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VERTICAL MODES IN LEVEL MODELS

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The purpose of the present note is to briefly examine how the vertical discretization of ocean level models modifies the propagation of hydrostatic internal inertia-gravity waves. The results obtained herein may be regarded, in a certain sense, as a generalization of the part of Smith's work that was devoted to vertical modes in level models (Ocean Modelling 56, April 1984).

We consider small perturbations of a motionless reference state in a constant depth (H) ocean, where the stratification is characterized by a uniform value of the Brunt-Väisälä frequency N. The variables are the three components of the velocity, and the deviations of pressure and density. For these variables, separation of the vertical dependency is assumed to be valid. This leads to an infinite number of vertical modes. The amplitudes of these modes obey equations that are similar to the shallow water equations where an equivalent depth is introduced, namely

$$H_n = \left(\frac{1}{\alpha_n^2}\frac{N^2 H}{g}\right) H, \quad n = 1,2,3,\ldots, \tag{1}$$

where g is the gravitational acceleration. The separation constant α_n is equal to $n\pi$. All this is well known (Moore and Philander, 1977; LeBlond and Mysak, 1978).

It is possible to show that the mode splitting referred to above may be straightforwardly applied to finite difference schemes used in ocean modelling. In particular, it is easily seen that the amplitudes of the numerical modes are governed by difference analogues of the shallow water equations. The equivalent depth of the numerical modes may however be very different from that corresponding to the continuous description.

In the vertical direction, we consider centred finite differences and a staggering of variables that is in agreement with that of the most classical ocean models (Bryan, 1969; Cox, 1984; Semtner, 1986). Index k identifies one of the K vertical grid boxes of constant height Δz. After some calculations, we arrive at the following difference equation

$$W_{k+3/2} - 2\frac{4-(\alpha^N/K)^2}{4+(\alpha^N/K)^2}W_{k+1/2} + W_{k-1/2} = 0, \quad k = 1,2,\ldots,K-1, \tag{2}$$

where α^N and W represent the numerical separation constant and the vertical dependence of the vertical velocity, respectively. Taking into account the impermeability conditions of the bottom and the surface of the sea, $W_{1/2} = 0 = W_{K+1/2}$, the solution of (2) is (Bender and Orszag, 1978)

$$W_{k-1/2} = i2A\sin[(k-1)\theta], \tag{3}$$

where A is a real constant and $i = \sqrt{-1}$. The admissible values of θ are

$$\theta_n = \frac{n\pi}{K}, \quad n = 1,2,\ldots,K-1. \tag{4}$$

Notice that there exist only $K-1$ discrete vertical modes. After some calculations, we get

$$\alpha_n^N = \frac{2K}{[1+(\cot\frac{n\pi}{K})^2]^{1/2} + \cot\frac{n\pi}{K}}, \quad n = 1,2,\ldots,K-1. \tag{5}$$

Combining (1) and (5) yields the numerical equivalent depth

$$H_n^N = \left\{\frac{\{[1+(\cot\frac{n\pi}{K})^2]^{1/2}+\cot\frac{n\pi}{K}\}^2}{4K^2}\frac{N^2H}{g}\right\}H, \quad n = 1,2,\ldots,K-1. \tag{6}$$

Of course, when n, the order of the mode, is small compared with the number of grid boxes in the vertical direction, it is possible to show that H_n^N is asymptotic to H_n. Indeed, when $(n\pi)/K \to 0$, $\cot[(n\pi)K] \sim K/(n\pi)$, so that $H_n^N \sim H_n$.

To compare H_n^N with H_n, we found it appropriate to examine the ratio

$$\frac{c_{p,n}^N}{c_{p,n}} = \frac{\sqrt{gH_n^N}}{\sqrt{gH_n}} = \frac{n\pi}{2K}\left\{[1+(\cot\frac{n\pi}{K})^2]^{1/2}+\cot\frac{n\pi}{K}\right\}, \quad n = 1,2,\ldots,K-1, \tag{7}$$

where $c_{p,n}^N$ is the phase speed of pure gravity waves related to numerical mode "n" in the case where the truncation errors associated with time and horizontal finite differencing are negligible, *e.g.*, for very long horizontal wavelengths. In (7), $c_{p,n}$ obviously denotes the phase speed of pure gravity waves in the continuous model.

The ratio (7) is calculated in Table 1 for all modes for which $2 \le K \le 10$. We see that $c_{p,n}^N/c_{p,n}$ is a decreasing function of n/K and that $c_{p,n}^N/c_{p,n}$ never exceeds 1.

When n decreases, the vertical length scale of the corresponding mode increases. Thus, the smaller n/K is, the better the corresponding mode is resolved by the vertical descretization.

To conclude, one may say that the vertical discretization results in a relative decrease of the equivalent depth that is all the more significant when the mode considered is less adequately resolved on a vertical grid. In the numerical model, this implies a slower propagation of the internal gravity and inertia-gravity waves. But, this slowdown depends in detail on the particularities of time and horizontal differencing. Therefore, with the exception of very special cases where the time and horizontal discretizations are very accurate, $c^N_{p,n}/c_{p,n}$ only gives a rough indication about the actual propagation speed error of numerical gravity waves.

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TABLE 1.

The ratio $c^N_{p,n}/c_{p,n}$ as a function of K and n. Notice that K and n are integers such that $K \geq 2$ and $1 \leq n \leq K-1$.

order of the mode: n

	1	2	3	4	5	6	7	8	9
2	.785	XX	XX	XX	XX	XX	XX	XX	XX
3	.907	.605	XX	XX	XX	XX	XX	XX	XX
4	.948	.785	.488	XX	XX	XX	XX	XX	XX
5	.967	.865	.685	.408	XX	XX	XX	XX	XX
6	.977	.907	.785	.605	.351	XX	XX	XX	XX
7	.983	.932	.844	.716	.540	.307	XX	XX	XX
8	.987	.948	.882	.785	.656	.488	.273	XX	XX
9	.990	.959	.907	.832	.732	.605	.445	.246	XX
10	.992	.967	.925	.865	.785	.687	.560	.408	.224