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The age, a tool for understanding complex fluid flows

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1. Introduction

To understand or interpret a fluid flow the processes taking place in it must be identified and investigated. To do so, appropriate gauges or sufficiently realistic numerical models must provide values of the state variables of the flow under consideration — i.e. velocity components, pressure, density, tracer concentration, etc. — with enough accuracy and space-time resolution so that they can be subsequently pictured by means of appropriate computer graphics software. However, most fluid flows, whether they occur in natural or artificial domains, are so complex that examining a number of graphical representations of state variables is not sufficient for gaining a profound insight into their functioning. This is why computer graphics must often be used in conjunction with other interpretation techniques, some of which demand that auxiliary variables be measured or computed. The age is one such auxiliary variable.

According to Ref. [4] the age of a parcel of a tracer, i.e. a substance dissolved in a fluid mixture, is defined to be "the time elapsed since the parcel under consideration left the region in which its age is defined to be zero." The nature of this region, which may be zero- to three-dimensional — i.e. a point, a curve, a surface or a volume — depends obviously on the flow considered and the purposes for which the age is introduced. The general theory of the age is outlined in Ref. [4] and set out in Ref. [3]. It is mostly intended for numerical models, to help understand flows simulated numerically and assess the functioning of the model itself.

The main features of the general theory of the age are as follows:

- the age of every constituent of a fluid mixture can be evaluated, as well as that of suitably-defined group of constituents;
- the age of every constituent depends on time and position;
- advection, mixing, and production/destruction processes are properly taken into account;
- the age may be computed in the Eulerian formalism by means of a numerical algorithm similar to that used for simulating the evolution of the concentration of a tracer.

Below the general theory of the age is summarised. Then, its application to the estimation of the ventilation rate in the World Ocean is outlined, with some emphasis on the difference between the age of various tracers and that obtained from the widely-used radioisotope dating technique. Finally, other possible applications of the general age theory are mentioned.

2. The general theory of the age

Consider a fluid mixture made up of I constituents that can be identified by the index i ($1 \le i \le I$). Let the Cartesian coordinates of the position vector $\mathbf{x}$ of any point in the domain of interest read (x,y,z). According to Ref. [4] the concentration distribution function of the i-th constituent, $c_i(t,\mathbf{x},\tau)$, is defined as follows: at time t, the mass of the i-th constituent contained in the volume

(1) $(x-\Delta x/2,\ y-\Delta y/2,\ z-\Delta z/2) \le (x,\ y,\ z) \le (x+\Delta x/2,\ y+\Delta y/2,\ z+\Delta z/2)$

with an age lying in the interval

(2) $\tau-\Delta\tau/2 \le \tau \le \tau+\Delta\tau/2$

tends to $\rho c_i(t,\mathbf{x},\tau)\Delta x\,\Delta y\,\Delta z\,\Delta\tau$, as Δx, Δy, Δz, and

$\Delta\tau$ tend to zero, where ρ is the density of the fluid. For the sake of simplicity, it is assumed that the variations of the latter are negligible, i.e. the Boussinesq approximation may be made. Further hypothesizing that the age is positive definite, the concentration at time t and location $\mathbf{x}$ of the i-th constituent is thus

$$(3) \qquad C_i(t,\mathbf{x}) = \int_0^\infty c_i(t,\mathbf{x},\tau)\, d\tau.$$

Therefore, at the same time and location, the mean age of the i-th constituent is given by

$$(4) \qquad a_i(t,\mathbf{x}) = \frac{1}{C_i(t,\mathbf{x})} \int_0^\infty \tau c_i(t,\mathbf{x},\tau)\, d\tau.$$

It is convenient to introduce an additional variable, namely the age concentration, which is defined to be

$$(5) \qquad \alpha_i(t,\mathbf{x}) = C_i(t,\mathbf{x}) a_i(t,\mathbf{x}).$$

From the mass budget of every constituent, it may be shown [4] that the concentration distribution function obeys the following partial differential equation:

$$(6) \qquad \frac{\partial c_i}{\partial t} = p_i - d_i - \nabla \bullet (\mathbf{u} c_i - \mathbf{K} \bullet \nabla c_i) - \frac{\partial c_i}{\partial \tau}.$$

where ∇ and $\nabla \bullet$ represent the gradient and divergence operators, respectively; p_i (≥ 0) and d_i (≥ 0) denote the rate of production and destruction of the i-th constituent, which may be caused by radioactive decay, chemical reactions, etc.; $\mathbf{u}$ represents the fluid velocity resolved in the model considered, while $\mathbf{K}$ is the symmetric, positive-definite, diffusivity tensor needed to parameterise in a Fourier-Fick manner the unresolved transport of the constituent under study, which is due to turbulent fluctuations and molecular-scale processes. The rightmost term in equation (6) is associated with ageing, i.e. the process by which the age of every parcel of the fluid tends to increase by a certain amount of time as time progresses by the same amount of time. This may be viewed as advection with a unit velocity in the age direction.

If relevant initial and boundary conditions are available, (6) may be solved so as to eventually obtain, from (3) and (4), the concentration and the mean age of every constituent of the fluid mixture under study. In most applications carrying out this task is unlikely to be easy, since the concentration distribution function depends on 5 independent variables, i.e. t, x, y, z, and τ. If no information is required about the distribution of the mass in the age direction, it is not necessary to solve the equation governing the age distribution: in accordance with (5), the mean age may be calculated as the ratio of the age concentration to the concentration. The evolution of these variables is governed by

$$(7) \qquad \frac{\partial C_i}{\partial t} = P_i - D_i - \nabla \bullet (\mathbf{u} C_i - \mathbf{K} \bullet \nabla C_i),$$

$$(8) \qquad \frac{\partial \alpha_i}{\partial t} = C_i + \pi_i - \delta_i - \nabla \bullet (\mathbf{u} \alpha_i - \mathbf{K} \bullet \nabla \alpha_i).$$

with

$$(9) \qquad [P_i - c_i(t,\mathbf{x},\tau=0),\ D_i,\ \pi_i,\ \delta_i] = \int_0^\infty (p_i,\ d_i,\ \tau p_i,\ \tau d_i)\, d\tau.$$

The equations (7) and (8) are derived from (6) by integration over the age and further appropriate manipulations, which may be found in Ref. [3]. It is worth stressing that the age concentration equation (8) includes advection and diffusion operators which are similar to those present in the equations governing the evolution of the concentration.

By manipulating equations (5), (7)-(9), it is readily seen that the age obeys

$$(10) \qquad \frac{\partial a_i}{\partial t} = 1 + \frac{\pi_i - a_i P_i}{C_i} - \frac{\delta_i - a_i D_i}{C_i} - \nabla \bullet (\mathbf{u} a_i - \mathbf{K} \bullet \nabla a_i) + \frac{2}{C_i} \nabla C_i \bullet \mathbf{K} \bullet \nabla a_i.$$

The last term in the right-hand side of this equation is not in flux form and has no equivalent in the equations governing the constituent concentrations. Thus, evaluating the age by solving (10) rather than having recourse to the age concentration would demand a specific numerical scheme, implying additional — unnecessary — numerical developments and diminishing the relevance of the age estimated in this way as a tool for understanding the functioning of the model under consideration.

In the next section the theory outlined above is used for estimating ages in the World Ocean.

3. The age in the World Ocean

According to Ref. [5] the "World Ocean circulation at its largest scale can be thought of as a gradual renewal or ventilation of the deep ocean by water that was once at the sea surface." Thus, estimating the age as the time elapsed since leaving the surface layers is likely to provide useful insight into the ventilation processes of the World Ocean. This is why the age is a popular diagnostic tool in this domain of interest (e.g. [1-2], [5], and [7]).

Diffusive transport is believed to be smaller than advection in the interior of the World Ocean, i.e. the whole World Ocean except the surface and bottom boundary layers, the thickness of which rarely exceeds a few hundreds of metres. Thus, in most of the domain of interest, neglecting diffusion may be considered as a fair approximation. This is actually the key assumption of radioisotope dating techniques, which yield estimates of the age termed "radio-ages" herein. Such an approach is widely used in studies in which the age is evaluated with the help of field data only (e.g. [1]), but can also be utilised in numerical models (e.g. [2]). A radio-age is different from the ages ensuing from the theory outlined in the present paper. This issue is briefly addressed below by tackling a relevant problem of tracer transport in the World Ocean.

In the present study, only one type of radio-age dating is considered, namely a carbon-14-like technique: a radioactive tracer and a passive tracer are needed. The former and the latter are identified by the subscripts "r" and "p", respectively. For these tracers, the source/sink terms included in equations (7) and (8) read

(11) $(P_p, D_p, \pi_p, \delta_p) = (0, 0, 0, 0)$,

and

(12) $(P_r, D_r, \pi_r, \delta_r) = (0, C_r/T, 0, \alpha_r/T)$.

where $T\log 2$ is the half-life of the radioactive tracer. At the initial instant t=0, all tracer concentrations and age concentrations are zero in the whole domain of interest. At any time t>0, all ages are prescribed to be zero at the ocean surface, while there is no tracer or age concentration flux across the other boundaries of the domain of interest. At the ocean surface, the tracer concentrations are constant at any time t>0.

As stated above, the radio-age is calculated under the key assumption that diffusion is negligible. In this case, a seawater parcel — i.e. an arbitrarily small volume of seawater — originating from the surface layer of the ocean would not exchange matter with its environment during its journey through the interior of the ocean, implying that all constituents contained in it would have the same history and, hence, the same age. In other words, the radio-age is a single approximate value of the age of all the constituents of a seawater parcel.

The trajectory $\mathbf{r}(t)$ of a seawater parcel obeys the differential equation $d\mathbf{r}(t)/dt = \mathbf{u}[t, \mathbf{r}(t)]$. Along such a trajectory, the radio-age increases as

(13) $\bar{a}[t^2, \mathbf{r}(t^2)] = \bar{a}[t^1, \mathbf{r}(t^1)] + (t^2 - t^1)$,

where t^1 is an arbitray value of time t, while t^2 denotes any subsequent instant. In the no-diffusion limit the relations

(14) $C_p[t, \mathbf{r}(t^2)] = C_p[t, \mathbf{r}(t^1)]$

and

(15) $C_r[t, \mathbf{r}(t^2)] = C_r[t, \mathbf{r}(t^1)]\, e^{-\frac{t^2 - t^1}{T}}$

hold true. Then, combining (14)-(15) yields

(16) $$\bar{a}[t^2, \mathbf{r}(t^2)] = \bar{a}[t^1, \mathbf{r}(t^1)] + T\log\frac{C_p[t^2, \mathbf{r}(t^2)]/C_p[t^1, \mathbf{r}(t^1)]}{C_r[t^2, \mathbf{r}(t^2)]/C_r[t^1, \mathbf{r}(t^1)]}.$$

Let the constant μ denote the ratio of the concentration of the radioactive tracer to that of the passive tracer at the ocean surface. Then, formula (16) may be transformed to provide the expression of the radio-age at location $\mathbf{x}$ and time t:

(17) $$\bar{a}(t, \mathbf{x}) = T\log\frac{\mu C_p(t, \mathbf{x})}{C_r(t, \mathbf{x})}.$$

As $C_r/C_p = \mu$ at the ocean surface, the radio-age remains zero on this boundary — as is desirable.

Inspired by Ref. [6], Ref. [3] demonstrated that, at any point $\mathbf{x}$ in the domain of interest and any time t>0, the age of the passive tracer, the age of the radioactive tracer and the radio-age are such that

(18) $$\boxed{0 \le a_r(t, \mathbf{x}) \le \bar{a}(t, \mathbf{x}) \le a_p(t, \mathbf{x}) \le t}.$$

In other words, all ages are generally different and obey at any time and position the same inequalities. This theoretical result, though rather powerful, does not give any clue as to the order of magnitude of the difference between the ages.

The difference between the ages may be illustrated by the numerical results such as those of the World Ocean model developed at the Institut d'astronomie et de géophysique of the Université catholique de Louvain (see [4] for appropriate references). Herein only steady-state results are examined. This implies that the concentration of the passive tracer is everywhere equal to the value imposed at the ocean surface.

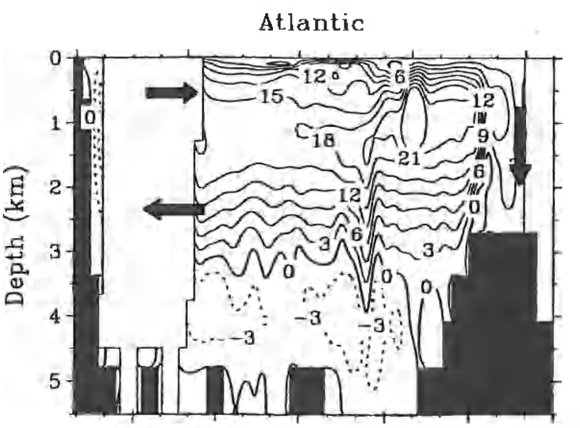

Figure 1. Illustration of the meridional transport in the Atlantic, which results from the integral of the meridional and vertical velocity over the longitude from the western side of the basin to the eastern one. This transport is divergence-free. Hence, it can be viewed as deriving from the streamfunction of which some isolines are displayed above. The contour interval is $3\times10^6 m^3 s^{-1}$. *The arrows indicate the direction of the transport.*

The Atlantic meridional circulation is as follows (Figure 1): water generally flows northward close to the surface; North Atlantic Deep Water (NADW) is formed by convective processes in regions around 60 °N; NADW returns southward at a depth of the order of 2-3 km, flowing on top of Antarctic Bottom Water, which was formed in the vicinity of the Antarctic.

At a steady state, the age of the passive tracer may be seen to be almost equal to that of the water [3]. The Atlantic zonal mean of the passive tracer (Figure 2) is visibly consistent with the meridional circulation as displayed in Figure 1.

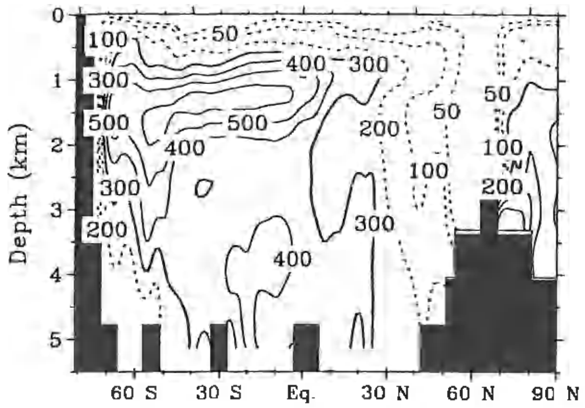

Figure 2. Contours of the zonal mean of the age of the passive tracer in the Atlantic at a steady state (in years).

	$T = 8{,}266$	$T = 1{,}000$
$< a_p - \bar{a} >$	51	260
$< \bar{a} >$	848	639
$< \bar{a} - a_r >$	47	151

Table 1. Steady-state averages (in years) over the World Ocean of the radio-age, and of the differences $a_p - \bar{a}$ and $\bar{a} - a_r$ as simulated for two different values of the radioactive decay timescale T (in years).

If the timescale of decay of the radioactive tracer is set to the Carbon-14 value, 8,266 years, the difference between the ages is of the order of a few tens of years, i.e. about 5% of the age of the water. Thus, this radio-age is a good estimate of the age of the water. The reason thereof is that the timescale of decay is sufficiently large that the concentration variations due to the radioactive decay are sufficiently small that the associated diffusive fluxes — which causes the ages to be different — are rather negligible. However, if the timescale of decay is decreased to a much smaller value, say 1,000 years, then the difference between the ages grows up to an order of magnitude comparable with that of the age of the water as is exemplified in Table 1.

4. Conclusion

The partial differential equations were introduced from which the age of every constituent of seawater, be it passive or not, may be derived easily. The relation between the age of a passive tracer, that of radioactive tracer and a Carbon-14-like radio-age were briefly examined from the results of a World Ocean model. The impact of the half-life of the radioactive tracer on the difference between various ages was alluded to.

The present theory of the age may be applied with minor modifications to all fields of research related to fluid mechanics, including problems in which compressibility must be taken into account.

Future work will focus on the analysis of the order of magnitude of the difference between various ages, the computation of the concentration distribution function and the study of the age in various marine regions, such as the northwestern European continental shelf. The usefulness of the age in ecological models will also be investigated.

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