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THE EXTENDED KALMAN FILTER APPROACH TO VPA

by

J. Kettunen
Finnish Game and Fisheries Research Institute, Fisheries Division,
P.O. Box 193, SF-00131 Helsinki 13, Finland

and

M. Hildén
Department of Limnology, University of Helsinki,
E-Building, Viikki, SF-00710 Helsinki 71, Finland

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Abstract

In fish stock assessment, it is of the utmost importance to make optimal use of existing stock assessment and fisheries data. The Kalman filter is suggested as an efficient algorithm for synthetizing available a priori and a posteriori information.

The principle of the linear Kalman filter is presented. The extended Kalman filter is introduced to solve the problem of combined state-parameter estimation.

The use of the extended Kalman filter in VPA modelling is illustrated with a simple numerical example.

The drawbacks and advantages of the extended Kalman filter are discussed.

Résumé

Un des problêmes primordiaux dans l'évaluation du stock de poisson est l'utilisation optimale des données existantes d'évaluation du stock et des pêcheries. On suggère que le filtre Kalman est un algorithme efficace pour synthêtiser les informations disponibles a priori et a posteriori.

On présente le principe essentiel du filtre Kalman linéaire. On introduit le filtre Kalman élargi pour résoudre le problème de l'estimation combinée état-paramètre.

L'utilisation du filtre Kalman élargi dans le contexte du modelage VPA est illustré par un exemple numérique simple.

Les problèmes et avantages de l'utilisation du filtre Kalman élargi sont discutés.

Introduction

Virtual population analysis VPA (GULLAND 1965) is the standard procedure for obtaining the information on stock size and exploitation needed for the management of fish stocks. In an ordinary VPA, assumed values are used for the terminal fishing mortalities and the natural mortality. These assumed values can introduce serious errors when the VPA is used for catch prediction (POPE 1977). This is clearly disturbing, since one of the main tasks of fishery science is to set catch quotas, i.e. to make catch predictions. It is also troublesome that the current procedure for estimating total allowable catches obscures the effects of noise in the data base on the catch predictions (POPE 1982).

New approaches to stock assessment (e.g. GUDMUNDSSON et al. 1982, NIELSEN 1982, POPE & SHEPHERD 1982) use additional data and clearly stated restricting assumptions in order to solve the problems of VPA. These methods also allow studies of the noise corruption in the data base, which represents a great improvement on ordinary deterministic VPA. Another promising method is the use of combined state and parameter estimation techniques developed within systems theory. In this paper we present an introduction to the topic by illustrating how the recursive state and parameter estimation technique known as the extended Kalman filter (EKF) (e.g. EYKHOFF 1974, BECK 1979, RINALDI et al. 1979, MAYBECK 1979) could be applied to a system which can be described by the VPA model.

When ordinary VPA is used, the available catch at age data are assumed to be exact and hence the values obtained for stock size and fishing mortality track the data base exactly. When the EKF is used, due account is taken of the noise in the different data bases and of the uncertainties involved in the model used. Thus the estimates obtained are the result of a weighting process. In the linear case the Kalman filter can be shown to yield optimal estimates with respect to many statistical criteria when the underlying assumptions concerning the nature of the noise terms are met (EYKHOFF 1974, MAYBECK 1979).

The linear Kalman filter

Suppose that the system under study can be represented by a linear, stochastic differential equation of the form:

$$\dot{x}(t) = F(t)x(t) + B(t)u(t) + G(t)w(t), \tag{1}$$

where $\dot{}$ denotes a time derivative, $x(t)$ an n-dimensional state vector, and $u(t)$ an r-dimensional deterministic control(-input). $F(t)$, $B(t)$ and $G(t)$ are known coefficient matrices featuring system behaviour, and the effects of the control and noise on the system, respectively. $w(t)$ denotes white Gaussian noise, with the properties:

$$E(w(t)) = 0 \tag{2a}$$
$$E(w(t)w^T(t)) = Q(t)\delta(t-t'), \tag{2b}$$

where $\delta(t-t')$ is the delta function with the property:

$$\begin{aligned} \delta(t-t') &= 1 \quad && \text{for } t=t' \\ \delta(t-t') &= 0 && \text{otherwise} \end{aligned} \tag{2c}$$

$Q(t)$ is a symmetric matrix expressing system noise covariances.

The solution of equation (1) has the general form:

$$x(t) = \phi(t,t_0)x(t_0) + \int_{t_0}^{t} \phi(t,\tau)B(\tau)u(\tau)d\tau + \int_{t_0}^{t} \phi(t,\tau)G(\tau)d\beta(\tau), \tag{3}$$

where the term $\beta(\tau)$ denotes Brownian motion with derivative $d\beta/dt = w(t)$. The term $\phi(t,t_0)$ is called the state transition matrix, having the following properties:

$$d(\phi(t,t_0))/dt = F(t)\phi(t,t_0) \tag{4a}$$
$$\phi(t_0,t_0) = I \quad (= \text{identity matrix}) \tag{4b}$$
$$\phi(t_3,t_1) = \phi(t_3,t_2)\phi(t_2,t_1) \tag{4c}$$

Equation (3) can also be expressed in discrete form.

$$x(t) = \phi(t,t-1)x(t-1) + B_d(t-1)u(t-1) + G_d(t-1)w_d(t-1) \tag{5}$$

$B_d(t-1)$, $G_d(t-1)$, $Q_d(t-1)$ and $w_d(t-1)$ are analogous to the terms in the continuous case. The subscript d denotes that the coefficient matrices refer to the discrete form.

When the state transition matrix is known, the system state can be calculated explicitly with equation (3), if the initial condition $x(t_0)$ is stated.

Usually $x(t_0)$ is not known exactly, but has a stochastic nature, with:

$$\text{mean:} \quad \hat{x}_0 = E(x(t_0)) \tag{6a}$$
$$\text{covariance:} \quad P_0 = E((x(t_0)-x_0)(x(t_0)-x_0)^T), \tag{6b}$$

where P_0 is a symmetric matrix. Superscript T denotes the transpose of the matrix.

The system output vector at discrete time instants t1, t2, t3, ... can be represented by the linear equation:

$$z(t_i) = H(t_i)x(t_i) + v(t), \tag{7}$$

where $z(t_i)$ is an m-dimensional observation vector, $H(t_i)$ is an mxn-dimensional matrix and $v(t)$ is white Gaussian noise with the statistics:

$$E(v(t_i)) = 0 \tag{8a}$$

$$E(v(t_i)v^T(t_j)) = \begin{cases} R(t_i) & \text{for } t_i = t_j \\ 0 & \text{otherwise.} \end{cases} \tag{8b}$$

$R(t_i)$ is a symmetric mxm-dimensional matrix with the diagonal elements decribing measurement accuracy.

There are two independent sources of information: The equations (3) and (6) represent the a priori information, whereas the observations give the a posteriori information.

To reach the best possible estimate of the system, a proper algorithm is needed. The accuracy of the estimate is also of interest.

A Bayesian approach, involves the following two tasks:

(i) $\max(p(x(t)|z(t-1)))$

(ii) $\max(p(x(t)|z(t)))$

To state the same verbally:

(i) Solve the maximal probability of the system state at time t, if observation at time t-1 is known, and a model of the system is available.

(ii) Determine the maximal probability of the state estimate just after the new observation at time t.

The solution to this problem was first derived by KALMAN (1960) and KALMAN and BUCY (1961). Comprehensive treatments of the subject have been given e.g. by EYKHOFF (1974), YOUNG (1974) and BECK (1979).

The solution of the problem stated above is the linear Kalman filter. It is a predictor-corrector-type recursive algorithm, which gives the optimal synthesis of a priori and a posteriori information. The algorithm can be reduced to the following 5 equations, which are given for both continuous and discrete time models:

(a) Prediction(time propagation)of the state and covariance to time t, when the observations at time t-1 is known:

State (continuous model):

$$\hat{x}(t|t-1) = \phi(t,t-1)\hat{x}(t-1|t-1) + \int_{t-1}^{t}\phi(t,\tau)B(\tau)u(\tau)d \tag{9a}$$

State (discrete model):

$$\hat{x}(t|t-1) = \phi(t,t-1)\hat{x}(t-1|t-1) + B_d(t-1)u(t-1) \tag{9b}$$

Covariance (continuous model):

$$P(t|t-1) = \phi(t,t-1)P(t-1|t-1)\phi^T(t,t-1) + \int_{t-1}^{t}\phi(t,\tau)G(\tau)Q(\tau)G^T(\tau)\phi^T(t,\tau)d\tau \tag{10a}$$

Covariance (discrete model):

$$P(t|t-1) = \phi(t,t-1)P(t-1|t-1)\phi^T(t,t-1) + G_d(t-1)Q_d(t-1)G_d^T(t-1) \tag{10b}$$

The estimates of both state and covariance can be updated after new observations with the following equations:

Kalman-gain:

$$K(t) = P(t|t-1)H^T(t)[H(t)P(t|t-1)H^T(t) + R(t)]^{-1} \tag{11}$$

State update (both continuous and discrete models):

$$\hat{x}(t|t) = \hat{x}(t|t-1) + K(t)[z(t) - H(t)\hat{x}(t|t-1)] \tag{12}$$

Covariance update (both model types):

$$P(t|t) = P(t|t-1) - K(t)H(t)P(t|t-1) \tag{13}$$

The superscript -1 denotes the inverse of a matrix.

State-parameter estimation

In the linear Kalman filter both the linear model and constant parameters are assumed. Therefore only the time propagation of the system uncertainty and state estimate have to be calculated. This approach cannot be used for VPA, where the fishing mortality has traditionally been treated as a discontinuous parameter of the system. The state evaluation of the VPA system can be represented by the equation:

$$\begin{pmatrix}\dot{N}\\ \dot{C}\end{pmatrix} = \begin{pmatrix}(-m-f) & 0\\ f & 0\end{pmatrix}\begin{pmatrix}N\\ C\end{pmatrix}, \tag{14}$$

where N denotes fish stock, C denotes cumulative catch of the cohort and m and f are the parameters natural and fishing mortality.

In reality both parameters of the VPA model vary in time. Thus the matrix F(t) is a function of time. Both system parameters and state have to be estimated simultaneously.

One approach to realizing this is to augment the state vector x(t) with the parameter vector $\alpha(t)$ and use the result as the new state vector. We obtain:

$$x^*(t) = \left(\frac{x}{\alpha}\right) \tag{15}$$

Now the problem is to specify the dynamics of the parameters. They might be specified as:

(a) time-independent: $\dot{\alpha} = 0$

(b) as varying in a random walk fashion: $\dot{\alpha} = \xi(t)$

Other definitions of vector α require more information on the parameters.

The extended Kalman filter (EKF)

The conbined state and parameter estimation leads to a new system coeffient matrix $F^*(\cdot)$, which consists of vector functions. As the functions include products of the elements of $x(t)$ and $\alpha(t)$, they are non-linear.

It is possible to construct an optimal non-linear filter for the case of combined state and parameter estimation. However this is often not practical because the calculations grow too laborious (RINALDI et al. 1979).

A possibility of solving non-linear problems is offered by the extended Kalman filter, whose derivation has been presented in detail, e.g. by MAYBECK (1982).

Briefly, the principles of the EKF are as follows (see BECK 1980):

(1) Linearization of the augmented state equations $x^*(t)$ about some nominal reference trajectory. For small perturbations a set of linear dynamic equations are obtained by taking first-order Taylor series expansion.

(2) Linearization of the non-linear observation equation. When the nominal measurement trajectory is defined, a linear small perturbation observation equation can be derived.

(3) Application of a linear Kalman filter to the perturbational equations.

(4) The choice of the reference trajectory is crucial to the operation of the filter. If it is inaccurate, the linearization is no longer a valid approximation. In the EKF the current state estimate is used as the reference trajectory.

A simple theoretical example

Consider the situation presented in figure 1. Low-noise measurements are available for fish stock, catch and fishing effort. The task is to use them and the VPA model to estimate the fish stock, catches and mortality parameters shown as the "true solution" in fig. 1.

For simplicity, suppose further that:

(a) Fish stock can be estimated from population indices using the formula:

$$N(t_i) = k(t_i)PI(t_i),$$

where $N(t_i)$ denotes the fish stock and $PI(t_i)$ the population index at time t_i. Assume further that $k(t_i) = 1$ for all values of t_i.

(b) Fishing mortality is linearly dependent on the fishing effort according to the formula:

$$E(t_i) = q(t_i)f(t_i),$$

where $E(t_i)$, $q(t_i)$ and $f(t_i)$ denote effort, catchability and fishing mortality, respectively. For simplicity let $q(t_i) = 5$ for all values of t_i.

(c) Suppose that the parameters are constant at each time-propagation step. The parameter dynamics equation has the form:

$$\dot{\alpha} = \begin{pmatrix} \dot{f} \\ \dot{m} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The estimation algorithm has the form:

State time propagation:

$$\hat{x}^*(t|t-1) = \phi(t,t-1)\hat{x}^*(t-1|t-1),$$

where:

$$\hat{x}^*(t-1|t-1) = \begin{pmatrix} N(t-1) \\ C(t-1) \\ f(t-1) \\ m(t-1) \end{pmatrix} \quad \text{and}$$

$$\phi(t,t-1) = \begin{pmatrix} -m-f & 0 & 0 & 0 \\ f & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Covariance time propagation:

$$P(t|t-1) = \phi_{lin}(t|t-1)P(t-1|t-1)\phi_{lin}^T(t|t-1) + Q_d(t-1),$$

where:

$$\phi_{lin}(t|t-1)=\begin{pmatrix} e^{-(f+m)} & 0 & -\frac{N(1-e^{-(f+m)})}{(f+m)} & -\frac{N(1-e^{-(f+m)})}{(f+m)} \\ \frac{f(1-e^{-(f+m)})}{(f+m)} & 1 & \frac{f\cdot N\cdot(1-e^{-(f+m)})}{(f+m)^2}+N(1-\frac{f}{(f+m)}) & \frac{f\cdot N\cdot(1-e^{-(f+m)})}{(f+m)^2}-\frac{f\cdot N}{(f+m)} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

ϕ_{lin} is derived from:

$$\phi_{lin}(t|t-1) = \mathcal{L}^{-1}\{(\frac{\partial F}{\partial x_i})^{-1}\} ,$$

where:

$\frac{\partial F}{\partial x_i}$ denotes the Jacobian matrix of the augmented state equation

$\mathcal{L}^{-1}$ denotes the inverse Laplace transform

$P(t-1|t-1)$ is a 4x4 matrix with diagonal elements describing state vector uncertainty and other elements cross correlation uncertainties.

<u>Kalman-gain</u>:

$$K(t) = P(t|t-1)H^{*T}(t)(H^*(t)P(t|t-1)H^{*T}(t) + R(t))^{-1},$$

where:

$H^*(t)$ denotes a linearized observation equation coeffient matrix:

$$H^*(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \end{pmatrix}$$

State and covariance update equations result directly from eqs. (12)-(13) by substition.

The criteria for choosing the initial values of the state and uncertainties $P(0|0)$ R and Q_d were the following:

(a) For the first observation vector the value of natural mortality was taken as m = .2.

(b) $P(0|0)$ was estimated by assuming a certain measurement noise for the 1st observation and by quessing the accuracy of the natural mortality estimate.
It should be noted that being a cumulative quantity, initial catch has the value C = 0 and is an absolute value.

(c) The $R(t)$ and $Q_d(t)$ matrices were supposed to be time-independent diagonal matrices.

The results for the example are shown in fig. 2. As can be seen, the further the calculation is processed, the closer the prognostic and observed values come to each other. The uncertainty of all the elements of the state vector, execept natural mortality, also diminishes during the estimation. The increase of natural mortality uncertainty is due to both the choice of Q_d and the fact that no measurement directly updates the m values. Comparison between figures 1 and 2 also shows that the true solution of N and c will be approached as processing continues.

The example is, however, only one illustration of the algorithm, and although it promises convergence with the true solution, extreme divergence from it can also easily be demonstrated.

Discussion

Use of the EKF algorithm in fish stock assessment necessitates statistical tests of the assumptions behind the filter algorithm. It will also be necessary to develop the filter further when a thorough performance analysis has been carried out.

First, the mathematical model upon which the filter is to be based must be shown to be adequate for the system (MAYBECK 1979). Thus there are no *a priori* reasons for using the VPA model. The VPA model does not necessarily show correctly how the fishing effort is related to stock size and mortality or how these are related both to each other and to the catch in the system under study (cf. BECK 1982). The choice of model does not, however, greatly change the procedure of applying an EKF algorithm and therefore the VPA is used as an example. Furthermore, if the main problem in fish stock assessment is the available data base rather than the population model (LUDWIG and HILBORN 1983), effort should be concentrated on tuning the filter and analysing its performance. This process also reveals a great deal about the underlying model (MAYBECK 1979). In principle, the EKF can also be used as a tool in system identification (BECK 1980).

The tuning of the filter involves the quantification of three matrices and one vector. These are the *a priori* estimation error covariance matrix $P(0|0)$, the system noise covariance matrix $Q(t)$, the measurement noise covariance matrix $R(t)$ and the vector of a priori state parameter estimates $\hat{x}^*(0|0)$ (BECK 1980). All of these affect the results of the filter and therefore the performance of the filter must be thoroughly analyzed.

The initial estimation error covariance matrix $P(0|0)$ is generally assumed to be diagonal (BECK 1980) and should reflect the uncertainty of the a priori state parameter estimates $x^*(0|0)$. For the case of the estimation of a constant signal $x(t)$ from white noise-corrupted signals, EYKHOFF (1974) shows that an erroneous assumption of $\hat{x}(0|0)$ in combination with a small $P(0|0)$ causes a slow approach to the true value. Thus it is probably better to overestimate than to underestimate the a priori estimator error covariance in the non-linear case as well.

The Q and R matrices are generally assumed to be diagonal (BECK 1980). The diagonal elements of the measurement noise matrix R display the variance of the available measurements. In principle, the system noise covariance matrix $Q(t)$ reflects the uncertainties of the model with respect to reality (BECK 1979). It depends, however, very much on subjective judgement. BECK (1980) suggests that the diagonal elements of the Q matrix for state parameter estimates might be evaluated from the accuracy of the model dynamics relative to the accuracy of the measurements. In the case of a VPA model this means an evaluation of how well a year-class is thought to follow an exponential decrease relative to the accuracy of the available population estimates in the form fo catch per unit effort (CPUE) or some other population index. If measurements of the fishing mortality (f) such as effort data are used, an evaluation of the underlying model ($f = g(E)$, where E = effort),relative to the measurement accuracy of E is needed. It should be noted that CPUE data used as a measure of stock size and the corresponding effort data used as a measure of fishing mortality are not independent observations and should therefore not be used together, since they would probably bias the results. On the other hand even quite noisy independent population index data from exploratory fishing programs or echo sounding could be used together with commercial effort data in order to improve the estimates obtained from the filter.

A further problem is the relation of the Q and R matrices to time. It is customary to assume that they are time-independent (BECK 1980). In the VPA application it is, however, conceivable that these covariance matrices vary with time. The quantification of the R matrix is not necessarily a problem, because estimates of measurement uncertainty can be obtained for each sampling instant.

The Q matrix is more of a problem. If young age groups are included in the assessment the assumption of an exponential decrease with constant parameters for one year might be a very crude model indeed, although a similar model can be a fairly good approximation for older ages. Similar difficulties can be encountered when modelling fishing mortality as a function of effort. This clearly shows that investigations of the validity of the models used in fish stock assessments are badly needed.

The statistical properties of the noise terms should be investigated. They are assumed to be white Gaussian, but the effects of non-white noise and the possibilities of using noise colouring filters in the process model should also be studied (EYKHOFF 1974).

The validity of the linearization in the VPA-EKF must be questioned, since there are no guarantees for convergence when the original model is non-linear (RINALDI et al. 1979, BECK 1980, 1982). It is possible that the perturbations around the reference trajectory, which is crucial in the development of the EKF, cannot be considered small if there are great and rapid changes in the parameter values, e.g. due to selective fishing gear. A possible solution to this problem is the use of shorter time steps than one year. If, for example, catch data are obtained more frequently than stock size or fishing mortality estimates, the Kalman filter can be designed to take into account the different sampling frequencies (MAYBECK 1979). Another possibility is to use a different linearization procedure than that around a reference trajectory (EVANS 1982). The sensitivity of different forms of the filter to perturbations can be analyzed through Monte-Carlo simulations. Covariance analysis will also be necessary in analyzing the performance of the filter.

Once a well performing EKF has been achieved, efficient use can be made of the available fish stock assessment data. Population index data and effort data can be used in the filter either directly or according to some specified function. Thus separate regression methods for off line tuning of VPA could be avoided. In this respect the EKF resembles the integrated models of NIELSEN (1982) and GUDMUNDSSON et al. (1982).

Analysis of the results obtained from the Kalman filter gives opportunities to test e.g. the assumption of time-invariant natural mortality (m). If the assumptions turns out to be invalid, m can be modelled as a parameter exhibiting random walk, or if the data are available, as a function of predator density, thus extending the VPA-EKF to the multispecies case.

For short time periods, both f and m can be considered constant and a linear Kalman filter can be used for catch (i.e. state) prediction, which is needed when the total allowable catches (TAC) are set at status quo. The validity of the assumption of time-invariant f and m can be tested and thus changes in the mortality rates can be detected. When a linear Kalman filter is used for catch prediction an estimate of the uncertainty involved is also obtained in the form of the covariance matrix $P(t|t-1)$. The Kalman filter can therefore be used to approach the problem of the variance of the TAC estimates recently addressed by POPE (1982). It must be noted, however, that the matrix $P(t|t)$ cannot be interpreted as an a posteriori measure of the true estimation error covariances (BECK 1980).

It can be concluded that the Kalman filter is an attractive tool in many fields of fishery science. As regards its application to the VPA model, it is not yet clear whether it ultimately creates more problems than it solves, but it clearly gives an alternative view of the model and opportunities to perform a thorough sensitivity analysis. The Kalman filter approach is also a way of arriving at a stochastic VPA, which shows the necessity of having not only good catch data but also information on the noise of the data. This is valuable, because improved knowledge of the noise in the data base is necessary for the rational management of fish stocks, whether one wishes to use the Kalman filter or not.

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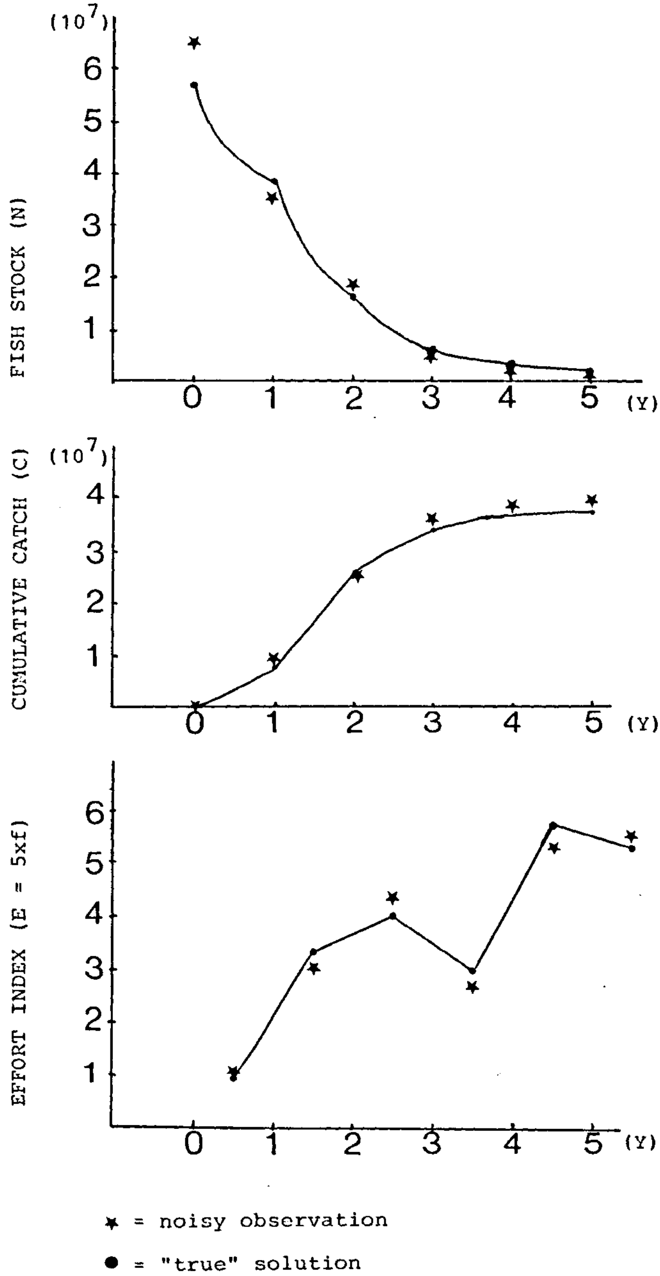

Figure 1. Numerical example.
"True" solution and measurements.

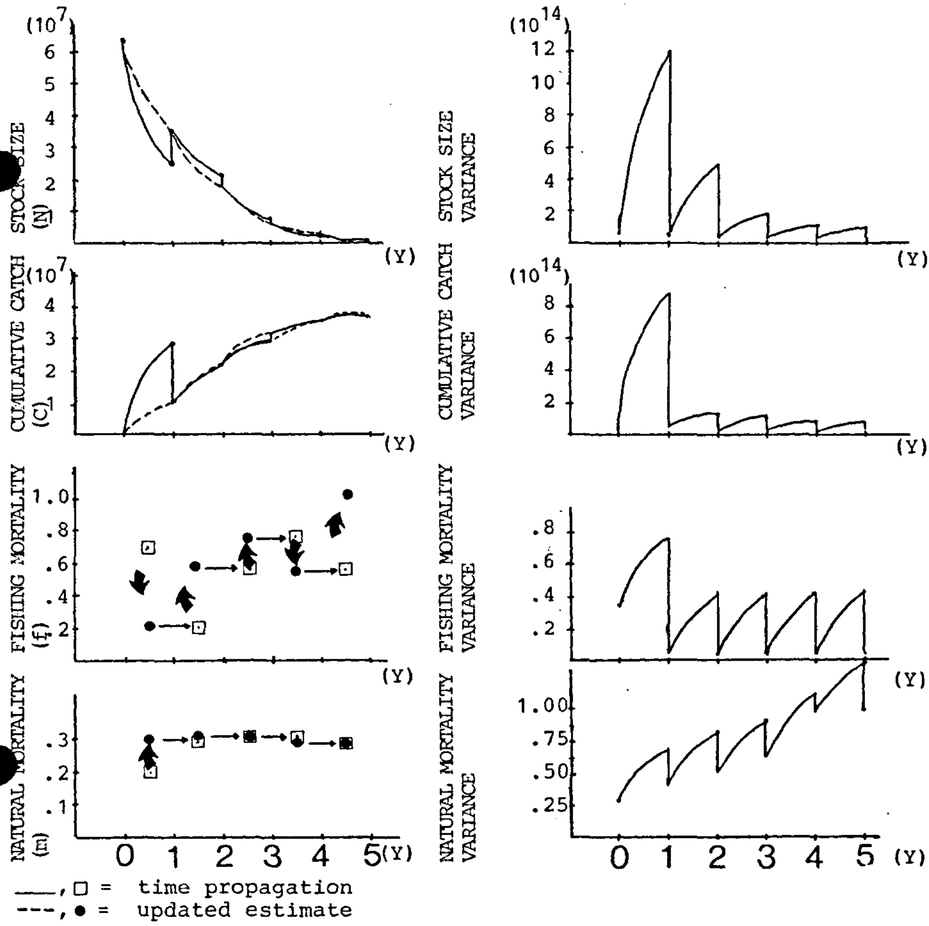

Figure 2. Numerical example.
Extended Kalman filter results.

Multiprint, Helsinki 1983