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# ON THE STABILITY OF DYNAMIC SURPLUS PRODUCTION MODELS

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## ABSTRACT

The present work had been initiated by the specific situation that arose in ICSEAF in the last years of its existence (1988 - 1990) when three kinds of dynamic surplus production models were used as official methods for total allowable catch (TAC) assessment. Nevertheless, now that the dynamic surplus production models are being introduced to TAC forecasting very broadly, the discussion of theirstability based on the results of analysis of the models used in ICSEAF for forecasting Cape hake TAC looks to be meaningful.

The stability of the Butterworth - Andrew (1984), the Lleonart - Salat - Roel (1985), the Babayan - Kizner (1988) dynamic surplus production models, as well as of some newer models by Kizner (1989, 1990) with the control through both catch and fishing effort, is discussed. As the models considered are discrete ones, direct analysis of the behavior of the perturbations is performed. Not only stationary states are considered, therefore a special notion of a critical boundary (dependent on the current fishing effort) is introduced.

The analysis carried out shows that the models with the control through fishing effort are preferable from the point of view of their stability (and therefore from the practical point of view) as the fishing effort acts as a stabilizing factor: as a matter of fact the population biomass never falls below the critical boundary. Models with the control through catch are also admissible, but when the biomass becomes lower than the critical level, the instability grows, and it is only after the solution transcends the critical level that the instability begins to decrease.

## NTRODUCTION

When dealing with the problem of stock and TAC assessment one has to operate sometimes with commercial fishery data which do not reflect the age structure of the exploitable population. In such a case a surplus production model can serve as a mathematical instrument of the investigation. While only traditional (i.e. equilibrium) models were in use it was possible to discuss the reliability and accuracy of the estimates obtained, but the question of their stability did not arise. But now that the dynamic surplus production models are being introduced to TAC forecasting the question becomes not only meaningful but a very important one.

One cannot state that an unstable model is, *a priori*, worse than a stable one. Nevertheless, the stability analysis, clarifying the conditions of the stability and the growth rates of unstable modes, make it possible to use surplus production models in TAC forecasting more deliberately and effectively.

The present work had been initiated by the specific situation that arose in ICSEAF in the last years of its existence (1988 - 1990) when three kinds of dynamic surplus production models were used as official methods for Cape hake TAC assessment. The stability of a number of models used in ICSEAF for forecasting Cape hake TACs is discussed here. Those are the Butterworth - Andrew (1984), the Lleonart - Salat - Roel (1985) (hereinafter referred to as Lleonart et al.), the Babayan - Kizner (1988) models, and three new models (one of them can be regarded as a modification of the Butterworth - Andrew model) suggested by Kizner (MS 1989, MS 1990). For simplicity of the following exposition it is convenient to examine two models by Kizner prior to the Babayan - Kizner one and the Butterworth - Andrew model along with its modification after the Babayan - Kizner model (regardless of the chronology).

It should be noted that the problem of the stability of stationary states of dynamic production models with different surplus production functions had been examined by many authors (see e.g. May, Beddington, Horwood, Shepherd, 1978); the technique of such an examination is a classic one and leads to an eigenvalue problem. The models which are considered below are discrete ones, therefore direct analysis of the behavior of the perturbations is performed, and not only stationary states are considered.

## THE LLEONART ET AL. MODEL

The model is a generalization of the traditional (equilibrium) surplus production Schaefer or Fox model expressed in terms of dependence of catch per unit effort (CPUE) upon the fishing effort; it is based on the concept of stock inertia:

$$u_{i+1} = gu_i + (1-g)F(f_i). \qquad (1)$$

Here

$u_i$ - CPUE for the year i,

$f_i$ - fishing effort in the year i (supposed to be known),

g - inertia coefficient ($0<g<1$),

F - a function relating CPUE to fishing effort in the traditional (equilibrium) models ($u = F(f)$); for the Shaefer- and the Fox-type models $F(f) = a - bf$ and $F(f) = ae^{-bf}$ respectively.

Supposing that a solution of equation (1), $u_i$, is a sum of a certain unperturbed solution U and of its perturbation $\varepsilon_i$,

$$u_i = U_i + \varepsilon_i, \qquad (2)$$

then substituting (2) into (1) and taking into account that $U_{i+1} = gU_i + (1-g)F(f_i)$, one gets:

$$\varepsilon_{i+1} = g\varepsilon_i. \qquad (3)$$

It is evident from (3) that the perturbation $\varepsilon_i$ does not grow with the increase of i (i.e. the passage of time): it is either constant for fully inertial stocks ($g = 1$) or decreases as a geometrical progression does in other cases (for $0<g<1$, to be more accurate).

So, the Lleonart et al. model is asymptotically stable for $0<g<1$, and neutrally stable at $g = 1$. The result obtained is valid for stationary states of the model as well.

## CATCH CONTROLLED MODEL

Two equations expressing balance of the stock biomass and the proportion between the biomass and CPUE:

$$B_{i+1} = B_i + P(B_i) - C_i, \qquad (4)$$

$$v_i = qB_i, \qquad (5)$$

will serve as a basis of the following construct* ions.

Here

$B_i$ - biomass at start of the year i,

$v_i$ - CPUE at start of the year i,

$C_i$ - catch in the year i,

P - production function: $P(B_i) = rB_i(1-B_i/K)$ and $P(B_i) = rB_i(1-\ln B_i/\ln K)$ according to Schaefer* and Fox respectively,

q, r, K - positive constants: q - catchability coefficient, K - carrying capacity, r - intrincic growth rate.

Here and below we operate only with 'model' (estimated) variables (except $C_i$ and $f_i$), thus there is no need to mark them with any special sign, such as ^, for example.

Substitution of (5) into (4) reduces the system (4), (5) to one equation with respect to CPUE:

* Such form of the production function or the 'rate of population increase' for describing continuouse dynamics of unexploited populations goes back to P.F.Verhulst.



$$v_{i+1} = v_i + qP(v_i/q) - qC_i. \qquad (6)$$

This model, suggested by the author (Kizner, MS 1989, MS 1990) can be called **the model with the control through catch** ($C_i$ which is supposed to be known for a period $i = 1,...,n$). The 'model' CPUE dynamics during the period of fishing history can be evaluated from the equation (6) if the start v, e.g. $v_2$ (see Annex), is given.

The equation (6) is a nonlinear one (with respect to $v_i$) and study of its finite perturbations would be a complicated problem, therefore only the linear stability analysis of the dynamic system will be carried out, i.e. the perturbations will be regarded as small ones. For the Schaefer production function the equation (6) takes the form:

$$v_{i+1} = v_i + rv_i(1-v_i/qK) - qC_i. \qquad (7)$$

After expanding $v_i$ into a sum

$$v_i = V_i + \nu_i, \qquad (8)$$

where $V_i$ is the unperturbed solution of (7) and $\nu_i$ is its small perturbation, and substituting (8) into (7) one gets:

$$\nu_{i+1} = [1 + r(1-2V_i/qK)]\nu_i. \qquad (9)$$

Now it is clear that there exists a critical level of the unperturbed solution $V_i$ of (7):

$$V_{CR} = qK/2;$$

in terms of the biomass it is $B_{CR} = K/2$, i.e. the maximum surplus production level, $B_{MSY}$, providing MSY. Indeed, $1-2V_i/qK \le 0$ when $V_i \ge V_{CR}$, and in this case the coefficient by $\nu_i$ in (9) (i.e. the expression in square brackets) is less than 1, and on the contrary, $1-2V_i/qK > 0$ and the coefficient by $\nu_i$ is more than 1 when $V_i < V_{CR}$.

It should be noted that though the concrete value of the parameter r does depend on the initial data as well as on the time scale accepted, usually $r < 1$ for the Schaefer-type model (e.g. for the Cape hakes r varies from about 0.3 to 0.5). Furthermore, as usual $V_i \le gK$: only in the early years of the period of intensive commercial fishery the CPUE can be close to qK or a bit higher. Therefore, in the context of the stability analysis of the models one undoubtedly can assume that $r(1-2V_i/qK) > -1$.

Thus,

$$\nu_{i+1}/\nu_i \le 1 \qquad \text{when } V_i \ge V_{CR}$$

(if $V_i = V_{CR}$ then $\nu_{i+1}/\nu_i = 1$),

$$\nu_{i+1}/\nu_i > 1 \qquad \text{when } V_i < V_{CR}$$

It is clear now that while $V_i \ge V_{CR}$ the perturbation does not grow with the growth of i, whereas it does grow monotonically wile $V_i < V_{CR}$. The perturbation grows or decreases no slower than a geometric progression if $V_i$ stays separated from $V_{CR}$ (i.e. $V_i$-$V_{CR}$ stays finite). One can say that when $V_i < V_{CR}$ **the zero mode is unstable.**

Because of generality of the conclusion made it remains valid for stationary states of the model under consideration.

For the Fox-type model the linear stability analysis (which requires to use the power expansion of the logarithm) gives qualitatively the same results. In particular, the critical level is the level of maximum surplus production $V_{CR}$ = qK/e (or $B_{CR} = B_{MSY}$ = K/e) providing MSY. Moreover, it can be shown that in the case of an arbitrary production function P with one maximum, $B_{MSY}$ is the critical level (in the terms of biomass) for the stability.

The procedure of TAC forecasting is the same for both new modifications of the model. Its stability will be discussed below.

## EFFORT CONTROLLED MODEL

Another model also suggested by Kizner (MS 1989, MS 1990) which will be called **the model with the control through fishing effort** comes from the previous one after replacing $C_i$ in (4) and (7) by $f_i(v_i + v_{i+1})/2$. In other words, now it is the fishing effort rather than the catch that becomes the external control action upon the stock. This considerably changes the type of the equation of the dynamics. When P is the Schaefer function, the governing equation is:

$$v_{i+1} = v_i + rv_i(1-v_i/qK) - qf_i(v_i + v_{i+1})/2,$$

which gives:

$$v_{i+1} = \frac{1 - qf_i/2 + r(1-v_i/qK)}{1 + qf_i/2}\, v_i. \qquad (10)$$

Acting in accordance with the above described scheme and substituting (8) into (10), one gets in the linear approximation the following equations for the perturbations:

$$\nu_{i+1} = \alpha_i + \rho_i(1-2V_i/qK)]\nu_i, \qquad (11)$$

where $\alpha_i = (1-qf_i/2)/(1+qf_i/2)$ and $\rho_i = r/(1+qf_i/2)$.

Strictly speaking, any critical level (i.e. a constant bound) does not exist in this case: the perturbation does not grow now when $V_i \geq [(\alpha_i+\rho_i-1)/\rho_i]qK/2 = (1-qf_i/r) \cdot qK/2$ (the upper boundary is not given because, as it was argued above, usually $V_i < qK$). That is why the term **'critical boundary'**, $V_i^b$, which is determined by the equation $V_i^b = (1-qf_i/r)qK/2$, is more appropriate here.

It is evident that the critical boundary itself and the values of $V_i$ providing an opposite inequality lie lower than qK/2; the higher the fishing effort is the lower the critical



boundary falls. E.g., for the Cape hake of the ICSEAF Divisions 1.3+1.4 the fitting of the model brings one to $K \approx 2.6 \cdot 10^3(\cdot 10^3 t)$, $r \approx 0.40$, $q \approx 0.42 \cdot 10^{-3}(\cdot 10^{-3} h^{-1})$, while $qf_i$ varies from 0.07 to 0.36. Hence the critical boundary lies lower than 0.41qK. In fact it lies lower than the actual and the estimated CPUE values for the majority of real intensively exploited stocks, and for the Cape hake among them (Figure 1), because the fishing effort acts as a stabilizing factor here. The instability could appear in the present model only in such a hardly probable situation when the stock was diminished to a very low level as compared to the carrying capacity and then the fishery intensity was reduced greatly.

The conclusion made is valid for every solution of the equation (10) including stationary states (i.e. for limit states achieved at $f_i$ = const).

TAC forecasts in both of the models described above are calculated as

$$TAC_{n+m} = f_{0.1}(v_{n+m} + v_{n+m+1})/2,$$

where

$$v_{n+k+1} = \frac{1-qf_{0.1}/2 + r(1-v_{n+k}/qK)}{1+qf_{0.1}/2}\, v_{n+k},$$

for k = 1,...,m (the first forecasted CPUE value, $v_{n+1}$, is determined by (7) or (11)). In view of the fact that the limit of the succession of the forecasted (with a fixed $f_{n+k} = f_{0.1}$ for all k) values of CPUE is higher than qK/2, the TAC forecasts are stable if started from a CPUE level which is not lower than qK/2 for the model with the control through catch, and $(1-qf_{0.1}/r)qK/2$ for the model with the control through fishing effort, i.e. practically always for the latter model.

The results obtained are valid in the case of the model with the Fox surplus production function too.

## THE BABAYAN - KIZNER MODEL

This model (the very model suggested by Babayan and Kizner in 1988) can be also called a **'central difference'** version of the model with the control through catch. It comes from equations (4) and (5) supplemented by the relationship

$$v_i = (u_{i-1} + u_i)/2. \qquad (12)$$

Equations (4), (5), (12) give the following principal equation of the model in the case of the Schaefer surplus production function:

$$u_{i+1} = u_{i-1} + r(u_{i-1}+u_i)[1 - (u_{i-1}+u_i)/2qK] - 2qC_i. \qquad (13)$$

In contrast to the governing equations of the two previous models, the CPUE change corresponding to the year i is in fact represented in (13) by the central difference $u_{i+1} - u_{i-1}$.

When analyzing the stability of the solution of the equation (13) the results of the examination of the first modification are useful. Indeed, substituting the expansion (2), where now $\varepsilon_i$ is a small perturbation, in (13) and using the following designations

$$(U_{i-1} + U_i)/2 = V_i, \qquad (\varepsilon_{i-1} + \varepsilon_i)/2 = \nu_i,$$

one can see that changes of the variable $\nu_i$ are governed by the equation (9). Since the behavior of the variable $\nu_i$ is known, it is easy to study the behavior of $\varepsilon_i$ from the relationships

$$\varepsilon_{i+1} = 2(\nu_{i+1} - \nu_i) + \varepsilon_{i-1}, \qquad \varepsilon_i = 2\nu_i - \varepsilon_{i-1}. \qquad (14)$$

It was found that $\nu_i$ decreases when $V_i > V_{CR} = qK/2$ or $qK/e$ (for the Fox-type model). Hence, if this condition is maintained sufficiently long, the ratio $\varepsilon_{i+1}/\varepsilon_{i-1}$ becomes close to 1 and $\varepsilon_i/\varepsilon_{i-1}$ to -1. In other words, a saw-tooth swing of the solution of the equation (13) must be observed, i.e. **the highest mode is unstable.**

When $V_i < V_{CR}$ the mean perturbation $\nu_i$ grows, hence (see (14)) sooner or later the perturbation $\varepsilon_i$ will become positive and growing. Thus, below the critical level at least **the zero mode is unstable.**

The conclusion made is valid for stationary states as well, but the TACs are forecasted here as $TAC_{n+m} = f_{0.1}u_{n+m}$, where

$$u_{n+k+1} = u_{n+k-1} + r(u_{n+k-1} + u_{n+k})[1-(u_{n+k-1} + u_{n+k})/2qK] - 2qf_{0.1}u_{n-k}$$

for k = 1,...,m (the first forecasted CPUE value, $u_{n+1}$, is determined by (13)).

This equation differs from (13), but the analysis of the following equation describing the behavior of the perturbation,

$$\varepsilon_{n+k+1} = [r(1-2V_{0.1}/qK) - 2qf_{0.1}]\varepsilon_{n+k} + [1 + r(1-2V_{0.1}/qK)]\varepsilon_{n+k-1},$$

where $f_{0.1} = 0.45r/q$, $V_{0.1} = 0.55qK$, shows that the modulus of the perturbation grows, and there is a saw-tooth swing. So, **the highest mode is unstable now.**

The same conclusion is valid for the model with the Fox surplus production function.

## THE BUTTERWORTH - ANDREW MODEL

The model is based on the equation (4) and the relationship

$$u_i = q(B_i + B_{i+1})/2. \qquad (15)$$

Here **the catch serves as the control action** upon the stock. That is the original version of the model, but one can modify it replacing (4) by

$$B_{i+1} = B_i + P(B_i) - f_i u_i. \qquad (16)$$

Expanding $B_i$ into a sum of an unperturbed solution $\mathcal{B}_i$ and of its small perturbation $\beta_i$,

$$B_i = \mathcal{B}_i + \beta_i, \qquad (17)$$



and substituting (17) into (4), one gets the equation for the variable $\beta_i$ which coincides with the equation (9) for the variable $\nu_i$, if P is the Schaefer function and with an analogous equation, if P is the Fox function.

Just as for the original version, after substituting (2) and (17) into (15) and (16) one can obtain the equation for the variable $\beta_i$ which coincides with the equation (11) for the variable $\nu_i$.

Hence, the stability conditions and the instability characteristics of the original and modified versions of the Butterworth - Andrew model are analogous to those of the two described above corresponding models with the control through catch and through fishing effort by Kizner.

## CONCLUSION

The TAC forecasts depend on the level of estimated CPUE for the end of the historical period, since it serves as a start level for the forecasts, and the errors in determining this level do depend on whether the solution describing the CPUE dynamics is stable or not. That is why when considering the reliability of the forecasts, one has to take into account not only the stability or instability of the forecasts themselves but the stability characteristics of the initial dynamic model as well.

The analysis carried out shows that the models with the control through fishing effort are preferable from the point of view of their stability (and therefore from the practical point of view) because the fishing effort acts as a stabilizing factor upon an exploited population. Models with the control through catch are also admissible, but when the solution falls below the critical level, the instability grows, and it is only after the solution transcends the critical level that the instability begins to decrease. The 'central difference' Babayan - Kizner model is the most unstable, which is why preliminary smoothing of the initial data series is required to be able to use this model, and forecasting only one or two years ahead can be recommended.

The linear Lleonart et al. model is always stable, not that it can pretend to give a substantial description of an exploited fish stock dynamics.

## ACKNOWLEDGEMENT

The presented stability analysis of a few kinds of models had been carried out for the ICSEAF Scientific Counsel, and the corresponding paper (Kizner, ME 1990) was bound to appear in the first volume of a new periodic edition 'Collection of Selected ICSEAF Scientific Papers'. Unfortunately, the dissolution of ICSEAF in 1990 caused cancellation of its planes of scientific publication.

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## ANNEX

It should be appropriate to describe here briefly the mode of fitting for the models with the control through catch and fishing effort by Kizner, so as to give a complete view of the usage of the models. The procedure is as follows.

First the initial (start) 'model' CPUE must be evaluated as

$$v_2 = (CPUE_1^{obs} + CPUE_2^{obs})/2,$$

where the actual (observed) CPUEs are provided by the marc 'obs'

Then the first approximations of the model parameters q, r, K, must be given (the values of the parameters of the corresponding :process error models can be taken) and the first approximations of the estimated $v_i$ (for i = 3,...,n+1) must be evaluated through (7) or (10) (or through analogous equations for the case of the Fox surplus production function).

Every next approximation of the estimates of the series $\{v_i\}$ and of the set of the model parameters must be found in the course of the iterative procedure of minimizing the functional

$$\sum_{i=2}^{n} [(v_i+v_{i+1})/2 - CPUE_i^{obs}]^2$$

if the error is supposed to be additive, or

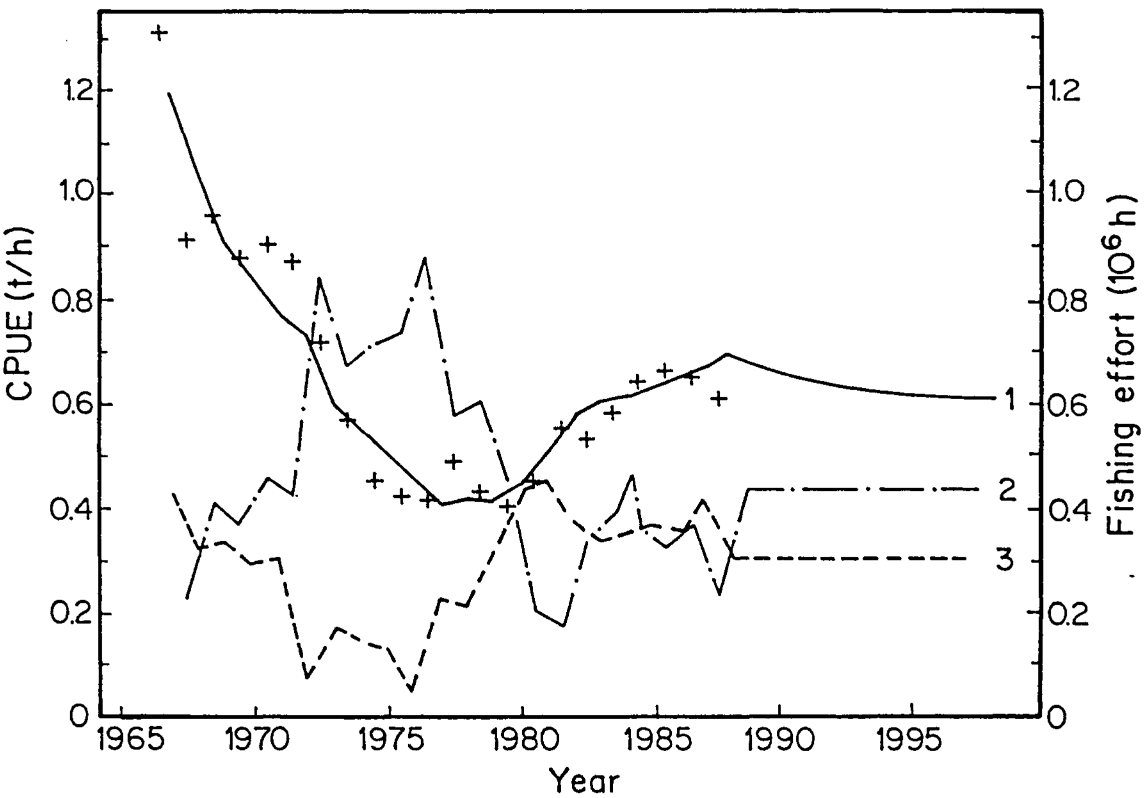
$$\sum_{i=2}^{n} [\ln((v_i+v_{i+1})/2) - \ln CPUE_i^{obs}]^2$$

if the error is supposed to be multiplicative.

On the output of the procedure described one has got the final estimates of q, r, K, as well as $v_i$ for i = 3,..., n+1.

## Figure Caption

Fig 1. Actual data, estimated CPUEs and the critical boundary for the model with the control through fishing effort (the second modification of the Babayan - Kizner model): crosses - actual CPUEs, 1 - estimated CPUEs, 2 - fising effort, 3 - critical boundary.



CPUE (t/h)
Fishing effort (10^6 h)
Year
1965
1970
1975
1980
1985
1990
1995
0
0.2
0.4
0.6
0.8
1.0
1.2
1
2
3


Fig. 1