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What is wrong with isopycnal diffusion in world ocean models?

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Abstract

Coarse grid Ocean General Circulation Models (OGCMs) take into account the effect of unresolved mesoscale eddies by means of a rotated mixing operator which diffuses tracers (such as temperature or salinity) along surfaces of constant potential density called isopycnals. In spite of its profitable physical aspects, the discrete version of the isopycnal mixing parameterization can produce oscillations in the tracer fields, which disagrees with the well-known properties of diffusion operators. The causes of this non-monotonic behaviour of a diffusion operator are highlighted. The location and magnitude of these over-/under-shootings are examined in the results of an OGCM.

Keywords: World ocean modelling; Mesoscale eddies; Isopycnal diffusion; Monotonic scheme; Over-/under-shootings

1. Introduction

Oceanic mesoscale eddies, the size of which typically lies between 10 and 100 km, play an important role in the transport of tracers [1–4]. Eddy-resolving models [5] require too large computational resources to be used for long simulations, such as those needed in climate studies. Therefore, most climate-oriented Ocean General Circulation Models (OGCMs) do not explicitly resolve the mesoscale eddies, implying that their effect must be parameterized appropriately. For an extended period of time, this has been achieved by means of horizontal diffusion terms [2–4].

However, it is generally believed that mesoscale eddies cause tracers, such as temperature or salinity, to mix predominantly along surfaces of constant potential density called isopycnals (density referenced at sea surface [1,6]) or neutral surfaces (density referenced locally at the mixing depth [7], rather than along horizontal surfaces). This is why Redi [1] and Solomon [6] suggested that a mixing operator leading to diffusion along isopycnals be used. A simplified form of the latter, based on relevant approximations, has been introduced by Cox [8] to enable a more practical numerical implementation. The formulation of Cox [8] was tested by several authors [2–4,9], who noticed significant improvements of the model results, such as a better representation of the intermediate water masses, a freshening of the deep ocean and a drastic reduction of convective activity.

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In spite of the improvements mentioned above, a number of problems remain unsolved. In particular, when the isopycnal mixing terms are discretised according to the scheme of Cox [8], it is generally necessary to retain a small, purely horizontal mixing term to avoid numerical problems such as over-/under-shootings of the tracer values [2–4,9].

Investigating the causes of this deficiency and understanding the role of the background horizontal diffusion are the main objectives of the present article. The Cox [8] operator is described in Section 2, while its usual discretisation is detailed in Section 3. Then, the reason why over-/under-shootings can arise is highlighted (Section 4). In Section 5, the amplitude of the latter is estimated in the results of an OGCM. Finally, conclusions are drawn in Section 6.

2. Isopycnal diffusion operator in the Cox approximation

Taking advantage of the smallness of the slope of the isopycnals ($\approx 10^{-3}$), of the ratio of the vertical to the horizontal scales of ocean basins ($\approx 10^{-3}$) and of the ratio of the cross-isopycnal to along-isopycnal eddy coefficients ($\approx 10^{-7}$), Cox [8] simplified the formulation of Redi [1], obtaining a parameterization implying smaller computational cost. The small-slope assumption of Cox [8] is well verified in the ocean because isopycnals are quasi-horizontal within the bulk of ocean volume except in a few regions corresponding to convective activity, upwelling and strong currents. It is worth stressing that the modified expression, in contrast to Redi's [1], does not contain any horizontal cross - derivatives, so that, without any loss of generality, the present discussion may be carried in out in a single vertical plane.

Let t, x and z represent time, the horizontal coordinate and the vertical coordinate – increasing upward – , respectively. Taking solely into account diffusion along surfaces of constant potential density ρ, the concentration C of a passive tracer obeys the following partial differential equation expressed in Cartesian coordinates

$$\partial_t C = -\partial_x F^x - \partial_z F^z \tag{1}$$

with

$$\begin{bmatrix} F^x \\ F^z \end{bmatrix} = -\begin{bmatrix} K^{xx} & K^{xz} \\ K^{zx} & K^{zz} \end{bmatrix} \begin{bmatrix} \partial_x & C \\ \partial_z & C \end{bmatrix}, \tag{2}$$

where ∂_t, ∂_x, ∂_z, F^x and F^z are the temporal derivative, the horizontal derivative, the vertical derivative, the horizontal and vertical diffusive fluxes, respectively. The coefficients K^{xx}, K^{zz}, K^{xz}, K^{zx} are the components of the Cox isopycnal diffusivity tensor

$$\begin{bmatrix} K^{xx} & K^{xz} \\ K^{zx} & K^{zz} \end{bmatrix} = \kappa_I \begin{bmatrix} 1 & \alpha \\ \alpha & \alpha^2 \end{bmatrix}, \tag{3}$$

where κ_I is the along-isopycnal diffusivity and $\alpha = -\partial_x\rho/\partial_z\rho$ denotes the slope of the isopycnals.

The evolution equation (1) may be reformulated in terms of equivalent horizontal and vertical diffusivities [10], $\overline{\kappa}_H$ and $\overline{\kappa}_V$

$$\partial_t C = \partial_x(\overline{\kappa}_H \partial_x C) + \partial_z(\overline{\kappa}_V \partial_z C) \tag{4}$$

with

$$(\overline{\kappa}_H, \overline{\kappa}_V) = \kappa_I \left[1 - r_C,\ \alpha^2\left(1 - \frac{1}{r_C}\right)\right], \tag{5}$$

where $r_C = \alpha/\alpha_C$ is the ratio of the slope α of isopycnals to the slope of $\alpha_C = -\partial_x C/\partial_z C$ of the iso-concentration lines. If the slope ratio r_C, is positive, then the equivalent diffusivities have opposite signs, so that one of the fluxes is down-gradient while the other is up-gradient, implying that numerical problems may arise when solving the discrete counterpart of the evolution equation above.

In spite of the potential occurrence of anti-diffusive fluxes, the isopycnal operator is clearly a well-conditioned diffusion operator, which verifies the following "min–max principle": in a domain V limited by insulating boundaries, the minimum concentration does not decrease as time progresses, while the maximum does not increase. To demonstrate the first part of this statement – i.e. that the minimum is not decreasing – , it is convenient to re-write $C(t, x, z)$ as

$$C(t,x,z) = \min[C(t_0,x,z)] + C_-(t,x,z) + C_+(t,x,z), \tag{6}$$

where

$$C_-(t,x,z) = \min\{0, C(t,x,z) - \min[C(t_0,x,z)]\}, \tag{7}$$

$$C_+(t,x,z) = \max\{0, C(t,x,z) - \max[C(t_0,x,z)]\}. \tag{8}$$

Combining Eqs. (1)–(3) and Eqs. (6)–(8), it is readily seen that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V C_-^2 \,\mathrm{d}x\,\mathrm{d}z = -2\int_V (\partial_x C_- + \alpha\partial_z C_-)^2 \,\mathrm{d}x\,\mathrm{d}z \leqslant 0. \tag{9}$$

Since $C_-(t_0, x, z)$ is zero in V, Eq. (9) indicates that C_- is zero in V as time progresses ($t \geqslant t_0$), implying that the concentration will not become smaller than the minimum observed at t_0. Demonstrating that the maximum concentration does not increase may be achieved similarly.

It will be seen that a standard discretisation of the isopycnal diffusion equation [8], set out in the next section, does not obey the above min–max principle.

3. Spatial discretisation of the mixing operator

It is believed that the main reason why the min-max principle may be violated is related to the spatial discretisation of the tracer equation, which is the reason why we will focus on this.

As indicated in Fig. 1, integer indices i and k are associated with the horizontal and vertical discretisations, respectively. So, $a_{i,k}$ represents the value of the variable a at $x = i\Delta x$ and $z = k\Delta z$, where Δx and Δz are the horizontal and vertical grid sizes, respectively. It is convenient to use the following discrete operators:

$$(\delta_x a_{i,k}, \delta_z a_{i,k}) = \left(\frac{a_{i+1/2,k} - a_{i-1/2,k}}{\Delta x}, \frac{a_{i,k+1/2} - a_{i,k-1/2}}{\Delta z}\right), \tag{10}$$

$$\overline{a_{i,k}}^{x,z} = \frac{a_{i+1/2,k+1/2} + a_{i-1/2,k+1/2} + a_{i-1/2,k-1/2} + a_{i+1/2,k-1/2}}{4}. \tag{11}$$

With the above notations, the discretised form of the isopycnal diffusion operator D (right-hand side of Eq. (1)), reads

$$D_{i,k} = -\delta_x F^x_{i,k} - \delta_z F^z_{i,k}, \tag{12}$$

where the fluxes are computed as

$$F^x_{i+1/2,k} = -\kappa_I\left(\delta_x C_{i+1/2,k} + \alpha_{i+1/2,k}\delta_z \overline{C_{i+1/2k}}^{x,z}\right), \tag{13}$$

$$F^z_{i,K+1/2} = -\kappa_I \alpha_{i,k+1/2}\left(\alpha_{i,k+1/2}\delta_z C_{i+1/2,k}\delta_x \overline{C_{i,k+1/2}}^{x,z}\right) \tag{14}$$

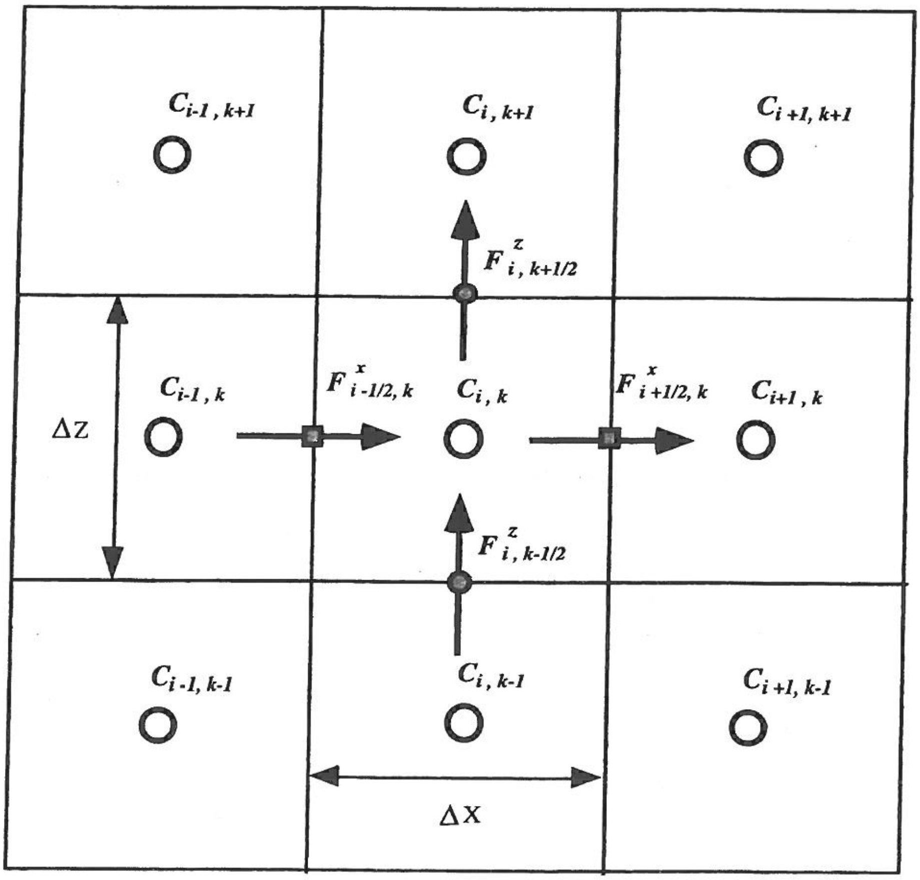

Fig. 1. Numerical grid in the vertical plane (indice k increases vertically downward). Horizontal and vertical fluxes of the tracer C are evaluated at square and circle hatched points, respectively. A double mean average is used when derivatives are needed where they are not naturally evaluated.

with

$$\left(\alpha_{i+1/2,k}, \alpha_{i,k+1/2}\right) = -\left(\frac{\delta_x \rho_{i+1/2,k}}{\overline{\delta_z \rho_{i+1/2,k}}^{x,z}}, \frac{\overline{\delta_x \rho_{i,k+1/2}}^{x,z}}{\delta_z \rho_{i,k+1/2}}\right). \qquad (15)$$

For consistency with the small-slope approximation and computational stability, Cox recommended that the absolute values of α evaluated according to Eq. (15) be not allowed to become larger than an appropriate threshold, typically 10^{-2}.

It is very important to discuss some consequences of this necessary limiting of slope. By definition, the isopycnal diffusion operator is expected to have no effect on a tracer which is homogeneous along isopycnal surfaces. This property is satisfied for the continuous Eqs. (1)–(3) and discrete Eqs. (12)–(15) cases. Indeed, it is readily seen that when the iso-ρ are parallel to the iso-C, replacing ρ by C in the definition of the isopycnal slope yields trivially a zero diffusive flux. However, the numerical scheme does not guarantee this in regions where the slope limitation is active. In such a case, the numerical scheme diffuses the tracer along isopycnal surfaces of slope α_{max} which is smaller than the actual slope. As discussed hereafter, this has important implications in the particular case where the tracer is potential density. Traditionally, ρ is diagnosed from the potential temperature T, salinity S and pressure fields by using an equation of state. If the latter may be linearised, the combination of the heat and salt equations yields an evolution equation for the density similar to Eq. (1). Hence, ρ is not affected during isopycnal diffusion of T and S because, by definition, there is no density gradient along isopycnals. In the regions where the slope is limited, the density evolution is such that buoyancy is somewhat mixed [4], as if a "hidden" convection scheme were at work.

4. Cause of numerical over-/under-shootings

Violations of the min–max principle were noticed repeatedly in large-scale ocean models [4,8,10–12] but the causes thereof were apparently not identified. This may be achieved rather easily by considering the following semi-discrete version of the tracer equation (1)

$$d_t C_{i,k} = D_{i,k}. \tag{16}$$

The discrete isopycnal diffusion operator may be written in terms of coefficients $\gamma_{l,m}$ obtained from expressions (12)–(15)

$$D_{i,k} = \sum_{l=-1}^{1}\sum_{m=-1}^{1} \gamma_{l,m}\, C_{i+l,k+m}. \tag{17a}$$

If, for example, the grid spacing and the isopycnal slope are constant, the $\gamma_{l,m}$ read

$$\gamma_{0,0} = -\frac{2\kappa_{\mathrm{I}}(1+r^2)}{\Delta x^2}, \tag{17b}$$

$$\gamma_{1,0} = \gamma_{-1,0} = \frac{\kappa_{\mathrm{I}}}{\Delta x^2} \geqslant 0, \tag{17c}$$

$$\gamma_{0,1} = \gamma_{0,-1} = \frac{\kappa_{\mathrm{I}} r^2}{\Delta x^2} \geqslant 0, \tag{17d}$$

$$\gamma_{1,1} = \gamma_{-1,-1} = -\gamma_{1,-1} = -\gamma_{-1,1} = \frac{\kappa_{\mathrm{I}} r}{2\Delta x^2}, \tag{17e}$$

where $r = \alpha\Delta x/\Delta z$ is the grid slope ratio (i.e. the ratio of the isopycnal slope α to the grid slope $\alpha_{\mathrm{G}} = \Delta z/\Delta x$). In any case, the consistency of the numerical scheme requires that $\gamma_{l,m}$ verify

$$\sum_{l=-1}^{1}\sum_{m=-1}^{1} \gamma_{l,m} = 0, \tag{18}$$

so that Eqs. (16) and (17e) may be transformed to

$$\partial_t C_{i,k} = \sum_{l=-1}^{1}\sum_{m=-1}^{1} \gamma_{l,m}\,(C_{i+l,k+m} - C_{i,k}). \tag{19}$$

If the monotonicity criterion were verified (i.e. if the coefficients $\gamma_{l,m}$ were positive except for $[l,m] = [0,0]$), then $\partial_t C_{i,k}$ would be positive if $C_{i,k}$ were the minimum of the tracer concentration, and would be negative if $C_{i,k}$ were the maximum, implying that the min–max principle would be obeyed [13]. Strictly speaking, this demonstration is not valid if the maximum or minimum is adjacent to a boundary of the computational domain. In such a case, more complex arguments must be appealed to, but the statement above would still hold true. Unfortunately, the discretisation of mixed derivative terms ∂_{xz} introduces two negative $\gamma_{l,m}$ (Eq. (17e)) – either $\gamma_{1,-1}$, $\gamma_{-1,1}$ or $\gamma_{1,1}$, $\gamma_{-1,-1}$ –, so that the solution of the semi-discrete isopycnal diffusion equation will not necessarily verify the min–max principle.

According to Jameson [13], the discussion above may be applied locally for each grid point showing that a local maximum or minimum may increase or decrease, respectively. For example, if $C_{i,k} \leqslant C_{i+l,k+m}$, $(l,m=-1,0,1)$, then it cannot be excluded that $d_t C_{i,k}$ may be negative, which would lead to a local under-shooting of the solution. Obviously, a similar conclusion may be drawn for over-shooting. Consequently, ripples could arise and grow, a surprising behaviour for the solution of a diffusion equation.

It is also useful to split the isopycnal diffusion operator $D_{i,k}$ in two contributions, $D^+_{i,k}$ and $D^\times_{i,k}$, which are associated with homogeneous (∂_{xx} or ∂_{zz}) and cross (∂_{xz}) derivative terms, respectively. The operator $D^\times_{i,k}$ which violates the monotonicity criterion has an *unmixing* behaviour while $D^+_{i,k}$ is a *mixing* operator which diffuses in the horizontal and vertical directions with diffusivities κ_I and $\kappa_I\alpha^2$, respectively. It is worth stressing that the non-monotonicity of $D^\times_{i,k}$ may be acceptable since it introduces the "negative" diffusion sufficient to re-orient the horizontal/vertical diffusion of $D^+_{i,k}$ in the isopycnal direction. Unfortunately, in contrast with the continuous case, the discretisation prevents the appropriate compensation between the $D^+_{i,k}$ and $D^\times_{i,k}$ from occurring because the associated $\gamma_{l,m}$ are not localised at the same place on the grid.

5. Analysis of min–max violations in an OGCM

The non-monotonic behaviour of the isopycnal mixing formulation could damage the distribution of any passive tracer by producing unphysical negative concentrations. Furthermore, min–max violations of temperature T, salinity S and therefore density ρ caused by the isopycnal mixing scheme are likely to produce unphysical water masses that may contaminate the world ocean through transport and diffusion. The impact of these spurious water masses on the degradation of model results is impossible to assess because we lack a reference monotonic isopycnal scheme to compare with. However, it is still possible to determine the location and amplitude of the over-/under-shootings in an OGCM. The measure of over-/under-shootings is taken in the steady state simulation since we are interested in the equilibrated circulation. It is worth stressing that in case of transient hydrodynamic simulations, the present over-/under-shooting measure could not be representative of what happens in the transient phase. However, in the scope of this study, this is not problematic since the basic goal is to illustrate and to localise the non-monotonic behaviour of the isopycnal diffusion operator.

5.1. Model set-up and experiments

The OGCM used in this study was developed at the Université Catholique de Louvain [14–16] (Belgium). This model has an horizontal resolution of 3° × 3° (typically $\Delta x \approx \Delta y \approx 300$ km) and a vertical discretisation with 15 levels ranging from $\Delta z \approx 20$ m at surface to $\Delta z \approx 800$ m for the deepest points (Fig. 2). A control run which has been integrated to the steady state is used as a starting point for the measure of min–max violations.

The amplitude ΔC_{SCH} of the over-/under-shootings of a tracer C (such as temperature T in °C, salinity S in g/l or density ρ in kg/m^3) caused by a numerical scheme SCH in an OGCM is obtained by integrating the model with solely the scheme SCH active (i.e. all other routines are switched off) during *one* iteration and by comparing the updated values of the tracer C at time $n+1$ to the extrema of C on the grid stencil at the previous time n. By extending the stencil of Fig. 1 to three dimensions, the definition of ΔC_{SCH} is

$$\Delta C_{\mathrm{SCH}} = \max\left[C^{n+1}_{i,j,k} - \max\left[C^n_{i+i',j-j',k+k'}\right], 0\right] - \min\left[C^n_{i,j,k} - \min\left[C^n_{i+i',j+j',k+k'}\right], 0\right], \qquad (20)$$

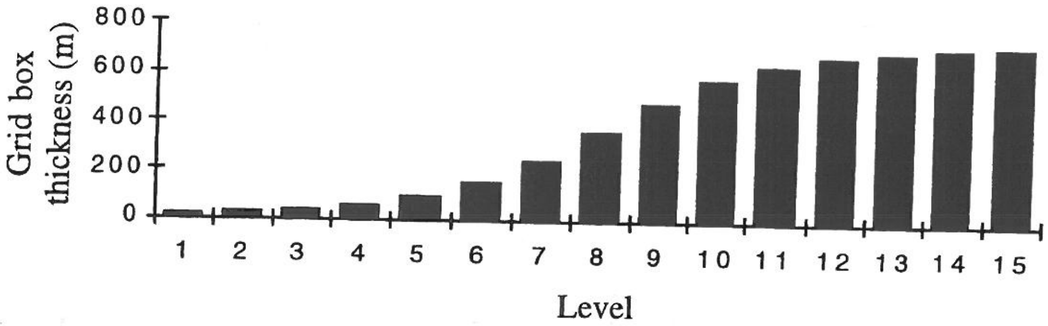

Fig. 2. Distribution of the grid thickness Δz (m) for each level k (increasing from surface to deep ocean) in our OGCM.

where the labels i',j' and k', refer to the points involved in the evaluation of fluxes of the scheme SCH. This definition is only valid for explicit schemes, we reduce the standard time step $\Delta t = 10^5$ s to $\Delta t = 10^3$ s so as to be in a position to resolve all fluxes explicitly without violating stability constraints.

The magnitude of min–max violations will be determined for the isopycnal mixing scheme – ΔC_{ISO} over 15 stencil points – and for a centered advection scheme – ΔC_{ADVC} over 7 stencil points –, allowing us to localise the "hot spots" of the over-/under-shootings of the isopycnal scheme and assess the relative importance of the over-/under-shootings caused by diffusion and advection. Note that the standard set of parameters ($\kappa_{\mathrm{I}} = 300$ m^2 s^{-1}, $\alpha_{\max} = 10^{-2}$) of the control run is employed to determine ΔC_{ISO}.

5.2. Distribution and magnitude of min-max violations of tracers

The salient feature of the distribution of ΔT_{ISO} and ΔS_{ISO} is that the extrema are close to the surface (Figs. 3 and 4 (a) and (b)). Indeed, the magnitude and the number of over-/under-shootings decreases rapidly with depth (Fig. 4(a) and (b)) from the surface, where the vertical grid spacing is fine and isopycnals slope steeply in the mixed layer, to the abyssal ocean, where the vertical resolution is coarser and isopycnal surfaces are almost horizontal. As the isopycnal diffusivity and the horizontal resolution are constant with depth, the slope ratio $r = \alpha/\alpha_{\mathrm{G}}$ appears to be the factor that controls most of the depth distribution of ΔC_{ISO}. This is illustrated by the anti-correlation between the grid thickness (Fig. 2) and the over-/under-shoot profile (Fig. 4(a) and (b)).

Another striking feature of the results is that the majority of over-/under-shooting points are adjacent to boundaries and especially corners (Figs. 3(a) and 5(b)), which is consistent with Gerdes et al. [12] (in their Appendix 2). Indeed, although their paper focused on the influence of numerical advection schemes on OGCM results, the authors noticed that some anomalies of T and S outside the physical range and confined to boundaries should be attributed to isopycnal diffusion. However, this problem was neither further investigated nor quantified.

This privileged position of over-/under-shootings (Fig. 5(b)) shows that a majority of over-/under-shooting points are adjacent to impermeable boundaries) could be explained by the fact that the zero flux boundary condition reduces the efficiency of the horizontal component of the mixing operator $D^{+}_{i,j,k}$ in damping the over-/under-shootings created by the unmixing operator $D^{\times}_{i,j,k}$. Another possible argument is based on the definition of P – i.e. the ratio of the number of points with $\gamma_{i',j',k'} < 0$ to the number of wet points involved in the spatial stencil – which shows that the tendency to over-/under-shoot is stronger in corners ($P = 4/9 \approx 44\%$) rather than in the open ocean ($P = 4/15 \approx 26\%$).

It is worth stressing that the properties of the isopycnal mixing formulation are responsible for special characteristics of the distribution of $\Delta \rho_{\mathrm{ISO}}$ compared to ΔT_{ISO} and ΔS_{ISO}. Indeed, $\Delta \rho_{\mathrm{ISO}}$ always corresponds to over-shootings and has a much less extended pattern relatively to ΔT_{ISO}

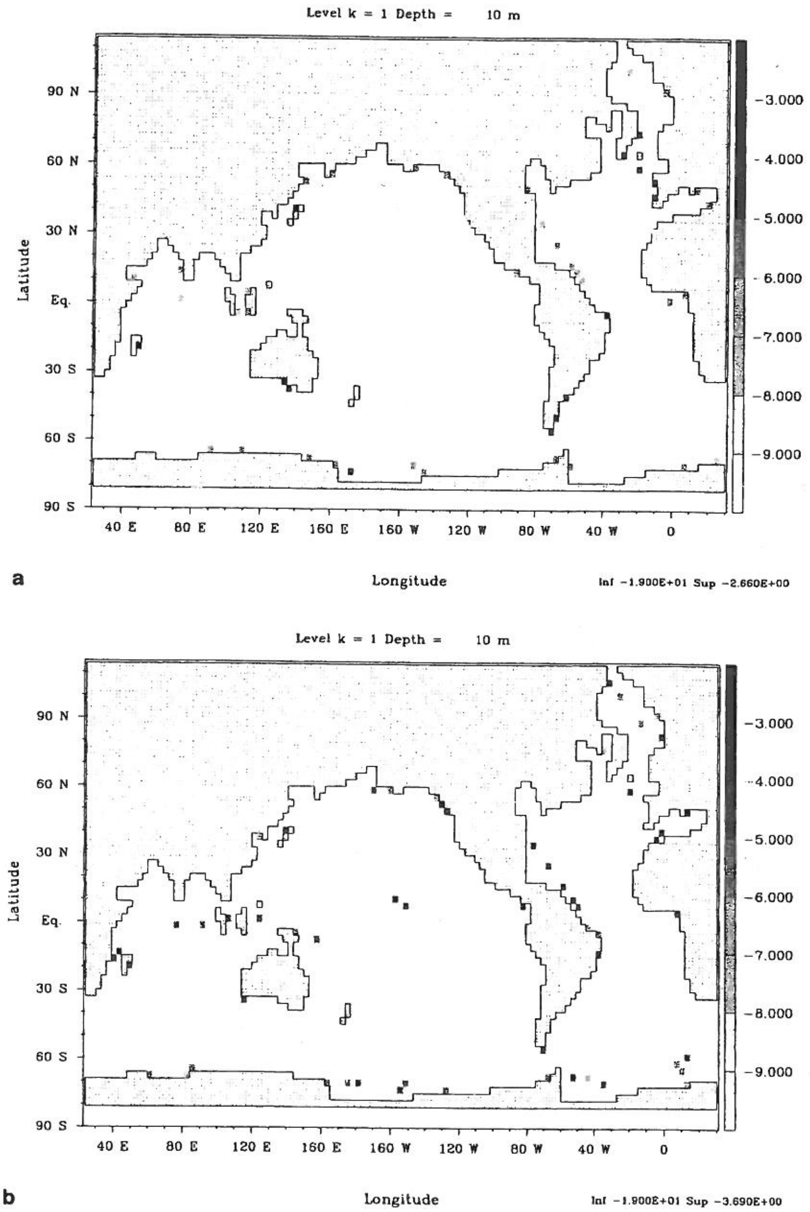

Fig. 3. Surface distribution of (a) ΔT_{ISO} and (b) ΔT_{ADVC} (in logarithmic scale) displayed according to a projection corresponding to the grid system of our OGCM, which consists of a spherical grid with its poles located on the equator to cover the Arctic – North Atlantic basin and a classic latitude–longitude grid for the rest of the ocean [16].

and ΔS_{ISO} (not shown here). This is consistent with the construction of the numerical scheme (17)–(19) which guarantees an exact compensation between T–S variations – of physical or numerical origin – except in the presence of slope limitations or non-linear effects in the equation of state which leads to densification of water [4,17].

5.3. Relative importance of over-/under-shootings

Although the volume associated with over-/under-shootings represents a very small fraction of the ocean and despite the smallness of the magnitude of ΔC_{ISO} (which is highly dependent on the

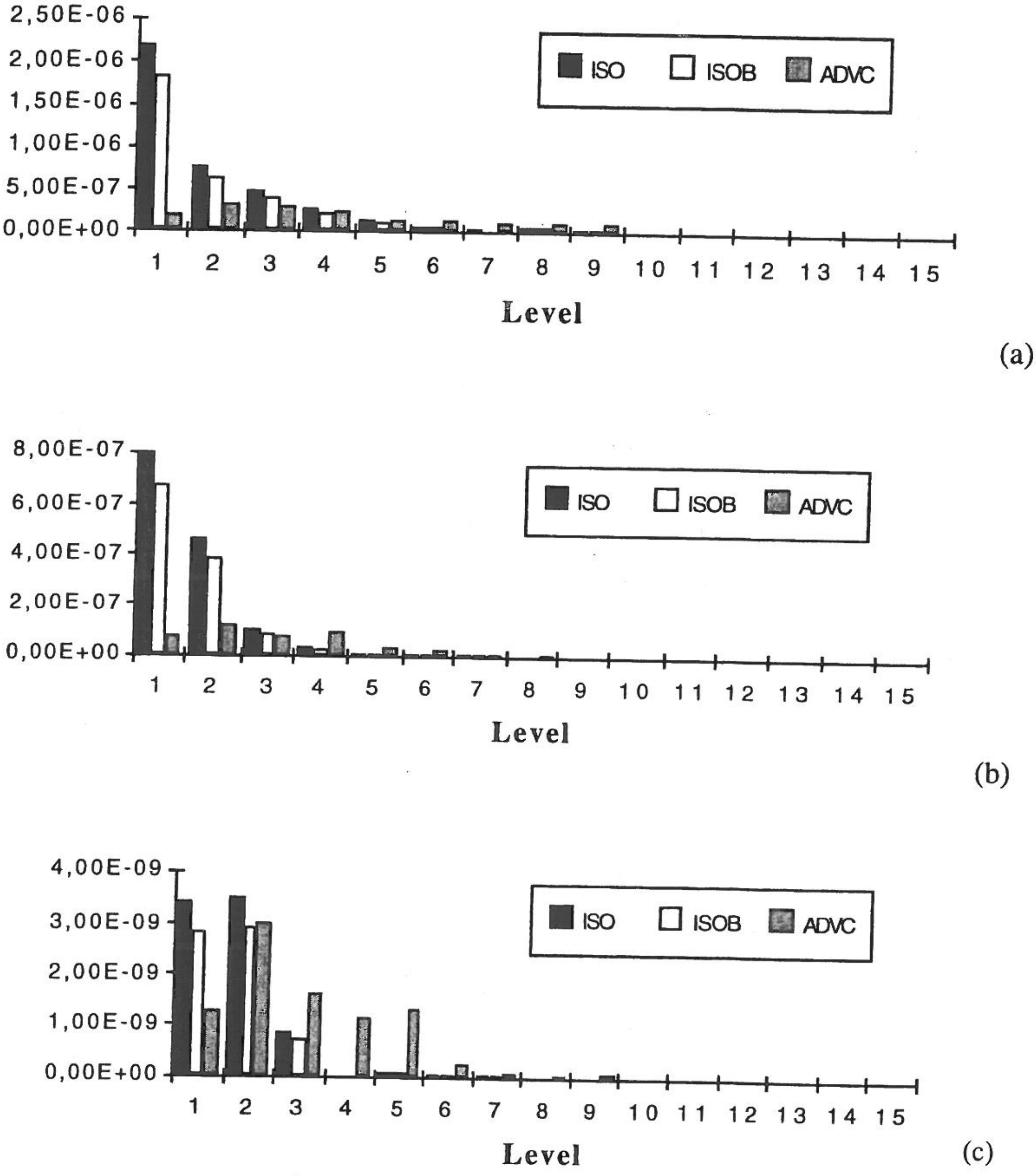

Fig. 4. Level distribution for each experiments of the extreme of over-/under-shootings of: (a) temperature (°C), (b) salinity (g/l) and (c) density (kg/m^3) divided by the time step $\Delta t = 10^3$ s (k increases from surface to deep ocean).

time step, the grid resolution and the isopycnal diffusivity), the cumulative effect of spurious water masses created at each time step could damage the model's climatic simulations after millions of iterations. As the impact of these spurious water masses cannot be assessed for want of a reference scheme for isopycnal diffusion, it is still instructive to compare ΔC_{ISO} and ΔC_{ADVC}.

Most extrema of ΔC_{ADVC} (Fig. 3(b)) are close to the surface in the regions of high velocity (western boundary currents, equatorial current, upwelling regions and convection zones) and are smaller than or of the same order of magnitude as ΔC_{ISO} (Fig. 4). However, the number of

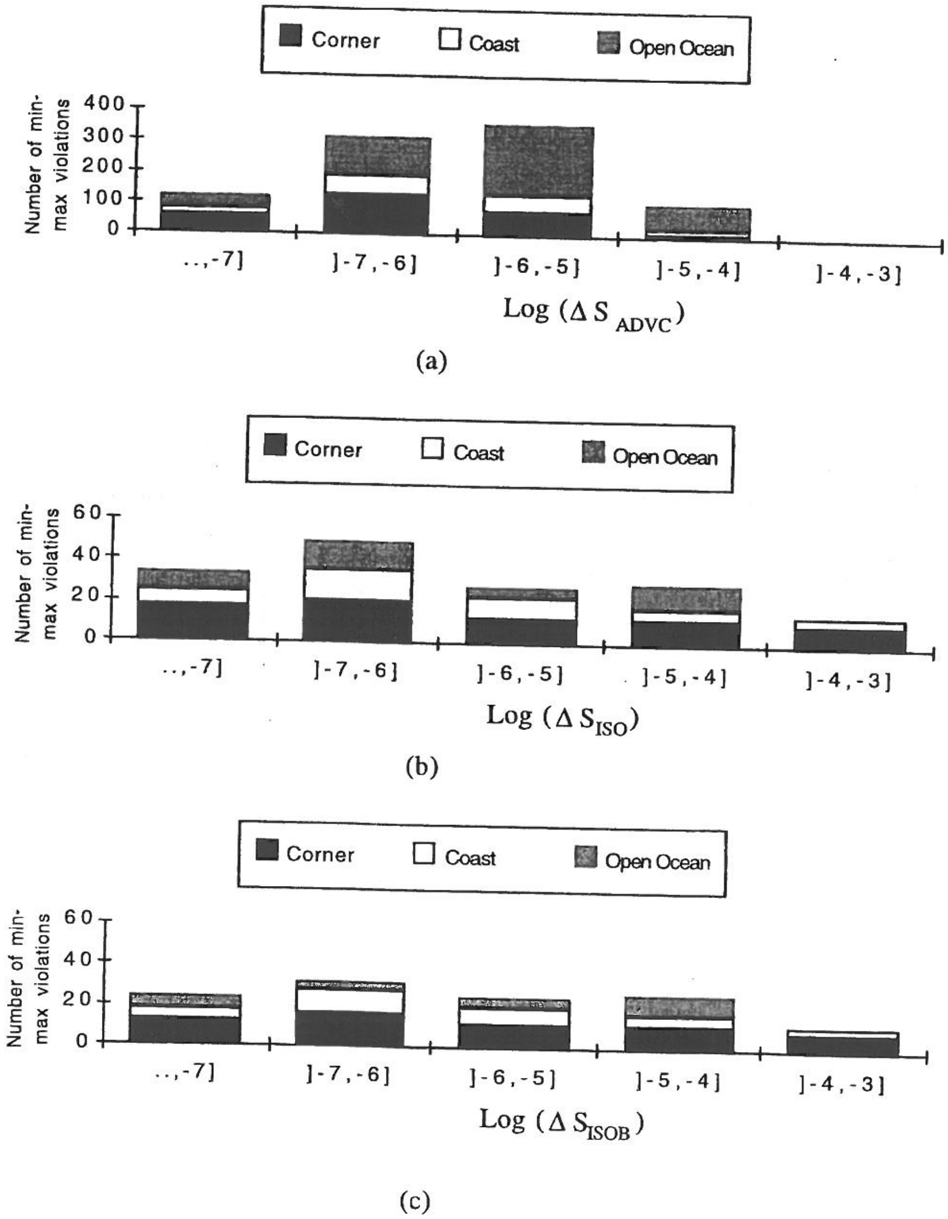

Fig. 5. Number of min–max violations in the salinity field for each experiment: (a) ADVC, (b) ISO and (c) ISOB. We distinguish the open-ocean points, the corner points and the point adjacent once to impermeable boundaries.

min–max violations associated with advection (Fig. 5(a)) is much larger than that due to isopycnal diffusion (Fig. 5(b)). If the advection scheme were improved [18] as in most models or if the isopycnal diffusivity were increased, the isopycnal mixing would certainly become the main source of over-/under-shootings in our OGCM, hence the need to improve the isopycnal scheme.

5.4. Horizontal diffusion as a partial remedy

As no monotonic isopycnal scheme has been developed yet [19], modellers [2–4,9,11,20] simply retain a certain amount of background horizontal diffusion term (with diffusivity κ_H) to alleviate the over-/under-shootings problem. However, despite its profitable impact on numerical noise, κ_H

must remain of modest magnitude because it introduces malignant effects like large spurious diapycnal diffusion of order $\kappa_H \alpha^2$ in regions of steeply sloping isopycnals [20–22] masking partly the effects of isopycnal diffusion.

The modified isopycnal over-/under-shooting amplitude estimate, ΔC_{ISOB}, is computed as ΔC_{ISO} except that a purely horizontal diffusion operator is added ($\kappa_I = 250$ m^2 s^{-1}, $\kappa_H = 50$ m^2 s^{-1}). The analysis of ΔC_{ISOB} illustrates well the efficient role of κ_H as a way of reducing the min-max violations (Fig. 5(b) and (c)). Indeed, with the value of κ_H considered, the number of points concerned by over-/under-shootings is reduced, especially in the open ocean. Unfortunately, contrary to the smallest magnitudes of ΔC_{ISO}, the extreme ones located adjacent to the continents are only slightly modified with κ_H (Fig. 4) maybe because the horizontal diffusion is less efficient near impermeable boundaries.

Using an artificial κ_H damps over-/under-shootings but does not eliminate them completely, since the extra γ-coefficients introduced by the discretisation of the additional horizontal diffusion have no direct effect on the negative coefficients (Eq. (17e)) introduced by the discretisation of the cross derivative terms. There is therefore no criterion that would determine the magnitude of the horizontal diffusivity that would be sufficient to prevent any min–max violations. However, by looking at the distribution of ΔC_{ISO}, it is suggested to use a κ_H that rapidly decreases to small values from the boundary to the interior ocean and from surface to depth. This approach is similar to Gerdes et al. [12] who used a depth-uniform κ_H but restricted it to a "boundary layer of one grid distance wide".

It is also important to stress that an explicit residual background lateral diffusivity in the interior ocean may be necessary to preserve the stability of advection schemes which have no implicit diffusion (such as the centered advection scheme).

6. Conclusions

The non-monotonic behaviour of the isopycnal mixing formulation of Cox [8] has been investigated. It was shown why the discretisation of the cross derivative terms is responsible for the occurrence of min–max violations in the tracer field. Over-/under-shootings of temperature, salinity and density were quantified in model results and their extreme values were found at surface, especially in the vicinity of permeable boundaries. The grid slope ratio r was shown to control most of the spatial distribution of min-max violations.

A comparison of the over-/under-shootings associated respectively with isopycnal diffusion and advection shows that as much effort should be devoted to preserving monotonicity of the isopycnal mixing formulation as to the search for sophisticated advection schemes. Furthermore, as the measure of over-/under-shootings is static (snapshot measure on one iteration), we underestimate the effective grid noise because we miss some harmful interactions between the active tracers which determines density and the density which determines the direction of diffusion [20].

The simplest way to reduce the noise is to retain a certain amount of background lateral diffusivity. However, this diffusivity was shown not to be entirely satisfactory because it does not guarantee monotonicity and furthermore introduces serious physical drawbacks. Therefore, the need for a monotonic isopycnal diffusion scheme seems to be justified.

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