expressing ∇ as the sum of the normal gradient and the tangential gradient:

$$\nabla = \mathbf{n}\frac{\partial}{\partial n} + \mathbf{t}\frac{\partial}{\partial t} \quad (11)$$

with

$$\left(\frac{\partial}{\partial n}, \frac{\partial}{\partial t}\right) = (\mathbf{n}\cdot\nabla, \mathbf{t}\cdot\nabla) \quad (12)$$

where **n** is the outer normal to the boundary; **t** is tangential to Γ with

$$\mathbf{t} = \mathbf{e}_3 \times \mathbf{n} \quad (13)$$

On Γ, the normal and tangential derivatives of $\delta\psi$ are independent quantities. Hence one must rewrite the second integral of (9) as

$$\int_\Gamma \mathbf{n}\cdot(\nabla\nabla\psi)\cdot\nabla(\delta\psi)\, d\Gamma = \int_\Gamma \left\{\mathbf{n}\cdot(\nabla\nabla\psi)\cdot\mathbf{n}\frac{\partial(\delta\psi)}{\partial t} - \frac{\partial}{\partial t}[\mathbf{n}\cdot(\nabla\nabla\psi)\cdot\mathbf{t}]\delta\psi\right\} d\Gamma \quad (14)$$

Strictly speaking, (14) is valid if Γ is a curve that is everywhere differentiable. If this is not the case, corrections are necessary at those points where the tangent to Γ is not unambiguously defined.

Combining (9) and (14), one has a natural boundary condition that is valid on Γ_1 and Γ_2:

$$\frac{\partial\mathbf{T}}{\partial n}\cdot\mathbf{n} = 0 \quad (15)$$

where

$$\mathbf{T} = \mathbf{e}_3 \times \mathbf{U} \quad (16)$$

Decomposing **U** on the boundary into its normal and tangential component,

$$\mathbf{U} = U_n\mathbf{n} + U_t\mathbf{t} \quad (17)$$

it may be shown that, along a rectilinear boundary segment, (15) simplifies to

$$\partial U_t/\partial n = 0 \quad (18)$$

which means that the normal derivative of the tangential transport is zero.

On Γ_1, ψ is prescribed so that no additional boundary condition is found. On Γ_2, however, (9) and (14) provide another natural condition,

$$\frac{c_D}{H^2}\left(\nu_* + \frac{|\mathbf{U}|}{H}\right)U_t = -\frac{V}{\gamma}\cdot\left[\frac{\alpha}{L^2}U_t + \frac{\partial}{\partial n}(\nabla\cdot\mathbf{T}) + \frac{\partial}{\partial t}\left(\mathbf{t}\cdot\frac{\partial\mathbf{T}}{\partial n}\right)\right] \quad (19)$$

which, along a rectilinear boundary segment, transforms to

$$\frac{c_D}{H^2}\left(\nu_* + \frac{|\mathbf{U}|}{H}\right)U_t = -\frac{V}{\gamma}\left[\frac{\alpha}{L^2}U_t + 2\frac{\partial^2 U_n}{\partial n\,\partial t} - \frac{\partial^2 U_t}{\partial n^2}\right] \quad (20)$$

In the limit $\gamma \rightarrow \infty$, (19) and (20) are equivalent to

$$U_t = 0 \;(\gamma \rightarrow \infty) \quad (21)$$

as y increases, the velocity tends to be more and more normal to the open boundary, provided no singular perturbation problem occurs. This is confirmed by PB's results.

Whether a flow that tends to be normal to an open boundary is actually realistic or not is far from clear. However, a few tens of kilometers away from the open boundaries, with the exception of the Eastern part of the Shpanberg Strait, PB's results are in excellent agreement with the observations [*Tripp*, 1985] and the flow patterns provided by other models [*Overland and Roach*, 1987; *Spaulding et al.*, 1987].

Adding integrals along Γ to the functional J, as defined by (2), would permit modification of the boundary conditions deriving from the minimum principle. This should be explored.

Acknowledgments. The author is indebted to Pierre Brasseur for the comments he provided on the first version of the manuscript.

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E. Deleersnijder, Institut d'Astronomie et de Géophysique G. Lemaître, Université Catholique de Louvain, Chemin du Cyclotron, 2, B-1348 Louvain-la-Neuve, Belgium.

(Received January 8, 1992;
revised March 24, 1992;
accepted March 26, 1992.)