III.- Shear effect and eddy diffusivity

Conservation equations, including hydrodynamical convection and diffusion, air-sea and sea-bottom interactions, external releases and chemical and biochemical interactions, can be written for a selected number of variables describing the state of pollution of any chosen region of the sea (e.g. Nihoul, 1971).[1]

In these equations, concentration variables are coupled with physical and hydrodynamical parameters which account for motions of the sea of an extreme complexity, characterized by different time scales and different length scales, from the quasi-steady currents to the rapid small turbulent eddies, interacting with each other and forming an intricated pattern amenable only, in most cases, to a statistical analysis.

Since an instantaneous description of the state of pollution is, in general, not required, it is thus convenient to average the evolution equations over some appropriately chosen length of time T.

This can be done by methods akin to that of Krylov - Bogoliubov and Mitropolsky and if $\overline{\rho_\alpha}$ denotes the average over T of any state variable measuring the concentration of a pollutant[2] and $\mathbf{v}$ the average velocity, the resulting equations further simplified by the introduction of eddy diffusivities, can be written (Nihoul, 1971)

(1)
$$\frac{\partial \overline{\rho_\alpha}}{\partial t} + \nabla \cdot \overline{\rho_\alpha \mathbf{v}} = \overline{Q_\alpha} - \nabla \cdot \overline{m_\alpha} + \sum_{\beta=1}^{s} \overline{P_{\alpha\beta}} + \frac{\partial}{\partial x_1} (\lambda_1 \frac{\partial \overline{\rho_\alpha}}{\partial x_1}) + \frac{\partial}{\partial x_2} (\lambda_2 \frac{\partial \overline{\rho_\alpha}}{\partial x_2})$$
$$+ \frac{\partial}{\partial x_3} (\lambda_3 \frac{\partial \overline{\rho_\alpha}}{\partial x_3})$$

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(1) For instance, the mathematical model established for the Belgian National Program for the Study of Sea Pollution (Nihoul, 1970) is based on the determination, in addition to physical and hydrodynamical variables, of selected elements' concentrations in a limited number of compartment or "phases" distinguished by the scale or the degree of refinement of their dispersion in the sea (for instance the phases of (i) all dissolved chemical substances, (ii) suspended particles, (iii) minute living organisms carried along by the sea, (iv) different groups of animals at different stages of the food chain, etc.).

(2) In the "phases" model, ρ_α would be for instance the concentration of one selected element (say mercury) in one specific phase (say the phase of all dissolved substances).

In this equation,

a. $\dfrac{\partial \overline{\rho_\alpha}}{\partial t}$ represents the time derivative of the "slowly varying"[1] average;

b. $\overline{\nabla . \rho_\alpha \overline{V}}$ represents the advection by the average current;

c. $\overline{Q_\alpha}$ represents the rate of production (or destruction) of α per unit volume due to external sources;

d. $\overline{m_\alpha}$ represents the "migration" flux of α with respect to the water masses (for instance, sedimenting particles have a vertical migration through the water);

e. $\overline{P_{\alpha\beta}}$ represents the specific rate of production (or destruction) of α by (chemical or biochemical) reaction β and the sum in (1) is over all reactions with a non zero budget [2];

f.
$$\frac{\partial}{\partial x_1} \left(\lambda_1 \frac{\partial \overline{\rho_\alpha}}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left(\lambda_2 \frac{\partial \overline{\rho_\alpha}}{\partial x_2} \right) + \frac{\partial}{\partial x_3} \left(\lambda_3 \frac{\partial \overline{\rho_\alpha}}{\partial x_3} \right)$$

represent the turbulent (+ molecular) diffusion.

1.- Vertically integrated equations

If one is mainly interested in the horizontal distribution of pollutants, it is convenient to integrate eq. (1) over depth. Let $x_3 = \zeta$ and $x_3 = -h$ be the equations of the mean free and bottom surfaces respectively and let

(2) $\qquad H = \zeta + h$

(3)(4) $\qquad V_1 = \displaystyle\int_{-h}^{\zeta} \overline{v_1}\ dx_3 \quad , \quad V_2 = \int_{-h}^{\zeta} \overline{v_2}\ dx_3 \ ,$

(5)(6) $\qquad S_\alpha = \displaystyle\int_{-h}^{\zeta} \overline{Q_\alpha}\ dx_3 \quad , \quad I_\alpha = \int_{-h}^{\zeta} \sum_{\beta=1}^{s} \overline{P_{\alpha\beta}}\ dx_3 \ ,$

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(1) i.e. ρ_α has a characteristic time of change much larger than the reference time T .

(2) Thus, in the "phases" model, all reactions which do not transfer an element from one phase to another give no contribution to the sum.

(7) $$M_\alpha = \int_{-h}^{\zeta} \left(\frac{\partial \overline{m_{\alpha,1}}}{\partial x_1} + \frac{\partial \overline{m_{\alpha,2}}}{\partial x_2} \right) dx_3 ,$$

(8) $$R_\alpha = \int_{-h}^{\zeta} \overline{\rho_\alpha} \, dx_3 .$$

One notes that, if ξ stands for any of the variables t, x_1 and x_2, one has

(8) $$\int_{-h}^{\zeta} \frac{\partial q(t,x_1,x_2,x_3)}{\partial \xi} \, dx_3 = \frac{\partial}{\partial \xi} \int_{-h}^{\zeta} q \, dx_3 - q(\zeta) \frac{\partial \zeta}{\partial \xi} - q(-h) \frac{\partial h}{\partial \xi} .$$

Furthermore, the kinematic boundary condition at the free surface gives

(9) $$\frac{d\zeta}{dt} \equiv \frac{\partial \zeta}{\partial t} + \mathbf{v} \cdot \mathbf{\nabla}_1 \zeta = \left[v_3 \right]_{x_3 = \zeta}$$

where

(10) $$\mathbf{\nabla}_1 = \mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2}$$

and similarly, if allowance is made for variation of depth with space and eventually with time in the case of a non totally rigid bottom

(11) $$\frac{dh}{dt} \equiv \frac{\partial h}{\partial t} + \mathbf{v} \cdot \mathbf{\nabla}_1 h = - \left[v_3 \right]_{x_3 = -h} .$$

Integrating (1) and using (8), (9) and (11), one sees that the terms that arise from interchange of the order of integration and differentiation cancel out with the exception of the eddy diffusion terms. In view of the fact that eddy diffusivity is a relatively crude approximation, the assumption is however generally made (Fofonoff, 1962) that the integrated diffusion terms can also be approximated with the help of new eddy diffusivity coefficients μ .

Integration of (1) then gives

(12) $$\frac{\partial R_\alpha}{\partial t} + \mathbf{\nabla}_1 \cdot \int_{-h}^{\zeta} \overline{\rho_\alpha} \, \overline{\mathbf{v}} \, dx_3 = S_\alpha - M_\alpha + I_\alpha + \frac{\partial}{\partial x_1} \left(\mu_1 \frac{\partial R_\alpha}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left(\mu_2 \frac{\partial R_\alpha}{\partial x_2} \right)$$

$$+ \left[\lambda_3 \frac{\partial \overline{\rho_\alpha}}{\partial x_3} - \overline{m_{\alpha,3}} \right]_{x_3 = -h}^{x_3 = \zeta}$$

The last term in the right-hand side accounts for air-sea and sea-bottom interactions.

The second term in the left-hand side represents the depth-mean average advection and it appears that it is not equivalent to the advection by the mean average current although such an approximation is very often made in the literature (e.g. Wolff et al, 1970).

Actually, as pointed out by Taylor (1953, 1954), Elder (1959) and Bowden (1965), in some cases, the mean advection may be regarded as giving rize to two contributions, the first of which represents the advection by the mean current and the second additional diffusive terms which may be quite important as compared with eddy diffusion.

This will be discussed in the next section.

2.- Shear effect diffusivity

Taylor (1953, 1954) and Elder (1959) pointed out, in the cases of pipe and open channel flows, respectively, that longitudinal dispersion could be produced by the combination of a transverse gradient of velocity with transverse turbulent mixing. This process called the "shear effect" (Bowles et al, 1958) was identified by Bowden in estuaries and coastal waters where it could be associated with tidal currents, density currents or wind-driven currents and Bowden showed how it could be interpreted by a generalization of Taylor's theory (Bowden, 1965).

Bowden's formula was used by Talbot (1970) and the results compared with measurements of diffusion made by the Fisheries Laboratory, Lowestoft, with the dye tracer Rhodamine B. Talbot showed that the process was indeed important in coastal and estuary regions and that it was well accounted for by Bowden's theory provided some care was taken in estimating the vertical eddy diffusivity.

The shear effect, however, was never considered in the open sea and Talbot did not put Bowden's theory to the test in discussing the observations of three other experiments realized in Central North Sea (under the auspices of ICES) in 1965 and Central Southern North Sea in 1969.

Some of the results obtained by the first experimental cruises organized in the scope of the Belgian National Program for the Study of Sea Pollution were not however inconsistent with a non negligible shear effect in the open sea; perhaps emphasized in the shallow irregular test

region (see figure 1) by the fairly large and frequent variations of depth.

This suggested that the applicability to the open sea of the Taylor-Bowden theory be examined and, in the following, it is shown that a vector generalization to two dimensions of their reasonning yields expressions for the horizontal shear effect diffusivities which allow predictions in fair agreement with observed dispersion effects.

In order to exhibit the diffusivity effect more clearly, attention will now be restricted to a passive, non migrating contaminant released at some initial time and let to be convected and dispersed by the sea. In this case, the right-hand sides of eq. (1) and (12) reduce to the eddy diffusivity terms (and the index α may be dropped as the coupling between different eventual contaminants vanishes). The situation applies to experiments on diffusion made with dye tracers and the theoretical results will be compared with experiment B reported by Talbot.

Let

$$(13) \qquad r = \frac{R}{H} \equiv \frac{1}{H} \int_{-h}^{\zeta} \overline{\rho} \ dx_3 \ ,$$

$$(14) \qquad u_1 = \frac{V_1}{H} \equiv \frac{1}{H} \int_{-h}^{\zeta} \overline{v_1} \ dx_3 \ ,$$

$$(15) \qquad u_2 = \frac{V_2}{H} \equiv \frac{1}{H} \int_{-h}^{\zeta} \overline{v_2} \ dx_3 \ ,$$

$$(16) \qquad \mathbf{u} = (u_1, u_2) \ .$$

The equation of continuity, for an incompressible ocean or for a stratified ocean in Boussinesq's approximation, may be written

$$(17) \qquad \mathbf{\nabla} . \overline{\mathbf{V}} = 0 \ .$$

Integrating over depth and using (2), (9) and (11), one obtains

$$(18) \qquad \frac{\partial H}{\partial t} + \mathbf{\nabla}_1 . \mathbf{V} = 0 \ .$$

If a prime refers to the deviation from their vertical mean of the average variables $\overline{\rho}$ and $\overline{\mathbf{V}}$, eq. (12) may be written using (18)

$$(19) \qquad \frac{\partial r}{\partial t} + \mathbf{u}.\mathbf{\nabla}_1 r + \frac{1}{H} \mathbf{\nabla}_1 . \int_{-h}^{\zeta} r' \, \mathbf{v}' \, dx_3 = \frac{1}{H} \mathbf{\nabla}_1 .(\mu \mathbf{\nabla}_1 r \, H) \ .$$

In eq. (19), the mean advection term has been split into a term representing the advection by the mean motion and an additional term representing the shear effect.

To estimate the latter, one shall assume, following Taylor and Bowden that, in eq. (1),

(i) vertical convection and horizontal diffusion can be neglected in first approximation as compared to vertical diffusion;

(ii) in axes moving with the mean average velocity $\mathbf{u}$, the distribution of contaminant may be approximated by a steady one;

(iii) squares of deviations from the vertical mean may be neglected.

With these assumptions, eq. (1) reduces to

$$(20) \qquad \mathbf{v}'.\mathbf{\nabla}_1 r = \frac{\partial}{\partial x_3} \left(\lambda_3 \frac{\partial r'}{\partial x_3} \right) \ .$$

This equation can be used to calculate r' in terms of the gradient of the mean concentration r . Multiplying the result by $\mathbf{v}'$ and integrating over depth, one obtains thus an estimate of the shear effect.

The result turns out to be fairly simple if one assumes (this may be legitimate in sufficiently shallow water) that

$$(21) \qquad \mathbf{v}' = \mathbf{u} \, \varphi(\eta) \ ,$$

$$(22) \qquad \lambda_3 = \kappa \, g(\eta)$$

where

$$(23) \qquad \eta = \frac{x_3 + h}{H} \ .$$

Eq. (20) gives then

$$(24) \qquad H \, (\mathbf{u}.\mathbf{\nabla}_1 r) \, \varphi(\eta) = \frac{\partial}{\partial \eta} \left[\frac{\kappa}{H} g(\eta) \frac{\partial r'}{\partial \eta} \right] \ .$$

Integrating twice, multiplying by $\mathbf{v}'$ and averaging over depth, one gets

$$(25) \qquad \int_0^1 r' \, \mathbf{v}' \, d\eta = \mathbf{u}(\mathbf{u}.\mathbf{\nabla}_1 r) \, \frac{H^2}{\kappa} \int_0^1 F(\eta) \, \varphi(\eta) \, d\eta$$

where

(26)
$$F(\eta) = \int \frac{d\sigma}{g(\sigma)} \int \varphi(\xi) \, d\xi \ .$$

Formula (25) may be regarded as the vectorial analogue of Bowden's result (eq. 9 of his 1965 paper). The integral of the right-hand side has been calculated by Bowden for various velocity profiles; the corresponding form of $g(\eta)$ being calculated under the assumption of "neutral stability" so that λ_3 may be taken as equal to the vertical eddy viscosity ν_3 .

Bowden's result may be written

(27)
$$\frac{\int_0^1 F(\eta) \, \varphi(\eta) \, d\eta}{\kappa} = - \frac{\alpha}{u \, H}$$

where α depends upon the type of profile considered and, for instance,
$$\alpha = 1.11$$
for Van Veen's profile.

Actually, besides the hypothesis (21) and the assumption of a Van Veen profile which may not be quite accurate, the very value of α is furthermore influenced by the Richardson number (measuring the degree of stratification) as both the ratio $\frac{\lambda_3}{\nu_3}$ and the eddy viscosity ν_3 depend upon it (Ellison, 1957).

In a perfectly well-mixed sea $(Ri = 0)$, Ellison's theory suggests that the eddy diffusivity may be 1.4 larger than the eddy viscosity. This argument would recommend

(28)
$$\alpha \sim \frac{1.11}{1.40} \sim 0.8 \ .$$

Combining (25) and (27) and substituting the result in (19), one obtains

(29)
$$\frac{\partial r}{\partial t} + \mathbf{u}.\mathbf{\nabla}_1 \, r = \frac{1}{H} \ \mathbf{\nabla}.(\alpha H^2 \ \mathbf{u} \ \frac{\mathbf{u}.\mathbf{\nabla}_1 r}{u}) + \frac{1}{H} \ \mathbf{\nabla}_1.(\mu \ \mathbf{\nabla}_1 r H) \ .$$

One notes that the velocity $\mathbf{u}$ which appears in formula (29) is the actual depth-mean average velocity over the chosen characteristic time T . Hence, if T is much larger than the tidal periods, $\mathbf{u}$ refers only to residual and average wind-driven currents (and H is
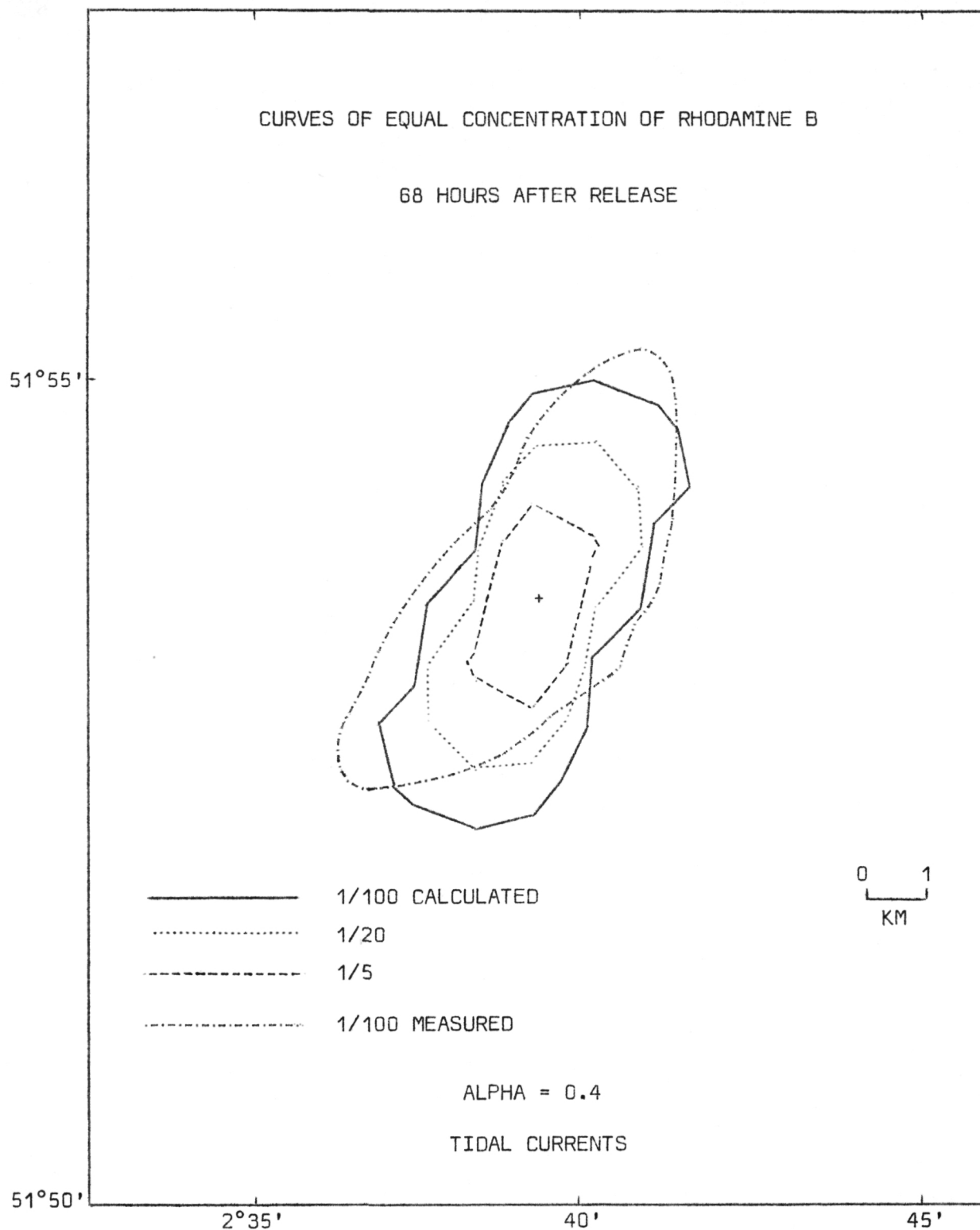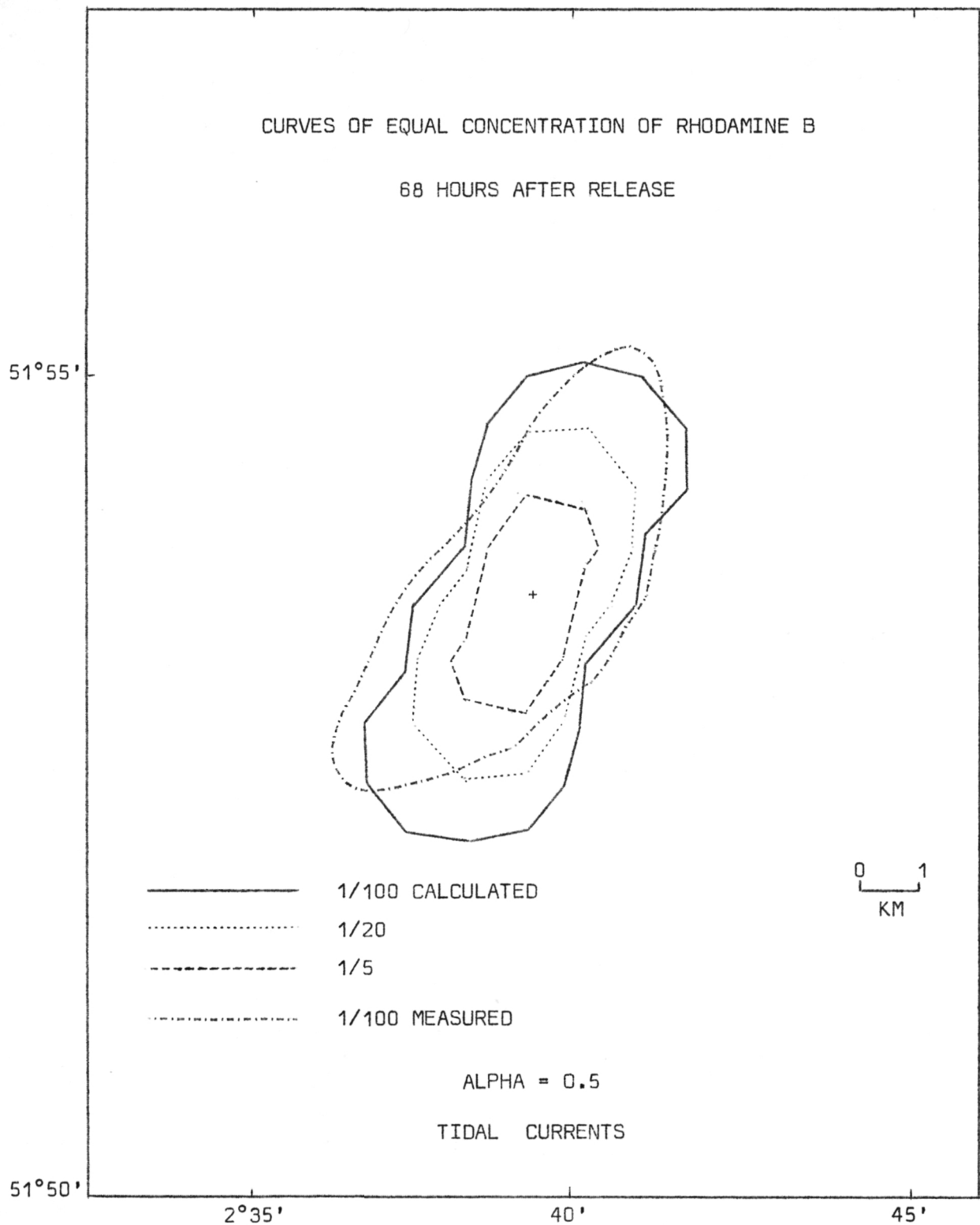
CURVES OF EQUAL CONCENTRATION OF RHODAMINE B

68 HOURS AFTER RELEASE

1/100 CALCULATED

1/20

1/5

1/100 MEASURED

ALPHA = 0.4

TIDAL CURRENTS

fig. 6.

CURVES OF EQUAL CONCENTRATION OF RHODAMINE B

68 HOURS AFTER RELEASE

fig. **7.**

CURVES OF EQUAL CONCENTRATION OF RHODAMINE B

68 HOURS AFTER RELEASE

——————— 1/100 CALCULATED

···················· 1/20

----------------- 1/5

—·—·—·—·—·—·— 1/100 MEASURED

ALPHA = 0.6

TIDAL CURRENTS

0   1
KM

51°55'

51°50'
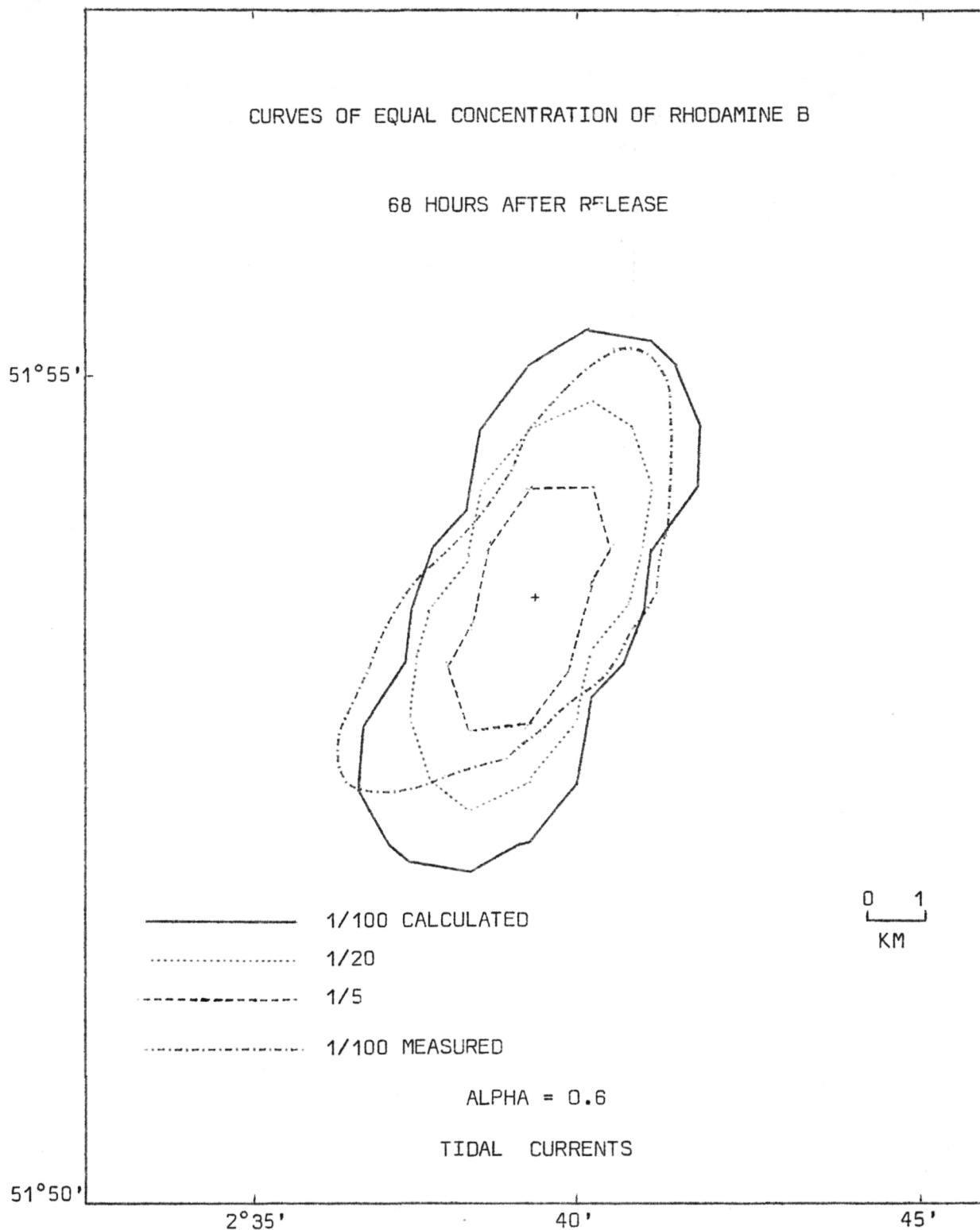
2°35'   40'   45'

fig. 8.

roughly a steady function of position). However the "eddy diffusivity" μ must than account for "tidal Reynolds stresses" and is presumably not isotropic as assumed for simplicity in (19) and (29).

It is thus better to take T smaller than the tidal periods in which case, $\mathbf{u}$ rotates (and H oscillates) with tides.

When the numerical integration is carried over several tidal periods, the net advection is therefore reduced to the effect of residual currents and the net diffusion effect is more important. This effect is however affected by the wind and sea conditions as the increase in vertical turbulence raises the average value of λ_3 and thus decreases α. (This was estimated to be the case in experiment B, for instance).

Figures 6, 7 and 8 show comparisons between concentrations of Rhodamine B calculated numerically with formula (29) and observed values after 68 hours in the case of experiment B.

Although the calculation relies on the estimation of meteorological conditions and tidal currents from insufficient chart data, a satisfactory agreement is found, suggesting that shear effect diffusion may be a cogent element in the dispersion of a pollutant in the Southern North Sea.

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