PROGRAMME NATIONAL SUR L'ENVIRONNEMENT PHYSIQUE ET BIOLOGIQUE

Pollution des Eaux

Projet Mer

Numerical solution of the complete shear-diffusion equation

by

Y. ADAM*

N 35

* Y. ADAM,
Institut de Mathématique,
Avenue des Tilleuls, 15
B-4000 LIEGE (Belgium)

1.- State of the art in shear diffusion computations

In the preceding reports (see the chapter : Numerical methods in *Modèle Mathématique – Rapport de synthèse I et II*) we have described the nature of the difficulties arizing from the typical form of the shear diffusion equation when it is to be solved by numerical methods, which are the only ones adapted to the complexity of the phenomena.

Let us remember the main problems :

- the differential operator modelling the shear effect diffusion is singular : this difficulty has been avoided to some extent by performing further KBM integration .
- in some range of the integration domain, *i.e.* in the neighbourhood of the blot (see Appendix A) the advection terms are one order of magnitude (at least) greater than the diffusion terms so that numerical methods adapted to the solution of the diffusion problem and not stable for any value of these advection terms are useless. In the first studies, the advection effect has been taken into account by assuming that the computation is run in a grid travelling with the center of the patch. It is evident that the pattern of the patch must be described in some way in order to predict the whole phenomenon.

Thus we selected several methods for computing the diffusion in a region of constant depth and uniform velocity field moving with the center of the patch. The A.D.I. methods (and particularly those of McKee) proved to be the only ones that combine accuracy, speed of computation and facility of programming. However in the North Sea, the spatial variation of depth and tidal currents is very large and cannot be neglected even in a small region (a few kilometers square); unfortunately, McKee's method is mainly designed for parabolic equations with constant coefficients, and it is very painful to adapt it to our particular problem. That is the reason we have investigated new methods. The idea has been found in a paper by Cha, Morton and Roberts (1972) and adapted to the shear diffusion equation. The essential feature of the latter method is an original way of dealing with the mixed derivative.

2.- Approximation of the mixed derivative

Let us write the expression of the gradient of any scalar in a direction β defined by the cosines (λ,μ)

(2.1) $$\beta.\nabla c = \lambda c_x + \mu c_y$$

(2.2) $$\beta.\nabla(\beta.\nabla c) = c_{\beta\beta} = 2\lambda\mu c_{xy} + \lambda^2 c_{xx} + \mu^2 c_{yy} .$$

We can thus compute c_{xy} from

(2.3) $$c_{xy} = \frac{1}{2\mu\lambda}(c_{\beta\beta} - \lambda^2 c_{xx} - \mu^2 c_{yy}) .$$

The operator

(2.4) $$L(c) = \alpha_1 c_{xx} + \alpha_2 c_{yy} + 2\alpha_3 c_{xy}$$

may now read

(2.5) $$L(c) = (\alpha_1 - \frac{\lambda}{\mu}\alpha_3)\ c_{xx} + (\alpha_2 - \frac{\mu}{\lambda}\alpha_3)\ c_{yy} + \frac{\alpha_3}{\lambda\mu} c_{\beta\beta}$$

where $c_{\beta\beta}$ is the second derivative in the β-direction. Now the mixed derivative has been replaced by a linear combination of second derivatives along the x, y and β axes.

The choice of β depends upon the requirement of positivity for given combination of α_1, α_2, α_3 as we shall see later.

As it is well known, the basic principle of A.D.I. methods is to discretize the equation

$$\frac{\partial c}{\partial t} = L(c)$$

in the following way :

- denoting by $c_{i,j}^{n+\frac{1}{2}}$ the first approximation of $c_{i,j}^{n+1}$ one computes (using boundary conditions $c_{i,j}^{n+\frac{1}{2}}$ by the implicit set of equations

(2.6) $$c_{i,j}^{n+\frac{1}{2}} - c_{i,j}^{n} = \Delta t\ (\alpha_1 - \frac{\lambda\alpha_3}{\mu})\ \delta_{xx} c_{i,j}^{n+\frac{1}{2}}$$

and then $c_{i,j}^{n+1}$ is computed at the second partial time step using

(2.7) $$c_{i,j}^{n+1} - c_{i,j}^{n} = \Delta t\ (\alpha_2 - \frac{\mu}{\lambda}\alpha_3)\ \delta_{yy} c_{i,j}^{n+1} + \frac{\Delta t}{\lambda\mu}\delta_{\beta\beta} c_{i,j}^{n+\frac{1}{2}}$$

where $\delta_{\beta\beta} c_{i,j}$ is the discrete expression for the second derivative in the direction β . It can be seen that the mixed derivative never appears in its usual form, therefore the computations are simpler.

3.- Optimum choice of the β - direction

As far as we have developped our new method, we have not defined which is the 'best' β - direction. First of all, let us remember that the indicatrix of the shear effect diffusion tensor

$$S = \begin{pmatrix} \alpha_1 & \alpha_3 \\ \alpha_3 & \alpha_2 \end{pmatrix}$$

is an ellipse

$$C^{nt} = \alpha_1 x^2 + \alpha_2 y^2 + 2\alpha_3 xy \ . \tag{3.1}$$

This can be reduced to a canonical equation in the coordinate system (ζ,η) with axes directed along the principal axes of the ellipse (fig.1).

$$C^{nt} = \lambda_1 \zeta^2 + \lambda_2 \eta^2 = A(\varepsilon\zeta^2 + \eta^2) \tag{3.2}$$

where λ_1 , λ_2 , $\lambda_1 < \lambda_2$ are the eigenvalues of the matrix

$$\begin{pmatrix} \alpha_1 & \alpha_3 \\ \alpha_3 & \alpha_2 \end{pmatrix}$$

and where φ and $\varphi + \frac{\pi}{2}$ are the directions of its orthogonal eigen-vectors.

One can easily realize that the diffusion in any direction is a combination of diffusions along the principal axes of the ellipse : from what it can be induced that a numerical solution of the equation might take this into account; this induction gives us an indication for the choice of the β - direction : it must be along one of the main axes, and better along the greater one, along the main direction of diffusion. Now we have a physical background for choosing the β - direction. Of course β may be the nearest possible direction to the principal axis

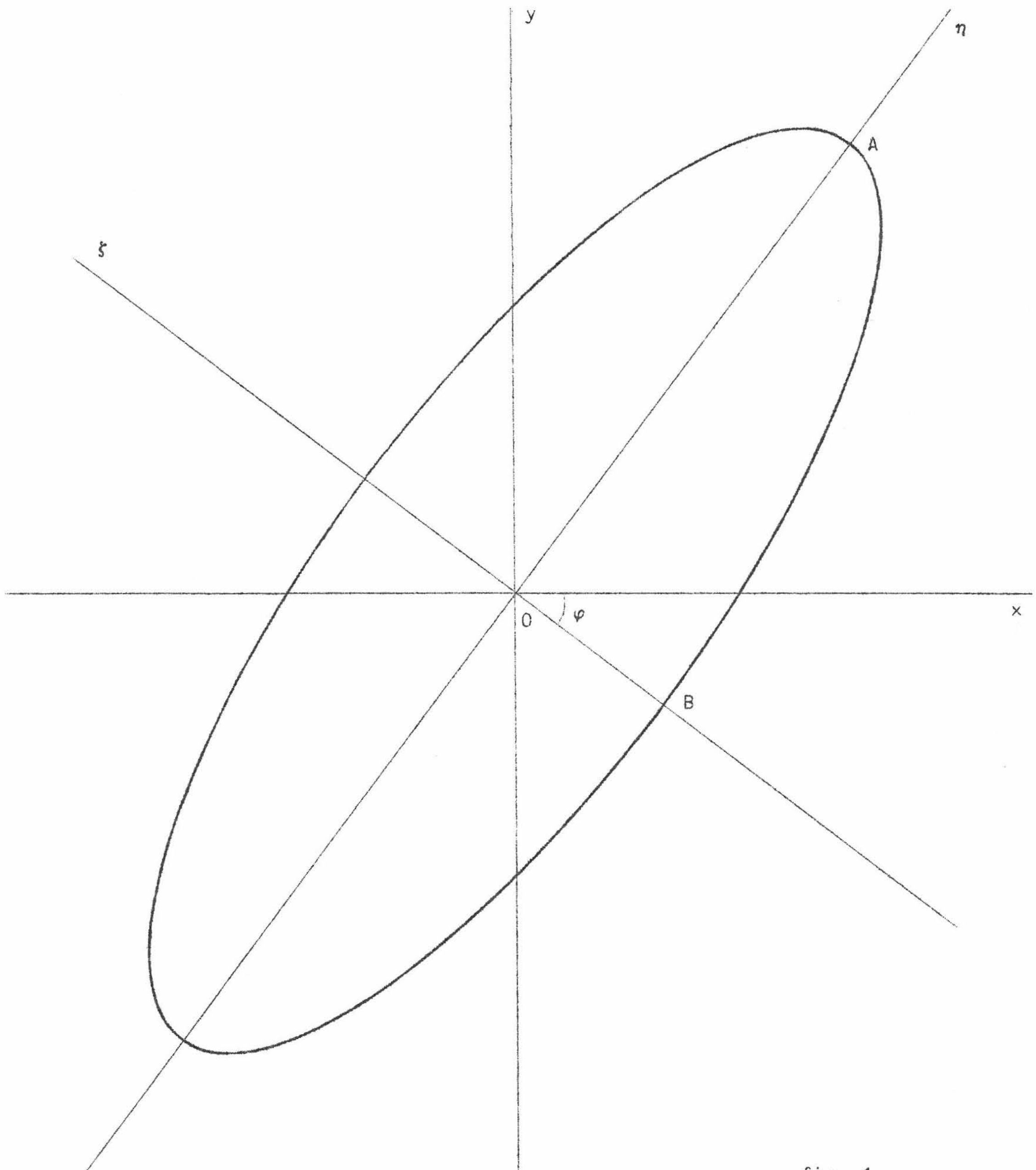

fig. 1.

Indicatrix of the tensor diffusivity

$OA = \lambda_1^{-\frac{1}{2}}$, $OB = \lambda_2^{-\frac{1}{2}}$

where λ_1 , λ_2 are the eigenvalues of the matrix associated to the tensor.

because, for practical reasons exposed later, the number of useful β directions is very limited.

4.- Numerical constraints on the β direction

The numerical solution of the diffusion equation discretized by the equations (2.6) and (2.7) is stable as far as the coefficients

$$(4.1) \qquad \alpha_1 - \frac{\lambda}{\mu}\alpha_3 \qquad \alpha_2 - \frac{\mu}{\lambda}\alpha_3$$

are positive.

This is a mathematical requirement that can be justified by the fact that, were one of the inequalities not satisfied, the 'corrected' diffusion coefficient would be negative, leading to negative diffusion.

Let us rewrite these conditions using

$$(4.2) \qquad \begin{aligned} \alpha_1 &= \lambda_1 \sin^2\zeta + \lambda_2 \cos^2\zeta \\ \alpha_2 &= \lambda_1 \cos^2\zeta + \lambda_2 \sin^2\zeta \\ \alpha_3 &= (\lambda_2 - \lambda_1) \sin\zeta \cos\zeta \qquad \text{and} \qquad \varepsilon = \frac{\lambda_1}{\lambda_2} \end{aligned}$$

(4.1) yields

$$(4.3) \qquad \begin{aligned} &\lambda_1 \sin^2\zeta + \lambda_2 \cos^2\zeta - \frac{\lambda}{\mu}(\lambda_2 - \lambda_1) \sin\zeta \cos\zeta > 0 \\ &\lambda_1 \sin^2\zeta + \lambda_2 \sin^2\zeta - \frac{\mu}{\lambda}(\lambda_2 - \lambda_1) \sin\zeta \cos\zeta > 0 \end{aligned}$$

or

$$(4.4) \qquad \begin{aligned} &\varepsilon \sin^2\zeta + \cos^2\zeta - \frac{\lambda}{\mu}(1 - \varepsilon) \sin\zeta \cos\zeta > 0 \\ &\sin^2\zeta + \varepsilon \cos^2\zeta - \frac{\mu}{\lambda}(1 - \varepsilon) \sin\zeta \cos\zeta > 0 \,. \end{aligned}$$

If

$$\frac{\mu}{\lambda} = \mathrm{tg}\,\varphi$$

i.e. if β has the direction of the great axis of the ellipse, both inequalities are always satisfied (they reduce to $\varepsilon > 0$); thus the coincidence of these important directions makes the solution well-behaved, as it had been guessed earlier from physical insight. If $\varphi = 0$ or $\varphi = \frac{\pi}{2}$, *i.e.* when the principal directions are the grid axes too, both inequalities hold.

For other values of $\frac{\mu}{\lambda}$, there may be points (ε, φ) such that *one* of the inequalities is not true.

Let us now refer to figure 2 showing a portion of the computational grid. If we want the computational scheme not to be too sophisticated, we may use only six discrete expressions for the derivative along the β axis $c_{\beta\beta}$.

$$(4.5) \quad \begin{cases} c_{\beta\beta} \sim \delta^0_{\beta\beta} c_{i,j} = \dfrac{1}{h_x^2 + h_y^2} (c_{i+1,j+1} + c_{i-1,j-1} - 2c_{i,j}) \\[2ex] c_{\beta\beta} \sim \delta^1_{\beta\beta} c_{i,j} = \dfrac{1}{2\, h_x^2 + h_y^2} (c_{i+2,j+1} + c_{i-2,j-1} - 2c_{i,j}) \\[2ex] c_{\beta\beta} \sim \delta^2_{\beta\beta} c_{i,j} = \dfrac{1}{h_x^2 + 2\, h_y^2} (c_{i+1,j+2} + c_{i-1,j-2} - 2c_{i,j}) \end{cases}$$

and the three symetric expressions, so that we have a limited set of six pairs (λ, μ) . We will choose one of the first three where $tg\ \varphi > 0$ and one among the others where $tg\ \varphi < 0$.

Let us assume $tg\ \varphi > 0$ and examine the case where λ and μ have respectively the values

$$\begin{cases} \lambda = h_x (h_x^2 + h_y^2)^{-\frac{1}{2}} \\[2ex] \mu = h_y (h_x^2 + h_y^2)^{-\frac{1}{2}} \end{cases}$$

or integer multiples of these expressions (because we must use for β one of the six directions defined on figure 2. Try to define the neighbourhood $V(\varepsilon, \varphi)$ such that the inequalities (4.4) hold. The limits of $V(\varepsilon, \varphi)$ are given by the equations :

$$(4.6) \quad \begin{cases} \varepsilon \sin^2\varphi + \cos^2\varphi = \dfrac{\lambda}{\mu} (1 - \varepsilon) \sin\varphi \cos\varphi \\[2ex] \sin^2\varphi + \varepsilon \cos^2\varphi = \dfrac{\mu}{\lambda} (1 - \varepsilon) \sin\varphi \cos\varphi \ . \end{cases}$$

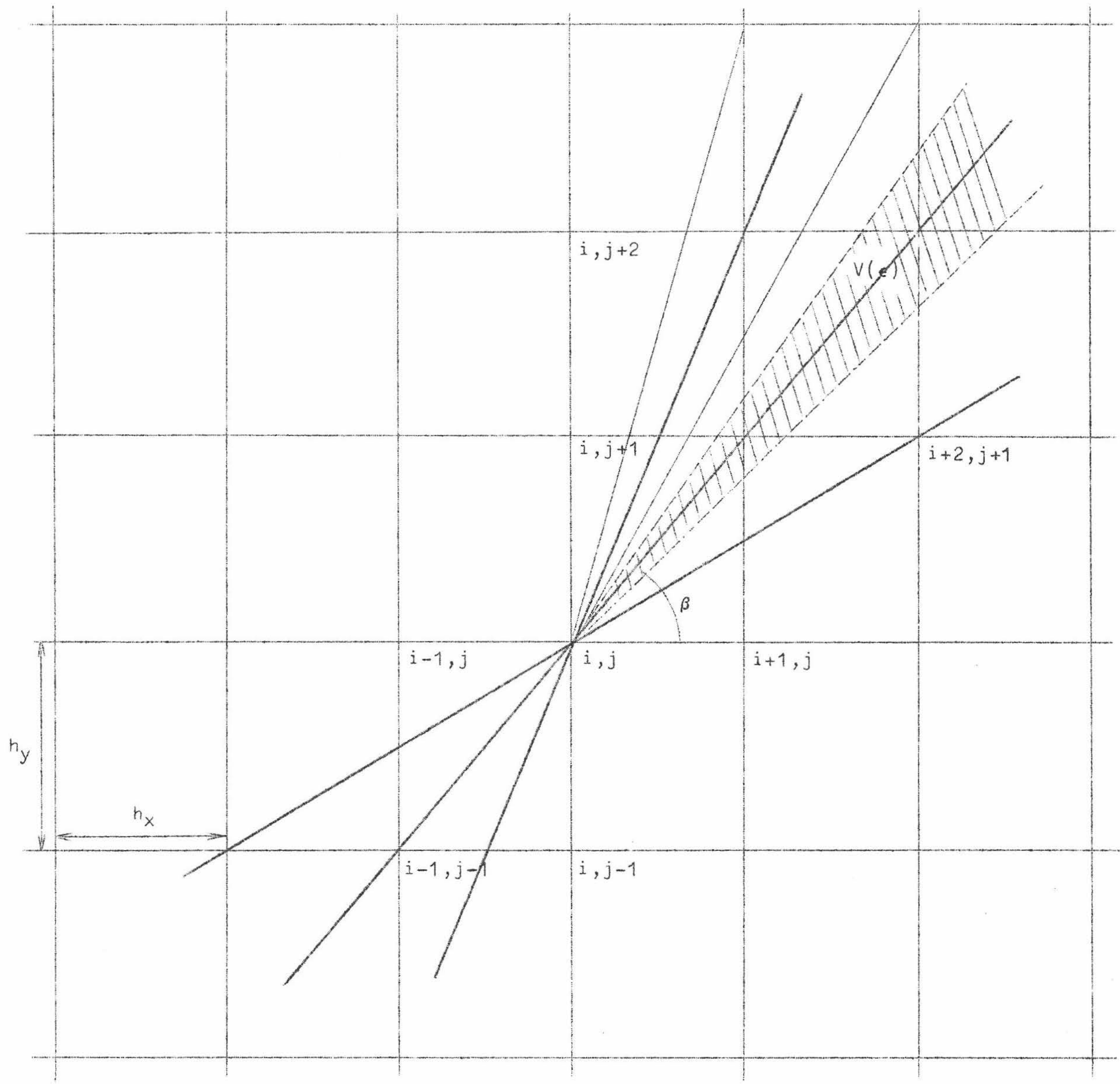

fig. 2.
Portion of the computational grid showing the useful β directions and rought representation of a neighbourhood $V(\epsilon,\varphi)$.

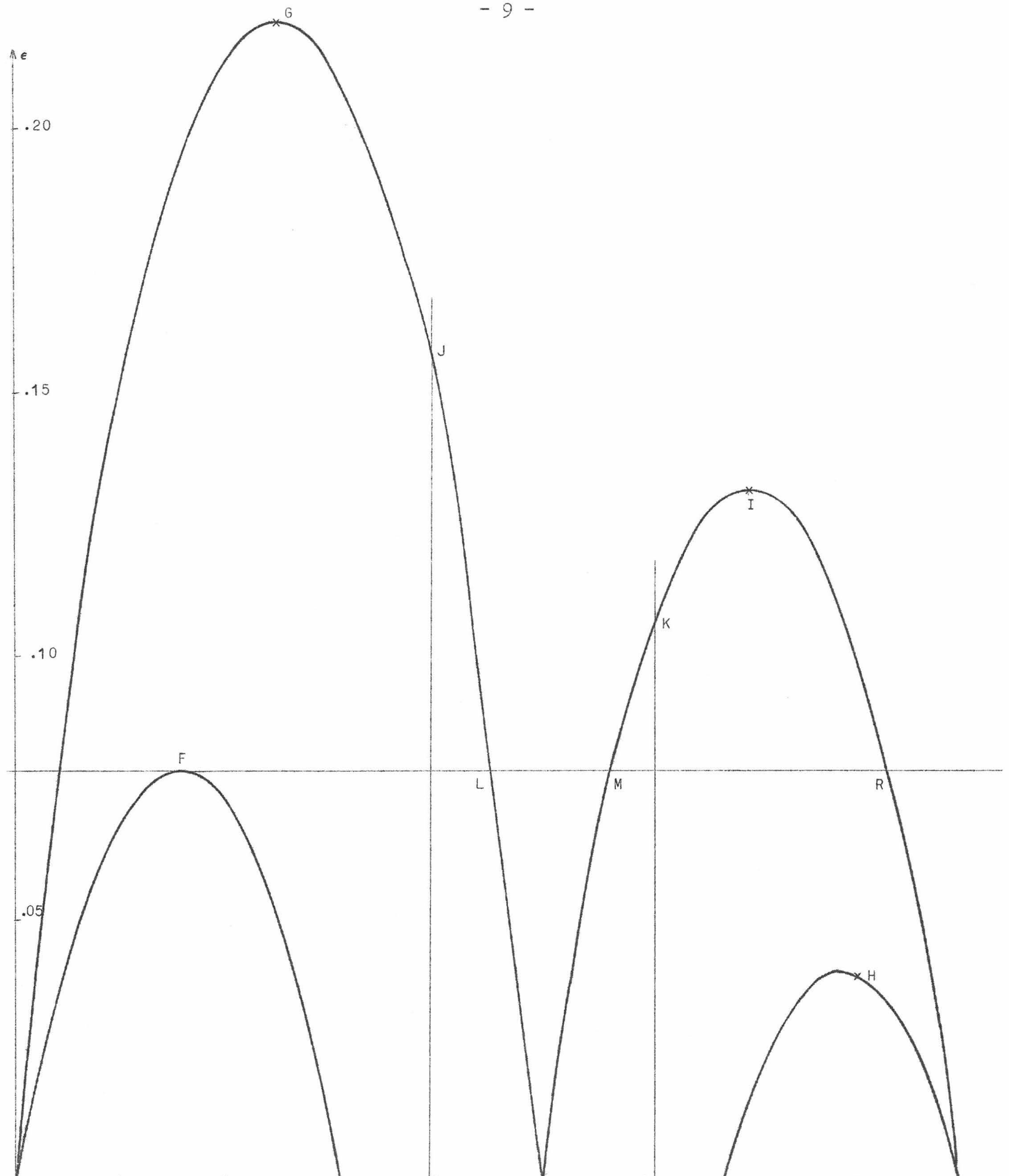
ε
.20
.15
.10
.05
G
J
I
K
F
L
M
R
H
A
10
20
30
B
N
40
φ_1
0
50
C
φ_0
P
60
Q
φ_2
D
70
80
90
E
φ

Figure 3 shows the functions $\varepsilon(\varphi)$ defining the convex region $V(\varepsilon,\varphi)$ (hachured region in fig. 2) for a ratio $\frac{\mu}{\lambda} \sim 1.2$ (effectively used for calculating the shear diffusion in the S.N.S.).

In the region CIE :

(4.7a) $$\varepsilon \leq \frac{\cos\varphi}{\sin\varphi} \; \frac{(\frac{\lambda}{\mu})\sin\varphi - \cos\varphi}{\sin\varphi + (\frac{\lambda}{\mu})\cos\varphi}$$

for $\varphi_0 < \varphi < 90°$; the diffusion coefficient $\alpha_1 - \frac{\lambda}{\mu}\alpha_3$ is < 0 .

In the region AGC :

(4.7b) $$\varepsilon \leq \frac{\sin\varphi}{\cos\varphi} \; \frac{(\frac{\mu}{\lambda})\cos\varphi - \sin\varphi}{\cos\varphi + (\frac{\mu}{\lambda})\sin\varphi}$$

for $0 < \varphi < \varphi_0$, the diffusion coefficient $\alpha_2 - \frac{\mu}{\lambda}\alpha_3$ is < 0 . ($\varphi_0 = \text{artg}\,\frac{\mu}{\lambda} \sim 50°$) .

It can easily be understood that when $\varepsilon > 0.22$, no problem arises, even though the only discrete approximation (4.5a) is used.

As our coefficients are such that ε is often < 0.22 , we have rather used the set of approximations

- (4.5a) when

$$\text{artg}\,\frac{2\,h_y}{3\,h_x} = \varphi_1 < \varphi < \varphi_2 = \text{artg}\,\frac{3\,h_y}{2\,h_x}$$

- (4.5b) when

$$0 < \varphi < \varphi_1$$

- (4.5c) when

$$\varphi_2 < \varphi < 90° .$$

The critical domain $[\complement V(\varepsilon,\varphi)]$ of the $\varepsilon - \varphi$ plane is then the region AFBNJCKIE and no negative coefficient ever appears when $\varepsilon > 0.16$.

A still wiser choice is

- (4.5a) when

$$46° < \varphi < 57°$$

- (4.5b) when

$$0 < \varphi < 46°$$

- (4.5c) when

$$57° < \varphi < 90° .$$

The critical domain is then AFBOLCMPDHE and negative diffusion can only occur when $\varepsilon < 0.08$ which is quite seldom.

5.- Extension of the method to the case of variable diffusion coefficients

The shear diffusion operator with variable coefficients reads

$$L(c) = (\alpha_1 c_x)_x + (\alpha_2 c_y)_y + (\alpha_3 c_y)_x + (\alpha_3 c_x)_y \tag{5.1}$$

$$\equiv \alpha_1 c_{xx} + \alpha_2 c_{yy} + 2\alpha_3 c_{xy} + \alpha_{2x} c_x + \alpha_{2y} c_y + \alpha_{3x} c_y + \alpha_{3y} c_x$$

The coefficients of second order derivatives remain unchanged and so remain the eigenvalues and eigenvectors of the matrix of the S tensor. The former discussion remains valid and the discretisation method may be extended to this particular problem. The discrete equations are now :

$$c^{n+\frac{1}{2}}_{i,j} - c^{n}_{i,j} = \frac{\Delta t}{h_x^2}\Big[\Big(\alpha_{1_{i+\frac{1}{2},j}} - \frac{\lambda}{\mu}\alpha_{3_{i+\frac{1}{2},j}}\Big)\Big(c^{n+\frac{1}{2}}_{i+1,j} - c^{n+\frac{1}{2}}_{i,j}\Big) - \Big(\alpha_{1_{i-\frac{1}{2},j}} - \frac{\lambda}{\mu}\alpha_{3_{i-\frac{1}{2},j}}\Big)\Big(c^{n+\frac{1}{2}}_{i,j} - c^{n+\frac{1}{2}}_{i-1,j}\Big)\Big]$$

(5.2)

$$c^{n+1}_{i,j} - c^{n}_{i,j} = \frac{\Delta t}{h_y^2}\Big[\Big(\alpha_{2_{i,j+\frac{1}{2}}} - \frac{\mu}{\lambda}\alpha_{3_{i,j+\frac{1}{2}}}\Big)\Big(c^{n+1}_{i,j+1} - c^{n}_{i,j}\Big) - \Big(\alpha_{2_{i,j-\frac{1}{2}}} - \frac{\mu}{\lambda}\alpha_{3_{i,j-\frac{1}{2}}}\Big)\Big(c^{n+1}_{i,j} - c^{n+1}_{i,j-1}\Big)\Big] + \Delta t\,\frac{\alpha_3}{\lambda\mu}\,\delta^0_{\beta\beta}\,c^{n+\frac{1}{2}}_{i,j}$$

where

1) $\delta^0_{\beta\beta} c^{n+\frac{1}{2}}_{i,j}$, $\frac{\lambda}{\mu}$, $\frac{\mu}{\lambda}$ may be changed with regards to the choice of the β axis;

2) $f_{i,j+\frac{1}{2}}$ is the value of f at a point between (i,j) and $(i\,,j+1)$. Generally, one approximates $f_{i,j+\frac{1}{2}}$ by $\frac{1}{2}(f_{i,j} + f_{i,j+1})$.

6.- Numerical treatment of the advection terms

Advection must be taken into account if we want to discribe completely the dispersion, particularly when the spatial variation is so large that the currents acting on the center of the pollutant patch are different from those acting on its edges. This happens in the S.N.S., mainly along the sand banks. It is thus a great interest to develop a numerical method to deal with advection terms.

The problem is rather difficult from a numerical viewpoint : we are treating a partial differential equation which is, theoretically, parabolic, but such that the first derivatives in it are (at least in some region) an order of magnitude greater than the diffusion terms, yielding a quasi-hyperbolic behaviour. This means that the discretization of these terms must be very carefully made, in order to avoid a too large unaccuracy (the error being greater than the diffusion terms) and instability (a crucial problem here, because we are trying to solve a quasi-hyperbolic problem by methods and with boundary conditions related to a parabolic system).

Fortunately, we have been able to adapt to our parabolic system a method of Russian meteorologists [Mardchuk (1970)] originally designed for hyperbolic systems, that is theoretically stable for any time step, centered in space and almost centered in time, mandatory requirement for a good accuracy.

7.- Principle of the numerical process

A great number of examples spread among the literature show some common features :

- a space-centered discretisation of first derivatives gives rise to parasite oscillations in the numerical solution;
- a backward or forward discretisation is very few accurate;
- stability requirements are very strong for fully explicit schemes.

In order to combine speed of computation and accuracy, one must find a scheme which is at least partly implicit and symetric. The following one is proposed (the example is given for the hyperbolic equation $c_t + uc_x + vc_y = 0$).

We will split the resolution into two parts along both axes and intc four steps :

First step :

$$c_{i,j}^{n+1/4} - c_{i,j}^{n} = \frac{\Delta t}{2h_x} u(c_{i+1,j}^{n+1/4} - c_{i,j}^{n+1/4}) = - \frac{\Delta t}{2h_x} u\, \delta_x^+ c_{i,j}^{n+1/4} \quad \text{for} \quad u < 0$$

(7.1)

$$= - \frac{\Delta t}{2h_x} u(c_{i,j}^{n+1/4} - c_{i-1,j}^{n+1/4}) = - \frac{\Delta t}{2h_x} u\, \delta_x^- c_{i,j}^{n+1/4} \quad \text{for} \quad u > 0$$

The various approximations of c_x depending on sign (u) are chosen such that the coefficient of $c_{i,j}^{n+1/4}$ in the implicit equations is always $1 + \frac{\Delta t}{h_x} |u|$.

As these approximations are only first order accurate, we perform a correction in the second step :

$$c_{i,j}^{n+\frac{1}{2}} - c_{i,j}^{n+1/4} = - \frac{\Delta t}{2h_x} u(c_{i,j}^{n+1/4} - c_{i+1,j}^{n+1/4})$$

(7.2)

$$= - \frac{\Delta t}{2h_x} u\, \delta_x^- c_{i,j}^{n+1/4} \quad \text{for} \quad u < 0$$

$$= - \frac{\Delta t}{2h_x} u\, \delta_x^+ c_{i,j}^{n+1/4} \quad \text{for} \quad u > 0 .$$

One computes that the correction yields second order accuracy for the space derivative.

The third and fourth steps are the respective equivalents of first and second ones, but in the orthogonal direction. It can be demonstrated that the complete scheme has second order accuracy in space, that it is stable for any value of $u \frac{\Delta t}{h_x}$ or $v \frac{\Delta t}{h_y}$. For our practical problem, the complete scheme is the following one.

$$c_{i,j}^{n+1/4} - c_{i,j}^{n} = - \frac{\Delta t\, u}{2h_x} \delta^+ c_{i,j}^{n+1/4} + \frac{\Delta t}{h_x^2} [(\alpha_1 - \frac{\lambda}{\mu} \alpha_3)_{i+\frac{1}{2},j} (c_{i+1,j}^{n+1/4} - c_{i,j}^{n+1/4})$$

(7.3a,b)

$$- (\alpha_1 - \frac{\lambda}{\mu} \alpha_3)_{i-\frac{1}{2},j} (c_{i,j}^{n+1/4} - c_{i-1,j}^{n+1/4})]$$

$$c_{i,j}^{n+\frac{1}{2}} - c_{i,j}^{n+1/4} = - \frac{\Delta t\, u}{2h_x} \delta^- c_{i,j}^{n+1/4}$$

$$c_{i,j}^{n+3/4} - c_{i,j}^{n+3/4} = - \frac{\Delta t\ v}{2h_y} \delta^+ c_{i,j}^{n+3/4} + \frac{\Delta t}{h_y^2} [(\alpha_2 - \frac{\mu}{\lambda} \alpha_3)_{i,j+\frac{1}{2}} (c_{i,j+1}^{n+3/4} - c_{i,j}^{n+3/4})$$

$$- (\alpha_2 - \frac{\mu}{\lambda} \alpha_3)_{i,j-\frac{1}{2}} (c_{i,j}^{n+3/4} - c_{i,j-1}^{n+3/4})]$$

$$+ \frac{\Delta t\ \alpha_3}{\lambda \mu} \delta_{\beta\beta}^0 c_{i,j}^{n+\frac{1}{2}} \qquad (7.3c,d)$$

$$c_{i,j}^{n+1} - c_{i,j}^{n+3/4} = - \frac{\Delta t\ v}{2h_y} \delta^- c_{i,j}^{n+3/4} .$$

The discrete equations are written in the case where $u < 0$, $v < 0$, $\varphi_1 < \varphi < \varphi_2$. The other approximations are very similar.

8.- Results

The computations have been run with that scheme on a grid covering the eastern part of the S.N.S. The spatial steps used were $h_x \sim 3500$ m and $h_y \sim 4200$ m . Several points of the C.I.P.S. grid lie on the computational grid. The scheme proved to be stable for time steps up to 1 hour (at least : we have not tried to use a larger Δt , because the scheme would not have represented correctly the variation of the tidal currents). The accuracy was however poor and a lot of negative concentrations appeared, some of which were very important ($\frac{1}{10}$ of the maximum concentration at the same time) in a region just around the blot.

This lack of accuracy may be due to several causes :

1) the velocity field has been extrapolated from tidal data which are not fully fiable. The elevation field has been computed from the former one in a very crude way and nothing insures that it is right; as a consequence, the whole set of basic parameters is very poorly represented. This seems to us to be the main limitation.

2) the apparition of large negative concentrations at the boundary of the patch, where steep gradients exist, seems to imply that these gradients are poorly represented by the large spatial steps. As the diffusion is rather slow, the steepness does not vanish rapidly and the occurence of negative concentrations is observed even after several tide periods.

In spite of all these problems, we have been able to draw the pattern and deformation of a pollutant patch (Fig. 4a,b,c,d,e,f). On each figure you can see the position of the C.I.P.S. points (5 6 7 8 12 13 14 15) the rough shape of the Belgian coast, and three isopleths of concentration, in the sequential order : full line, dotted line, hyphen line. The hyphen line of a former figure and the full line of the following one represent the same isopleth, so that one can visualize the movement of the patch. The time interval between two isopleths is one hour. A complete tidal period is thus represented.

9.- Conclusions

One could object that the spatial steps used are too large to yield a correct representation of the steep gradient appearing at the boundary of the pollutant patch. Much smaller steps would thus be useful, but this requirement is not easy to reach. In order to prove this let us assume steps ten times smaller. It is a very maximum size as we shall see in the following example.

A blot of pollutant (defined by the $\frac{1}{100}$ of maximum concentration) issued from an instantaneous release, has, after one tidal period in the usual conditions of the S.N.S. the shape of an ellipse of which the smaller axis is 600 m and the larger one 1900 m long. As the gradients are very steep, at least 3 points of the grid are needed to represent the function $c(x,y,t)$; thus $h_x \sim h_y \sim 300$ m . To cover the whole region, an array of about 150000 points is necessary (600 K of core memory) and the computation time is at least 100 times the present one (3 minutes for a tidal period).

It might be possible to use a less extended grid (covering say a 15 km × 15 km region).

But when advection is taken into account, a simple computation shows that a residual current of 0.1 m/s transports the center of the patch out of the grid after 3 or 4 tidal periods. At some points, the residual current is 0.4 m/s , and the tidal currents are still stronger!

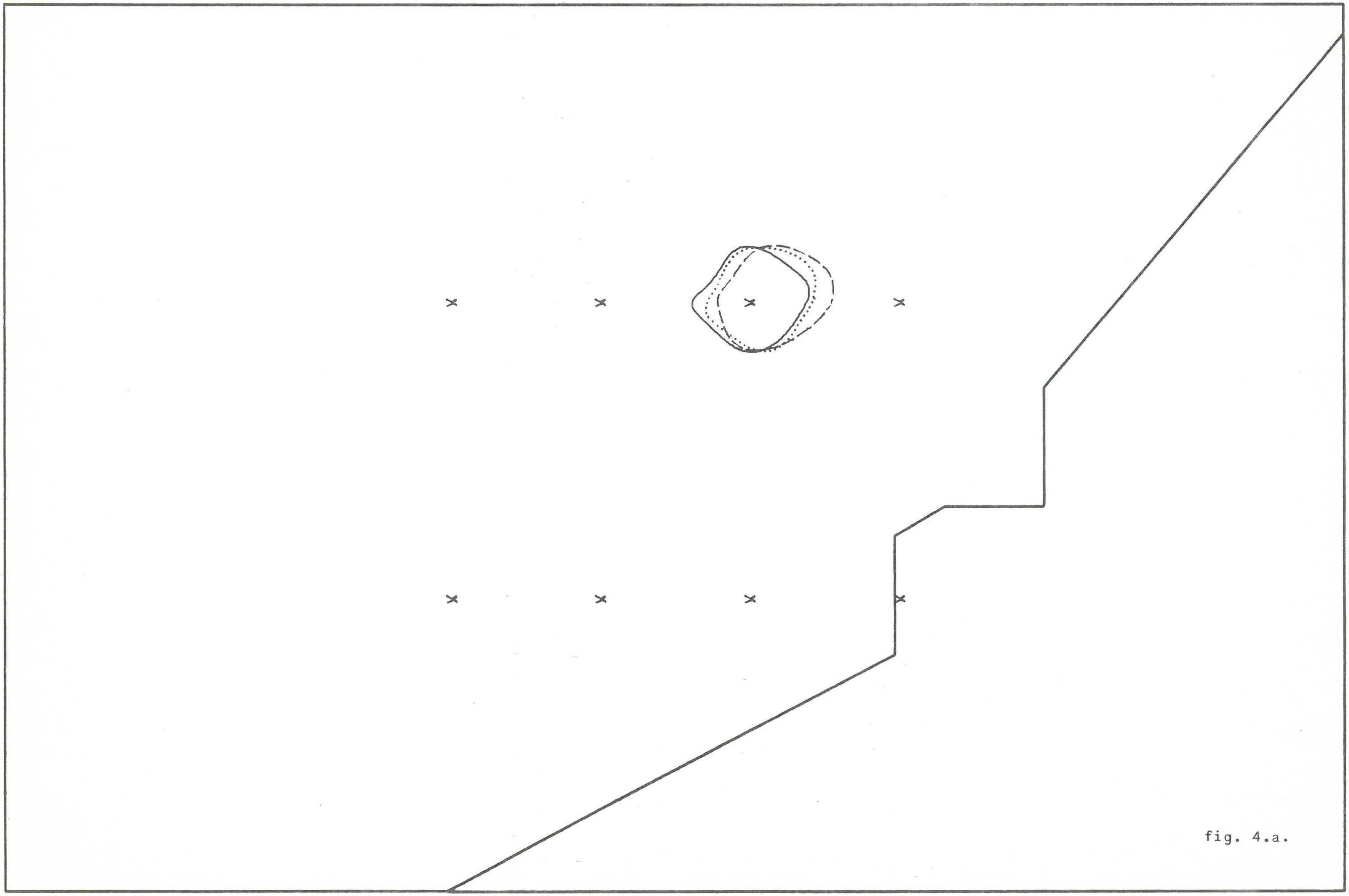

fig. 4.a.

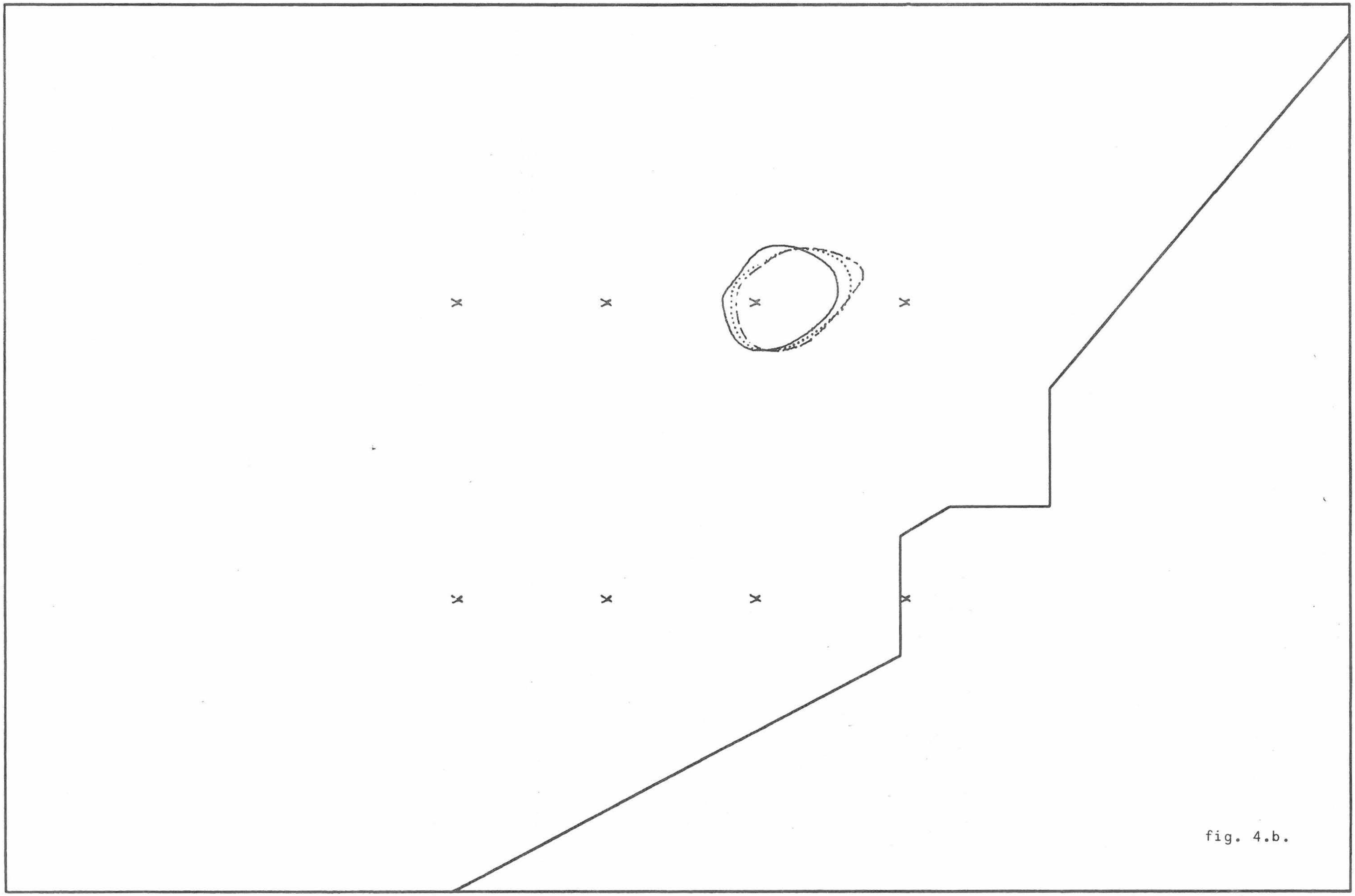

fig. 4.b.

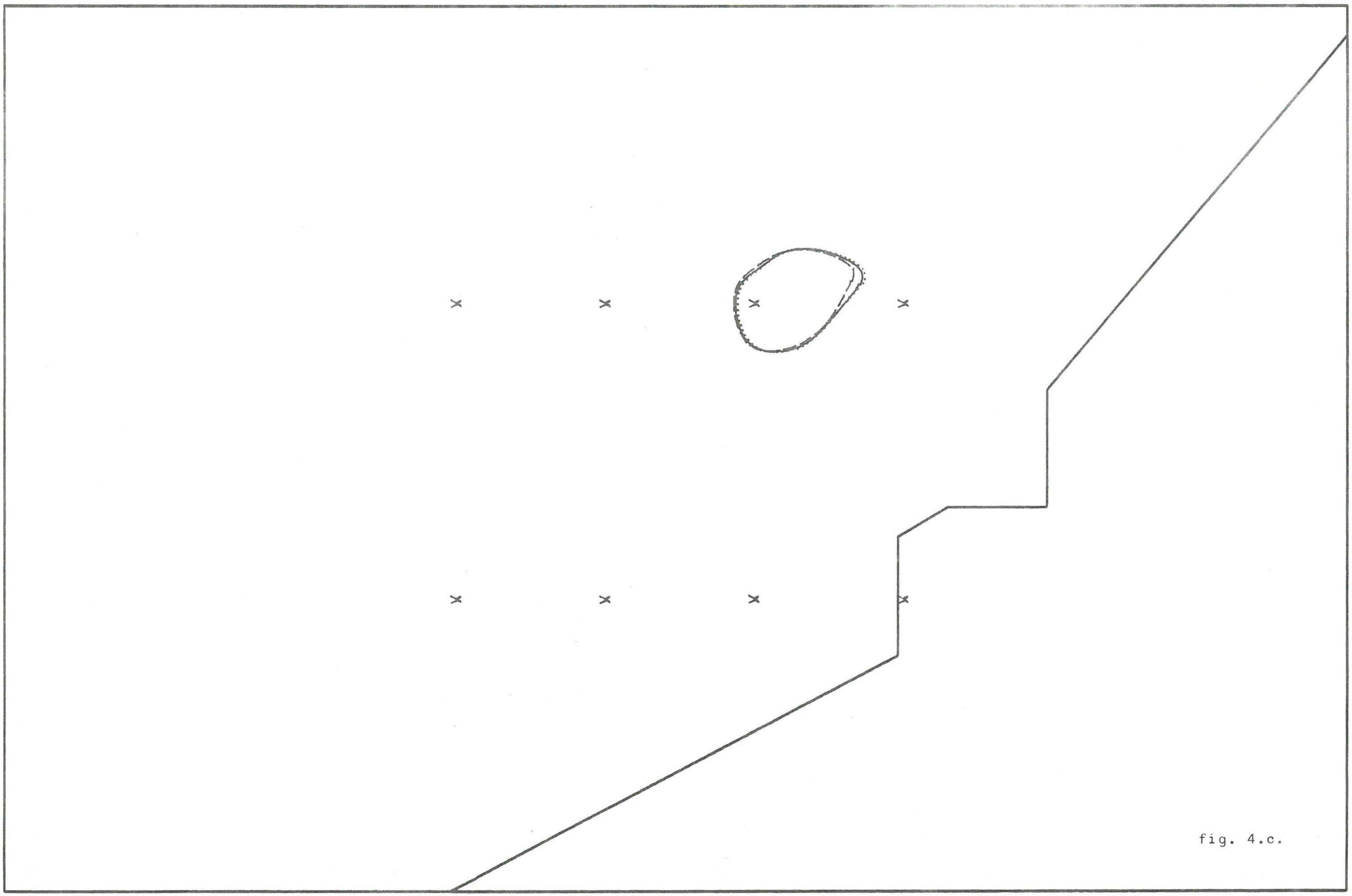

fig. 4.c.

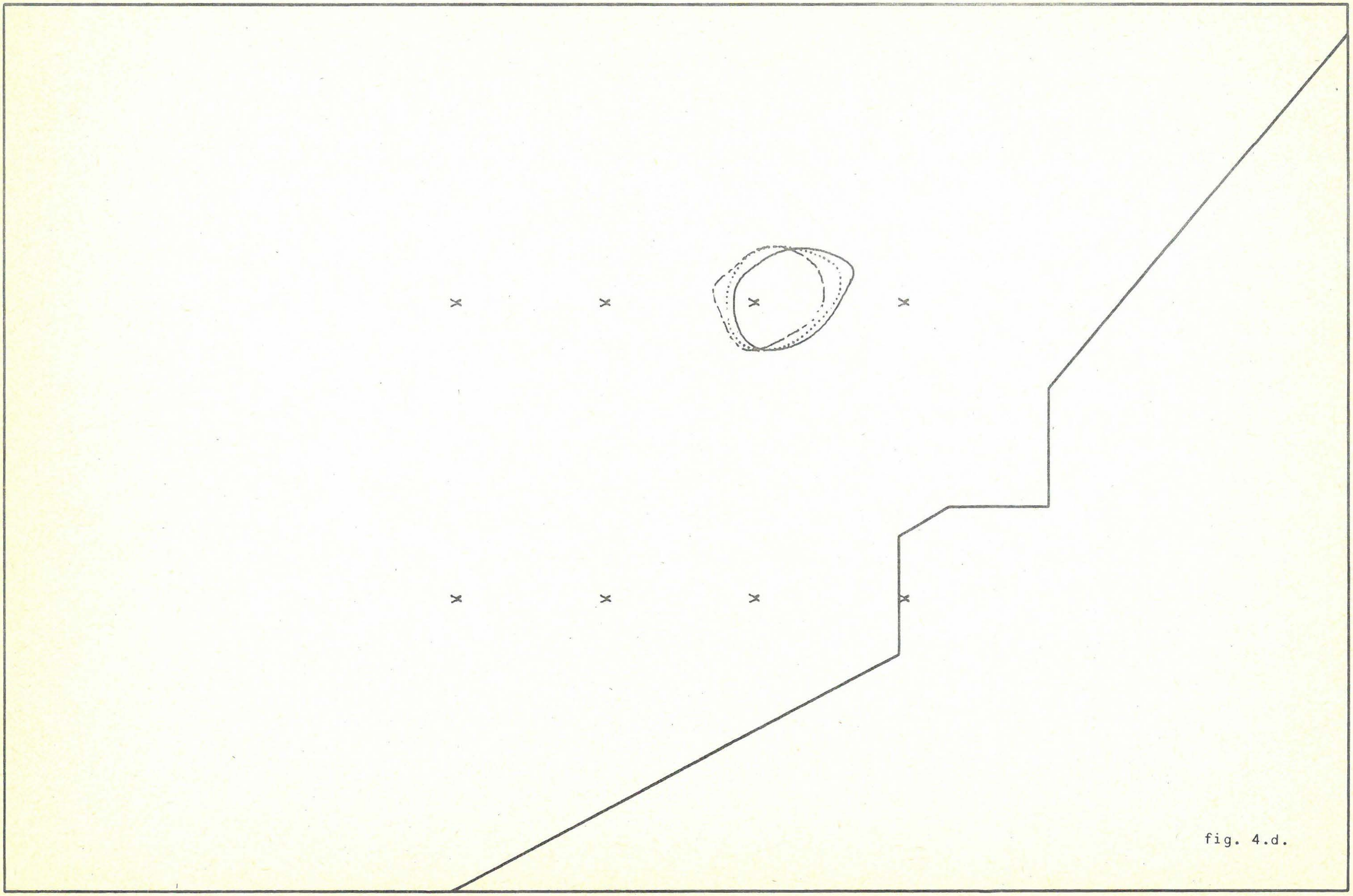

fig. 4.d.

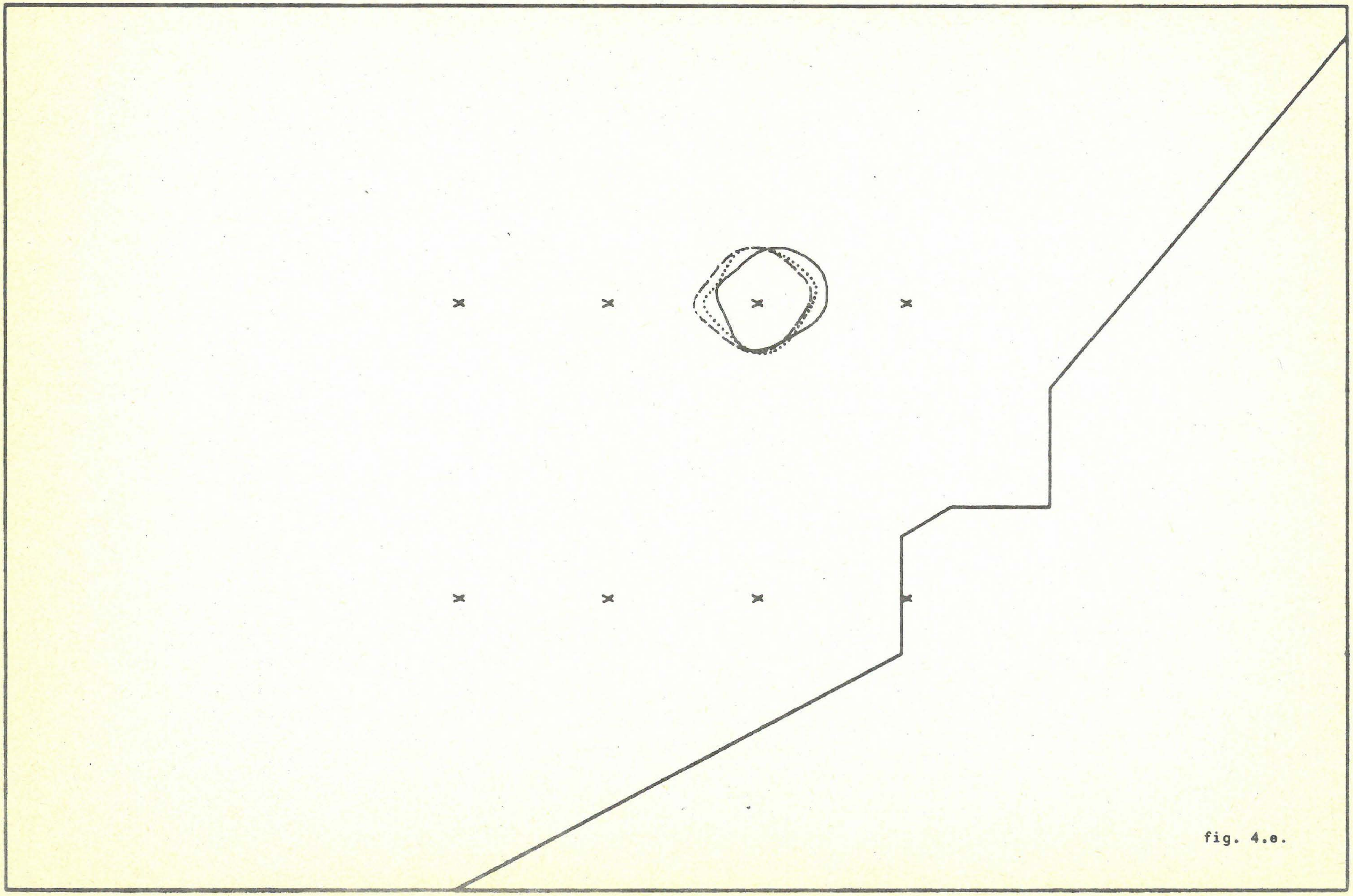

fig. 4.e.

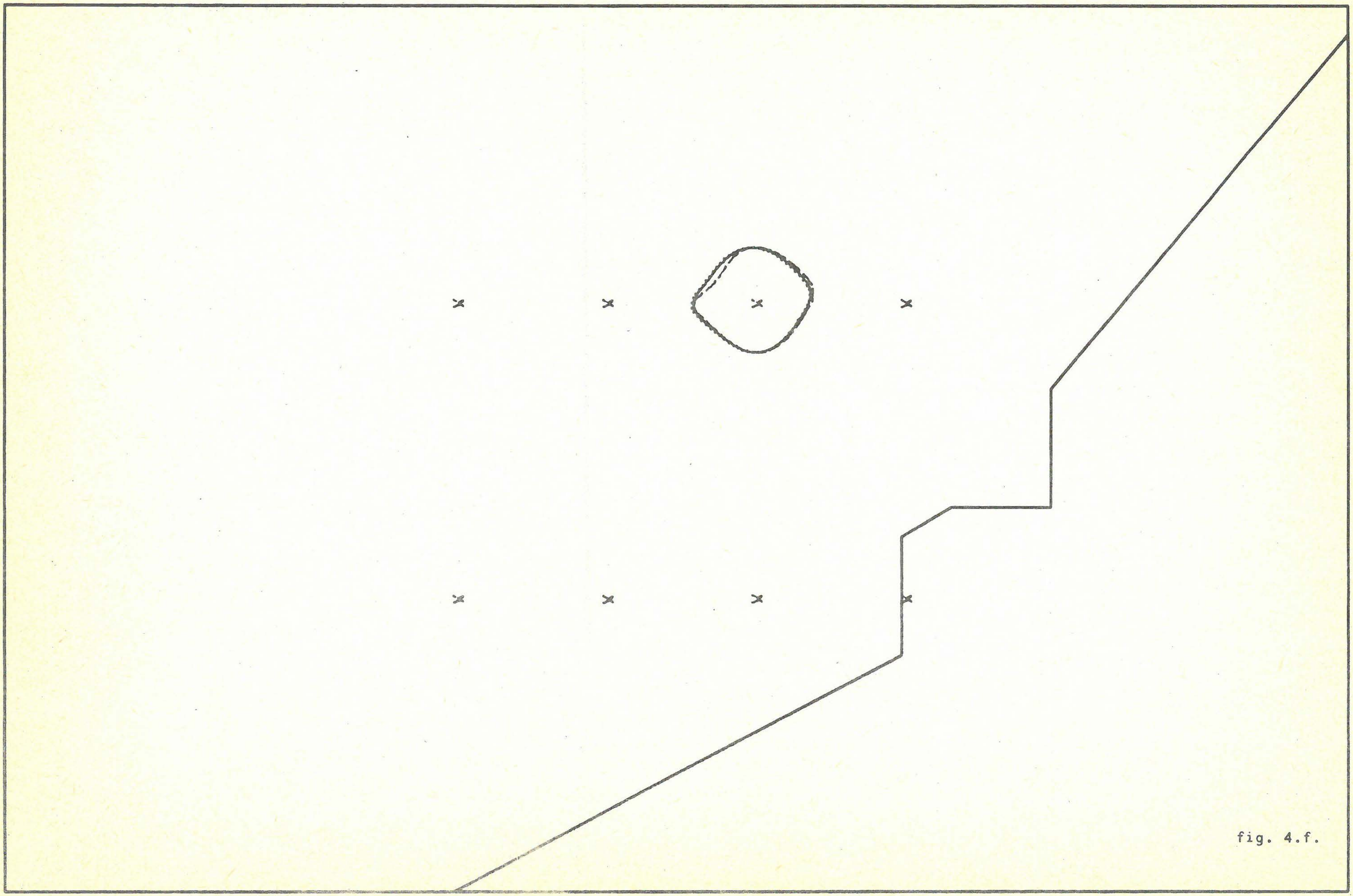

fig. 4.f.

The solution could be the following one : run the computations in a travelling grid following the displacement of the center of the patch. This would together reduce the memory requirements and the computation time by switching points which are obviously too far from the center to influence its pattern. However, a lot of difficulties arise, from numerical reasons (moving boundaries and boundary conditions) and from the programming point of view. Altogether, these difficulties need time before being solved.

Appendix A

Let us show which kind of difficulties arise in treating the diffusion problem with advection.

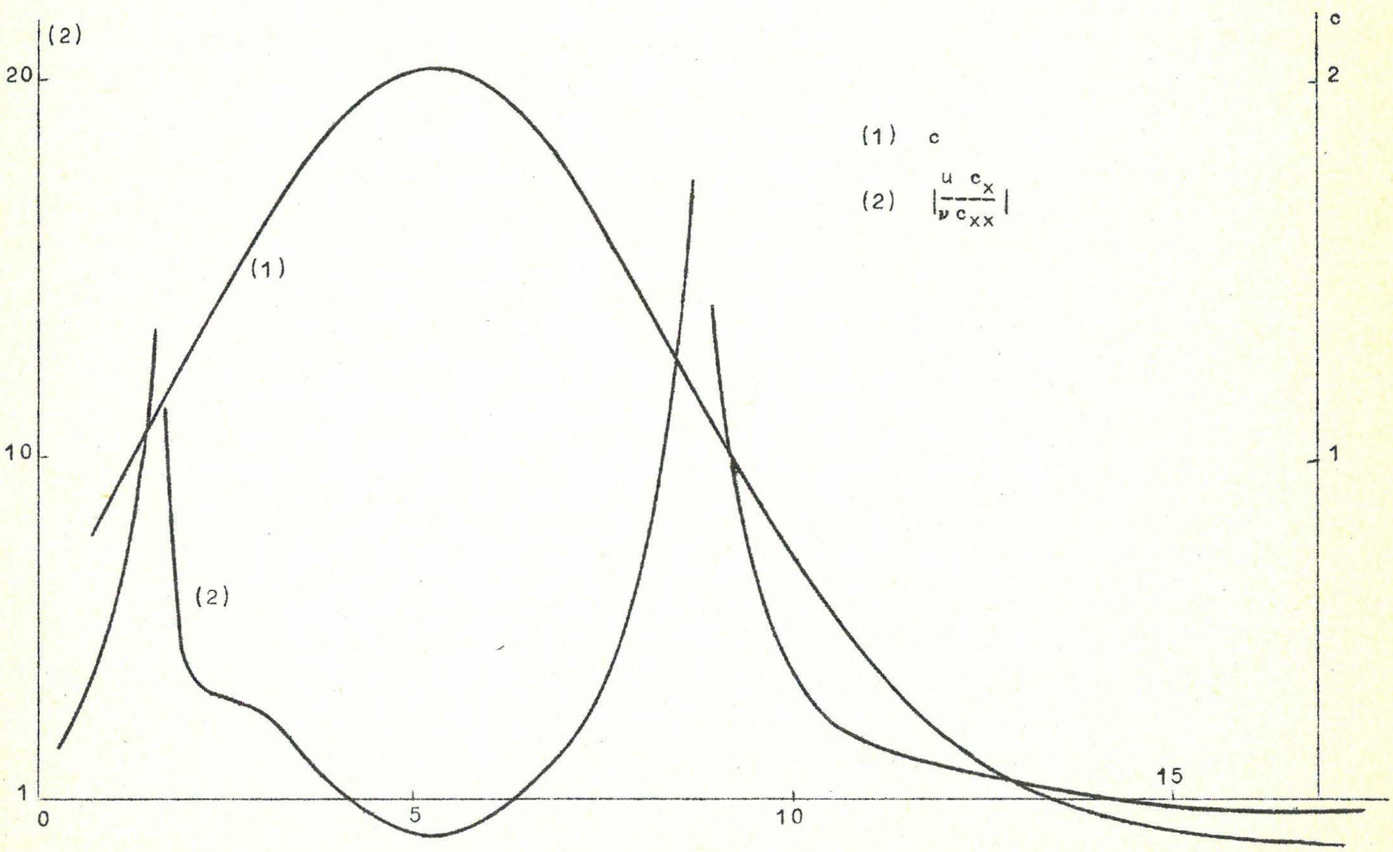

fig. 5.

Curve (1) is the solution of the equation $u_t + u\ c_x = \nu\ c_{xx}$ at one particular time.
Curve (2) is the ratio $\left|\frac{u\ c_x}{\nu\ c_{xx}}\right|$ at the same time.

For the simplicity of reasoning, let us consider the one dimensional equation :

$$c_t + uc_x = \nu\, c_{xx} \; .$$

For an initial $\delta(0)$ distribution, its solution is

$$c(x,t) = \frac{A}{t} \exp\left[\frac{x^2}{2\,\nu t} + \frac{u}{2\,\nu}\left(x - \frac{ut}{2}\right)\right] \; .$$

Figure 5 shows the ratio $\left|\frac{uc_x}{\nu c_{xx}}\right|$ for typical values of u and ν in the Southern North Sea. One can easily see that in the region of steep gradient, the ratio is of order $O(10)$, which means that in this region, the equation is of a hyperbolic type.

References

ADAM, Y. and RUNFOLA, Y., (1972). *Numerical methods*, in *Modèle mathématique, Rapport de Synthèse I, II*, Programme national sur l'environnement physique et biologique, Pollution des Eaux, Projet mer.

CHU, C.K., MORTON, K.W. and ROBERTS, K.V., (1972). *Numerical studies of the heat equation with highly anisotropic tensor conductivity*, in *Proceedings of the Third International Conference on Numerical Methods in Fluid Mechanics*.

McKEE, S. and MITCHELL, A.R., (1970). Alternating direction methods for parabolic equations in two space dimensions with a mixed derivative, *The Computer Journal*, *13*, 1.

MARDCHUK, G.I., (1970). *Méthodes numériques pour la prévision du temps*, Paris, Interscience, Armand Colin.