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Projet Mer

EROSION OF COHESIVE SOILS

by

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Synopsis

The erosion of visco-plastic cohesive soils under the action of a turbulent fluid flow is studied. The equations describing a sediment bed are discussed. To solve them the density distribution in the bed must be known at some initial time. Experimental field studies and results from the ultracentrifuge provide a clear picture of the density distribution in sediment layers.

The boundary condition between the bed-fluid interface is obtained from experimental observations. It is inferred that the instantaneous physico-chemical composition at the top of the bed is uniquely related to the bed shear stress during erosion.

The solution for the stationary erosion flux under a constant shear stress is obtained. The erosion flux depends on the properties of the bed and the shear stress induced by the turbulent fluid flow.

The theory is shown to agree very well with the experiments performed by Partheniades (1965).

1. Introduction

Sedimentation means the transport of matter through a solution due to an external force field. Thereby sediment layers are built up at the bottom region of rivers, estuaries and oceans. Depending on the friction velocity or shear stress acting at the layer, deposition or erosion occurs.

Knowledge of the erosion flux under varying flow conditions is important in such diverse areas as estuary maintenance, channel design and pollution dispersal in rivers and oceans.

The results provided by mathematical marine models are only truthworthy when correct boundary imputs are known, Nihoul (1973), i.e. when the erosion flux is related to the fluid flow.

Many investigations have been carried out on the transport of sand beds (which is not a pollutant) but relatively little has been done on muds. Particularly in pollution studies it is mandatory to have information on the settling and scouring rates of fine grained particles, McCave (1972).

In this paper we concentrate on the physical behaviour of the visco-plastic sediment layer and derive the expression for the erosion flux by considering the action of the turbulent flow.

In section 2 , the mass conservation law for a sediment bed is formulated. In section 3, the boundary condition at the bed-fluid interface is discussed ; use is made of the experimental data of Migniot (1968). The density distribution is discussed in section 4 and the interface mass flux continuity in section 5. In section 6 , the analytic solution for the stationary erosion flux is obtained which for a fine grained sediment can be simplified (section 7). In section 8, the theory is shown to describe very accurately the experimental studies performed by Partheniades (1965).

2. Mass conservation law for the sediment bed

Consider a dynamically smooth and isotropic viscoplastic sediment layer (sea bed) in a gravitational field acted on by a two dimensional turbulent fluid flow.

For simplicity we consider a two component non charged chemically non reacting system, for example water and a single component fine grained sediment as clay.

The solid particle or sediment mass flux $\underline{J}^s$ with respect to the barycentric velocity $\underline{v}$ of the bed is defined by

$$\underline{J}^s = \rho^s(\underline{v}^s - \underline{v}) \tag{1}$$

where ρ^s and $\underline{v}^s$ are the solid particle density and velocity.
The mass conservation equation for the sediment component reads

$$\frac{\partial \rho^s}{\partial t} = - \nabla \, . \, (\rho^s \, \underline{v}^s) \tag{2}$$

Substituting of (1) into (2) gives

$$\frac{\partial \rho^s}{\partial t} = - \nabla \, . \, \underline{J}^s - \nabla \, . \, (\rho^s \, \underline{v}) \tag{3}$$

We shall neglect temperature gradients in the sediment layer.
Further we consider throughout the layer constant diffusion, D^s , and sedimentation, S^s , coefficients. The sediment mass flux with respect to the barycentric velocity is in this case given by :

$$\underline{J}^s = - D^s \, \nabla \, \rho^s + S^s \, \rho^s \, \underline{g} \tag{4}$$

Substituting (4) in (3) results in

$$\frac{\partial \rho^s}{\partial t} = D^s \, \nabla^2 \, \rho^s - S^s \, \underline{g}.\nabla \, \rho^s - \nabla \, . \, (\rho^s \, \underline{v}) \tag{5}$$

These equations are formally the same as for a fluid. However, the sediment layer differs from a fluid - the latter cannot by definition support a shear stress at equilibrium - in that it shows viscoplastic behaviour, i.e., possesses a yield stress. That is to say, under an applied shear stress less or equal to the yield shear stress, it behaves like an elastic solid while above this stress it shows a rate of deformation which is a function of the difference between the applied and yield stress. The yield stress is a function of the solid component density which varies through the depth of the sediment layer.

Introduce a coordinate system wherein the z axis points downwards normal to the interface between the fluid and flat sediment layer. We consider a sediment layer in which the solid particle density depends only on z and t . The fluid flow above it is turbulent and two dimensional. Apart from a hydrostatic pressure the fluid exerts then only a shear stress to the top of the sea or river bed.

Consequently plastic shear deformation in the sediment layer is, if it occurs, parallel to the x-y plane. A reasonable approximation inside the layer is then to assume that the barycentric velocity component in the z direction is negligible small while its other two components are functions of z and t only. Under these restrictions (5) reduces to

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial z^2} - S g_z \frac{\partial \rho}{\partial z} \tag{6}$$

wherein we have omitted the superscript s referring to the solid contamitant.

The z component of (4) is

$$J_z = - D \frac{\partial \rho}{\partial z} + S \rho g_z \tag{7}$$

3. The boundary condition at the bed-fluid interface

Accordingly to Migniot (1968) the shear stress whereby erosion finds place is uniquely related to the yield stress. Tentatively we shall assume this to be true. Thus for a given sediment layer, with solid particle density ρ_c at the top, erosion finds place when the shear stress exerted on it by the fluid reaches a certain critical value.

To make this statement explicit let us reproduce the experimental results found by Migniot.

The yield shear stress τ_y is, for a great variety of muds, experimentally found to be related to the solid density ρ by

$$\tau_y = n\ \rho^m \qquad (8)$$

where n and m are constants depending on the soil. When τ_y is expressed in N/m^2 and ρ in g/ℓ Migniot found m to be close to 5 for all cohesive soils examined while n varies from 10^{-12} to 10^{-5}, depending on the particular sediment.

The yield stress at the top of the sediment layer, where $\rho = \rho_c$, is thus

$$\tau_y = n\ \rho_c^m \qquad (9)$$

Further it is found experimentally that the critical friction velocity U_* acting at the sediment layer, whereby erosion finds place is related to the yield stress by

$$\left.\begin{array}{ll} U_{*c}(cm/s) = \tau_y^{1/4}\ (dynes/cm^2) & \text{for } \tau_y \leq 15\ dynes/cm^2) \\ U_{*c}(cm/s) = 0.5\ \tau_y^{1/2}\ (dynes/cm^2) & \text{for } \tau_y \geq 15\ dynes/cm^2) \end{array}\right\} \qquad (10)$$

In M.K.S. units, one has :

$$\left.\begin{array}{ll} U_{*c}(m/s) = 0{,}01778\ \tau_y^{1/4}(N/m^2) & \text{for } \tau_y \leq 1{,}5\ N/m^2 \\ U_{*c}(m/s) = 0{,}016\ \tau_y^{1/2}\ (N/m^2) & \text{for } \tau_y \geq 1{,}5\ N/m^2 \end{array}\right\} \qquad (10.a)$$

Figure 1 taken from Migniot (1968) shows how the relations (10.a) compare with the experimental points obtained for a number of sediments.

There is a remark in place here. We infer from the experiments performed by Migniot that τ_y appearing in (10) is the average shear yield stress of the bed in the upper region (say 1 to 10 cm) of the layer. The contact with the fluid will lower the solid particle density and thus by (4) also the yield stress at the top of the sediment layer by a factor G , which we expect to have a value between 1 and 5 .

Taking this into account we obtain from (8) and (10.a)

$$U_c^*(m/s) = 0{,}0178 \quad G^{m/4} \tau_y^{1/4} (N/m^2) \text{ for } \tau_y \leq \frac{1.5}{G^m} N/m^2$$

$$U_c^*(m/s) = 0{,}016 \; G^{m/2} \tau_y^{1/2} (N/m^2) \quad \text{for } \tau_y \geq \frac{1.5}{G^m} N/m^2 \qquad (10.b)$$

where τ_y is now the yield shear stress at the top of the bed. It is clear from (10.a) and (10.b) that Migniot's figure 1 describes correctly the relation (10.b) when the absciss in that figure is replaced by $G^m \tau_y$.

The relations (10.b) may be written as :

$$U_c^* = p_i \; \tau_y^{q_i} \qquad (i = 1,2) \qquad (11)$$

where i = 1 in region 1, defined for $\tau_y \leq \frac{1.5}{G^m} N/m^2$, and i = 2 in region 2, defined for $\tau_y \geq \frac{1.5}{G^m} N/m^2$.

The critical friction velocity is related to the shear stress exerted by the fluid whereby erosion finds place by

$$U_{*c} = \sqrt{\frac{\tau}{\rho_v}} \qquad (12)$$

wherein ρ_v is the fluid density in the viscous sublayer.

Combining (9), (11) and (12) gives the relation between the shear stress whereby erosion finds place and the instantaneous value of the solid particle density at the top of the bed , ρ_c , as

$$\left.\begin{array}{l} \tau = E_i \; \rho_c^{B_i} \qquad (i = 1,2) \\ \text{wherein} \\ E_i = \rho_v \; p_i^2 \; n^{2q_i} \qquad B_i = 2 \; mq_i \end{array}\right\} \qquad (13)$$

The relations (10) and hence (13) have been determined under quasi-stationary experiments and may consequently not apply to rapidly varying bed shear stress.

Partheniades (1965) has criticized the existence of a relation between a critical friction velocity or shear stress whereby erosion occurs and the yield stress at the top of the bed.

From experiment he found that dissolving iron oxides into a sediment layer does not noticeably change the macroscopic shear stress but changes the bed stress whereby erosion occurs. In what follows it will become clear that it is only essential that a relation of the type (13) exists while a relation as (11) need not be true. That is to say, we demand that the instantaneous physico-chemical composition at the top of the sediment bed determines uniquely the bed shear stress whereby scouring (erosion) occurs. Observe that the bed shear stress necessary for erosion depends also on the particle density in the viscous sublayer via ρ_v appearing in (13). However the variation of ρ_v with particle density is rather small. This agrees with the observation of Partheniades that the erosion rate is practically independent of the average particle density in the fluid for the range up to 12 g/ℓ attained in his experiments.

Suppose that after some time during which a sediment layer has built up by deposition the shear stress is rather suddenly raised, say, due to the occurence of a storm. It is clear from (14) that in a short time the sea bed will be eroded to the depth where the solid concentration corresponds with the critical density. During such change one expects a high scouring rate or even the formation of mud pebbels entering the fluid. This has indeed been observed (Migniot, Partheniades).

We focus our attention to the calculation of the erosion flux which occurs from there on, when the applied shear stress is either constant (steady state) or changes quasi-stationary. Thereto (6) has to be solved under the appropriate initial and boundary conditions.

4. The density distribution in a cohesive sediment bed.

The initial condition of the sediment layer depends strictly speaking on the whole previous deposition and erosion history.

Nevertheless one can form a reasonable good idea of the solid particle density through the sediment layer from experiments in the ultracentrifuge where processes are speeded up enormously. Figure 2 taken from Fujita (1962) shows how the solid particle density distribution ρ changes in a centrifuge with inner radius r_1 and outer radius r_2 . The (constant) angular velocity is ω and initially the two component fluid is homogeneous with sediment density ρ_o .

The important thing to observe is that for a rather long time one observes a sediment density distribution having a "plateau" region and that the change in plateau density value varies only slowly with time. Migniot (1968) who performed experiments on the settling of solid particles in a gravitational field found also the distribution with a near "plateau" region.

Guided by these results we assume a solid particle density distribution as shown in fig. 3 and assume that the density ρ_p at the "plateau", at $z = \delta$, remains constant during erosion. The density at the fluid-bed interface at $z = X$, is denoted by ρ_c . As long as erosion occurs its value is determined by (13) i.e. by the shear stress acting at the interface.

The vertical profile shown in fig. 3 corresponds with the experimentally observed dissolved silica concentration in the North Sea by Wollast and Van Der Borght (1974).

For $X \leq z \leq \delta$ a parabolic sediment density distribution of the following form may be assumed

$$\rho = \rho_p + (\rho_c - \rho_p) \left(1 - \frac{z-X}{\delta-X}\right)^2 \quad X \leq z \leq \delta \tag{14}$$

which satisfies :

$$\text{at } z = X \quad , \rho = \rho_c$$

$$\text{at } z = \delta \quad , \rho = \rho_p \quad \text{and} \quad \frac{\partial \rho}{\partial z}(\delta,t) = 0 \tag{15}$$

5. The interface mass flux continuity

The introduced X and δ in (14) are, of course, unknown functions of the time. To determine them with (6) we need one additional boundary condition. Thereto we set up the mass balance equation at the interface. Denote by J_+^* the sediment flux with respect to the moving interface in the sediment layer arbitrarily close to X(t) and by J_-^* its value in the fluid arbitrarily close to X(t).

By definition one has

$$\underline{J}_-^* = \rho_c^s \left(\underline{v}_-^s - \frac{d\underline{X}}{dt}\right) \; ; \; \underline{J}_+^* = \rho_c^s \left(\underline{v}_+^s - \frac{d\underline{X}}{dt}\right) \tag{16}$$

where $\underline{v}_+^s$ and $\underline{v}_-^x$ are the velocities of the sediment contamitant in the sediment layer and fluid respectively and $\frac{d\underline{X}}{dt}$ is the velocity of the interface.

The z components of these fluxes are :

$$J_{-z}^* = \rho_c^s \left(v_-^s - \frac{dX}{dt}\right) \quad , \quad J_{+z}^* = \rho_c^s \left(v_+^s - \frac{dX}{dt}\right) \tag{16.a}$$

The sediment flux, in the sediment layer arbitrarily close to the interface,with respect to the barycentric or center of mass velocity $\underline{v}_+$ is by definition

$$\underline{J} = \rho_c^s \left(\underline{v}^s - \underline{v}_+\right) \tag{17}$$

Its z component is

$$J_z = \rho_c^s \left(v^s - v_+\right) \tag{17.a}$$

Combining (16.a) and (17.a) gives for the erosion flux counted positive downwards

$$J_z = J_{+z}^* + \rho_c^s \left(\frac{dX}{dt} - v_+\right) \simeq J_{+z}^* + \rho_c^s \frac{dX}{dt} \tag{18}$$

The approximation expresses that the barycentric velocity in the sediment layer is negligible compared to $\frac{dX}{dt}$.

As there are no sources or sinks at the bed-fluid interface the normal sediment flux accross it measured with respect to the interface velocity must be continuous, so that

$$J_{-z}^* = J_{+z}^* \equiv J_{z,v}$$

Hence (18) can be written, omitting the superscript s :

$$J_z = J_{z,v} + \rho_c \frac{dX}{dt} = \sigma \rho_c - D \frac{\partial \rho}{\partial t} (X,t) \quad (19)$$

Here, (7) has been used and we have put

$$\sigma = s \, g_z$$

where σ is the settle particle velocity in the sediment layer.

To determine the flux $J_{z,v}$ which crosses from the fluid to the sediment layer we calculate the diffusion and sedimentation fluxes through the viscous sublayer.

The flow in the viscous sublayer is laminar-like in that the mean velocity profile is identical to the linear velocity profile of a plane parallel laminar flow with zero pressure gradient, Monin (1965). Nevertheless the flow is three dimensional and unsteady, Kline et al (1967). Streaks were observed by them to waver and oscillate and to pass sometimes rapidly to the outer edge of the boundary layer. The continuous formation and break up of low speed streaks fluctuations, which increase the diffusion coefficient to a higher value than the molecular one, presumably become more frequent and larger with increasing Reynolds number.

We denote the solid particle density (mass of sediment per unit fluid volume) at the top of the viscous sublayer, where it meets the logarithmic boundary layer, by ρ_b (see fig. 3).

The diffusion flux through the viscous sublayer with respect to the interface velocity, counted positive downwards, may be expressed by

$$\hat{J}_{z,diff} = - L (\rho_c - \rho_b) \quad (20)$$

wherein the transport coefficient L is defined by

$$L = \frac{D_v}{\delta_v} > 0 \quad (21)$$

where D_v is the diffusion coefficient and δ_v the thickness of the sublayer.

In (20) we have assumed that the solid particle density varies linearly through the viscous sublayer.

In addition to this diffusion flux there is a sediment flux through the viscous sublayer. Its average, counted positive downwards is

$$\hat{J}_{z,sed} = \frac{1}{\delta_v} \int_0^{\delta_v} J_{z,sed}\, dz = \frac{1}{\delta_v} \int_0^{\delta_v} \sigma_v\, \rho\, dz = \frac{1}{2}\, \sigma_v(\rho_b+\rho_c) \qquad (22)$$

The total flux through the sublayer, counted positive downwards, is found by adding (20) and (22)

$$J_{z,v} = L(\rho_b - \rho_c) + \frac{1}{2}\, \sigma_v(\rho_b + \rho_c) \qquad (23)$$

The viscous sublayer thickness appearing in (21) is related to the critical friction velocity U_{*c}, the fluid kinematic viscosity and the fluid density ρ_v by (Monin, 1965)

$$\delta_v = \alpha_v \frac{\nu}{U_{*c}} = \alpha_v\, \nu \sqrt{\frac{\rho_v}{\tau}} \qquad (24)$$

where τ is the critical shear stress acting at the interface between fluid and sediment layer and α_v is a universal constant of order unity.

Hence

$$L = \frac{D_v U_{*c}}{\alpha_v\, \nu} = \frac{D_v}{\alpha_v\, \nu}\, \rho_v^{-1/2}\, \tau^{1/2} \qquad (25)$$

The relation (19) with $J_{z,v}$ given by (23) and the L appearing herein by (25) determine the additional boundary condition to (15). With them we shall proceed to solve the differential equation (6) ; its solution is approximated by assuming the parabolic distribution given by (14) and yields the erosion flux.

6. The solution for the erosion flux

Integration of (6) between $z = X$ and $z = \delta$ results in

$$\int_{z=X(t)}^{z=\delta(t)} \frac{\partial \rho}{\partial t}\, dz = -\,\sigma \int_{X(t)}^{\delta(t)} \frac{\partial \rho}{\partial z}\, dz + D \int_{X(t)}^{\delta(t)} \frac{\partial^2 \rho}{\partial z^2}\, dz$$

$$= - \sigma \left(\rho_p - \rho_c\right) + \frac{2D}{\delta - X} (\rho_c - \rho_p) \tag{26}$$

where we have made use of $\frac{\partial \rho}{\partial z} (\delta, t) = 0$ and

$$\frac{\partial \rho}{\partial z} (X,t) = - \frac{2}{\delta - X} (\rho_c - \rho_p) \tag{27}$$

which follows from (14) .

Now by Leibnitz's rule we have

$$\frac{d}{dt} \int_{z=X(t)}^{z=\delta(t)} \rho(z,t)dz = \int_X^\delta \frac{\partial \rho}{\partial t} dz + \rho(\delta,t) \frac{d\delta}{dt} - \rho(X,t) \frac{dX}{dt} \tag{28}$$

Combining (26) and (28) and using (15) gives

$$\frac{d}{dt} \int_X^\delta \rho dz = \rho_p \frac{d\delta}{dt} - \rho_c \frac{dX}{dt} + \sigma (\rho_c - \rho_p) + \frac{2D}{\delta - X} (\rho_c - \rho_p) \tag{29}$$

We proceed to evaluate the left hand side of this equation by means of (14). We find

$$\int_X^\delta \rho dz = \frac{\delta - X}{3} (2 \rho_p + \rho_c) \tag{30}$$

We have pointed out that it is a good approximation to assume that ρ_p is constant, see fig. 2. For a steady erosion flux which requires a constant applied shear stress at the top of the viscoplastic sediment layer one finds from (13) that ρ_c is constant. Hence

$$\frac{d}{dt} \int_X^\delta \rho dz = \left(\frac{d\delta}{dt} - \frac{dX}{dt}\right) \left(\frac{2 \rho_p + \rho_c}{3}\right) \tag{31}$$

Thus for this case (29) reduces to :

$$\frac{2 \rho_p + \rho_c}{3} \left(\frac{d\delta}{dt} - \frac{dX}{dt}\right) = \rho_p \frac{d\delta}{dt} - \rho_c \frac{dX}{dt} + \sigma(\rho_c - \rho_p) + \frac{2D}{\delta - X}(\rho_c - \rho_p) \tag{32}$$

In addition to this relation we obtain a second one when (27) is substituted in (19)

$$J_z = J_{z,v} + \rho_c \frac{dX}{dt} \qquad (33.a)$$

$$= \sigma \rho_c + 2D \frac{\rho_c - \rho_p}{\delta - X} \qquad (33.b)$$

In the stationary state J_z and $J_{z,v}$ are constant. Hence it follows from (33.a) that $\frac{dX}{dt}$ must be constant. It follows now from (33.b) that $\delta - X$ must also be constant.

Thus

$$\frac{d\delta}{dt} = \frac{dX}{dt} \qquad (34)$$

Whence for the stationary state we obtain from (32)

$$(\rho_c - \rho_p)\frac{dX}{dt} = \sigma(\rho_c - \rho_p) + \frac{2D}{\delta - X}(\rho_c - \rho_p) \qquad (35)$$

Eliminating $\frac{2D}{\delta-X}(\rho_c - \rho_p)$ between (33.b) and (35) gives

$$\frac{dX}{dt} = \sigma - \frac{J_{z,v}}{\rho_p} \qquad (36)$$

Substituting this result in (33.a) gives for the stationary erosion flux

$$J_z = \rho_c \sigma + (1 - \frac{\rho_c}{\rho_p}) J_{z,v} \qquad (37)$$

The erosion flux, J , for the stationary state counted as <u>positive upwards</u> , i.e. , $J = - J_z$, is finally found when (23) is substituted into (37) as

$$J = - \rho_c \sigma + \left(\frac{\rho_c}{\rho_p} - 1\right) \left\{L(\rho_b - \rho_c) + \frac{1}{2}\sigma_v(\rho_b + \rho_c)\right\} \qquad (38)$$

wherein

$$\rho_c = \left(\frac{\tau}{E_i}\right)^{1/B_i} \qquad (39)$$

as follows from (13). L which is also a function of τ is given by (26).

The equation (38) has been derived under the assumption that erosion occurs. The auxiliary condition herefore is as follows : when the instantaneous solid particle density at the top of the bed is ρ_c then erosion finds place if and only if the bed shear stress is larger of equal to the critical value τ determined by (39).

This loading condition can also be referred to the initial bed state. Consider a fluid flow such that no erosion finds place. (Either sediment deposition finds place or no deposition and no erosion). Denote the sediment density at the top of the sediment bed in this state by ρ_1 . The minimum shear stress necessary for the onset of erosion follows from (13) as

$$\tau_1 = E_1 \, \rho_1^{B_1} \tag{40}$$

For a shear stress which is an arbitrary small fraction less than τ_1 no erosion occurs. Thus (39) gives the steady state erosion flux for monotonically increasing bed stress τ such that

$$\tau \geq \tau_1 \tag{41}$$

It is clear that the analysis implies that when the bed shear stress is decreased,erosion will cease at once. If under this lower stress,deposition finds place,the sediment layer will build up whereby the density at the top of the bed diminishes until eventually a value is reached which corresponds with the new ρ_c determined by (39). At that time scouring will reoccur its rate being determined by (38). Hence lowering of the shear stress will after a certain time, determined by floculation and deposition, gives rise to a lower erosion flux which is independent of the previous history as observed by Partheniades (1965).

7. Some special cases

A simplification of the relation (38) is possible when the defloculated solid particle diameter is very small. During stationary erosion the solid particles break continuously from the bed. Thus unlike deposition during which the particles are floculated, the diameter of the particles passing during erosion through the viscous sublayer to the bulk fluid is likely to be very small. Therefore the settle velocity in the viscous sublayer approaches a negligible value. This is in accordance with the observations that single clay particle can be carried in suspension by Brownian motion.

The settle velocity, σ_v , during erosion we expect therefore to be much smaller than $L = \frac{D_v}{\delta_v}$.

The latter is of the order 10^{-9} m/sec (The molecular diffusion $D_v \simeq 10^{-11}$ m^2/sec while $\delta_v \simeq 0.01$ m)

Hence, for fine grained cohesive soils

$$L >> \sigma_v \quad , \quad L >> \sigma \tag{42}$$

The solid particle density at the top of the sediment layer, ρ_c, is during erosion related to the bed shear stress τ by the relation (39). The condition

$$\frac{\rho_c}{\rho_p} << 1 \tag{43}$$

is therefore expected to be satisfied for sufficiently small bed shear stresses. Under the conditions (42) and (43) , (38) reduces to :

$$J_i = \frac{L}{E_i^{1/B_i}} \left(\tau^{1/B_i} - \rho_b E_i^{1/B_i}\right) \quad \text{for} \quad \tau \geq \tau_1 \ (i = 1,2) \tag{44}$$

As $J_1 = 0$ for $\tau = \tau_1$ we have

$$\tau_1 = \rho_b \, E_1^{1/B_1}$$

Hence we may write

$$J_i = \frac{L}{E_i^{1/B_i}} \left(\tau^{1/B_i} - \tau_i^{1/B_i}\right) \qquad \text{for } \tau_c \geq \tau_i \ (i = 1,2) \tag{45}$$

where τ_2 would be the bed shear stress whereby $J_2 = 0$ if only region 2 is present. That is to say, when only the straight line describing the second region in fig. 1 is considered with the understanding that this line is prolonged to cut the absciss.

Substituting (21), (24) into (45) yields for the erosion flux

$$J_i = A_i \left(\tau^{1/B_i} - \tau_i^{1/B_i}\right) \tau^{1/2} \qquad \text{for} \quad \tau \geq \tau_i \ (i = 1,2) \tag{46}$$

where

$$A_i = \frac{D_v}{\alpha_v \nu \, \rho_v^{1/2} E_i^{1/B_i}} \tag{47}$$

The constants appearing herein differ in the first region (i=1) and second region (i=2) accordingly to (10.b)-(13).

It is interesting to observe that for $B_i = 1$, (46) reduces to the bed-load relation derived by Bagnold (1956), Monin (1965), for a sand bed. The quantity A_1 is then not determined from the theory but taken from an experimental graph where it is plotted against sand particle diameter.

Of course, the equations we have derived should apply equally to the flow of air over a dust layer, at least when the latter has similar properties as the sediment layer we considered above.

8. Comparison with experiment

Partheniades (1965), (1972), conducted laboratory erosion experiments in an open rectangular channel. We quote him for the following facts.

"The experiments were conducted in an open flume with recirculating water at ocean salinity. The bed material, sampled from the San Francisco Bay and known as Bay mud, contained approximately equal amounts of silt and clay and some traces of fine sand and organic matter. It is a highly plastic soil with a liquid limit of 99 %, and a plasticity index (PI) of 55 %. The clay portion consists predominantly of montmorillonite and illite. Fig. 4 shows the erosion rate as function of the average bottom shear stress. Series I correspond with a bed remolded at its field moister of 110 % with a shear strength of about 20 p.s.f ($\simeq$ 1000 N/m^2) at ultimate failure and 11 p.s.f. at the yield point. The minimum bed shear strength at which erosion was first observed was about 0.002 p.s.f. ($\simeq$ 0.1 N/m^2). The bed in series II was formed by releveling that of series I. However, due to some unavoidable water entrainment, it displayed a higher water content and lower shear strength. The observed increase of the resistance to erosion was attributed to cementation due to iron oxides dissolved in the water. The fact that the increased interparticle bonds were not reflected in the macroscopic shear strength of the bed indicates that the soil cohesion, as determined by conventional shear strength tests, is not a representative or a unique measure of the soil resistance to erosion".

We have commented on page 6 that a unique relation between the critical shear stress and yield stress is not essential for the theory. The essential thing is that the instantaneous physico-chemical composition at the top of the sediment layer (at the sediment-fluid interface) determines uniquely the shear stress whereby erosion finds place.

To apply the relation (46) to the experiments performed by Partheniades we shall assume that the Bay mud has properties which corresponds closely to the data given by Migniot for the Provins clay. For that mud $m = 5$ and $n \simeq 10^{-13}$.

The shear stress whereby erosion occurs is related to the yield stress at the top of the sediment layer by

$$\tau = \rho_v \, p_1^2 \, \tau_y^{2q_1} \qquad (48)$$

as follows from (11) and (12)

Series I

There is a clear change in the nature of series I , fig. 4 , for a bed shear stress of about $\tau_c = 0.011 \text{ lbs/ft}^2 = 0.53 \text{ N/m}^2$. Using this value in (46) and $\rho_v \simeq 1000 \text{ kg/m}^3$ and the values $p_1 = 0.0178 \, G^{5/4}$, $q_1 = 0.25$ found by Migniot for the first region*, we find for the corresponding yield stress $\tau_y = \frac{2.7}{G^5} \text{ N/m}^2$.

By inspection of fig. 1 it is seen that this value corresponds very well to the intersecting point of the two straight lines if the line for the higher yield stress range (region 2) is drawn through the experimental points (⊗) of the Provins clay. The equation for this second line is (m=5)

$$U^* = p_2 \, \tau_y^{q_2} = 0.0139 \, G^{5/2} \tau_y^{0.5}$$

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* For completeness we shall carry the correction factor introduced in (10.b) along.

Accordingly, we shall use the values

$$\left.\begin{aligned} p_1 &= 0.0173\ G^{5/4} \quad , \quad q_1 = 0.25 \\ p_2 &= 0.0139\ G^{5/2} \quad , \quad q_2 = 0.5 \end{aligned}\right\} \qquad (49)$$

with (49) it follows from (48) that the two regions are separated at the bed shear stress $\tau = 0.53\ {}^{N}/_{m^2}$.

From (49) and (13) it follows that

$$B_1 = 2.5 \quad , \quad B_2 = 5 \qquad (50)$$

The erosion relation (46) becomes now

$$J_1 = A_1(\tau^{0.4} - \tau_1^{0.4})\tau^{0.5} \quad \text{for } 0.12\ {}^{N}/_{m^2} \leq \tau \leq 0.53\ {}^{N}/_{m^2} \qquad (51.a)$$

$$J_2 = A_2(\tau^{0.2} - \tau_2^{0.2})\tau^{0.5} \quad \text{for} \quad \tau \geq 0.53\ {}^{N}/_{m^2} \qquad (51.b)$$

The initial critical bed shear stress reported by Partheniades whereby erosion occurs is $\tau_1 = 0.0025\ {}^{lbs}/_{ft^2} = 0.12\ {}^{N}/_{m^2}$.

Using the experimental values $J_1 = 0.435\ {}^{gr}/_{ft^2\text{-hr}} = 1.3 \times 10^{-6}$ kg/m^2-sec for $\tau = 0.53\ {}^{N}/_{m^2}$ and $J_2 = 2.46\ {}^{gr}/_{ft^2\text{-hr}} =$ $7.37 \times 10^{-6}\ kg/m^2$-sec when $\tau = 0.025\ {}^{lbs}/_{ft^2} = 1.197\ {}^{N}/_{m^2}$ obtained from fig. 4 ; we determine the constants in (51) as

$$\left.\begin{aligned} \tau_1 &= 0.12\ N/m^2 \ , \quad \tau_1^{0.4} = 0.43 \quad , \quad A_1 = 4.96 \times 10^{-6} \\ \tau_2 &= 0.39\ {}^{N}/_{m^2} \ , \quad \tau_2^{0.2} = 0.83 \quad , \quad A_2 = 33 \times 10^{-6} \end{aligned}\right\} \qquad (52)$$

The calculated curve in fig. 4 shows that with these values,(51) gives a remarkable good description of the erosion flux observed in series I. In particular note that the change in slope is predicted by the theory.

To confirm the theory we must however check if, with the A's given by (52), the diffusion coefficient in the viscous sublayer calculated with (47) corresponds in order of magnitude with the molecular diffusion coefficient ($10^{-10} - 10^{-11}\ m^2/sec$) .

For the two regions we obtain from (13) , taking $\rho_v = 1000\ kg/_{m^3}$:

$$E_1 = 10^{-7} G^{5/2} \quad , \quad E_2 = 1.93 \times 10^{-14} G^5$$

Using $\alpha_v \simeq 10$, $\nu = 10^{-5}$ $^m/sec$ we find from (47)

$D_v = 2.4 \times 10^{-11}\ G\ m^2/sec$ in region 1

$D_v = 1.9 \times 10^{-11}\ G^2 m^2/sec$ in region 2

These diffusion coefficients are of the same order. For a correction factor $G = 5$, the average diffusion coefficient in region 2 is about four times that in region 1. As the turbulent fluctuations in the viscous sublayer become presumably more frequent with increasing shear stress it must be expected that the diffusion coefficient becomes larger.

Series II.

Partheniades reports that the increased resistance against erosion in series II is believed to be caused by disolved iron oxides.

Fig. 4 shows that a strong increase in the erosion rate occurs for this II bed by a shear stress of about $\tau = 0.031$ $^{lbs}/_{ft^2}$ = 1.48 $^N/_{m^2}$.

According to (48) this corresponds with a yield stress of

$\tau_y G^5 = 21.8$ $^N/_{m^2}$ $(m = 5)$

This value should correspond with the yield stress whereby the dependence of U_* on $\tau_y G^m$ changes strongly. However, from fig. 1 we see that this value is much too high. In conclusion we may say that the physical properties of the bed in series II have changed so much that Migniot's data are not applicable.

Nevertheless we have made an attempt to fit series II. By trial we found that if the value of B_1 and B_2 in (46) are lowered, a slightly better fit could be obtained. In particular for $B_1 = 1$ and $B_2 = 2$ and $m = 5$, it follows from (13) that $q_1 = 0.1$ and $q_2 = 0.2$.

By fitting two points in the first region of series II (fig. 4) and another point in region 2 we obtain for A_1 , A_2 in (46) in M.K.S. units

$$A_1 = 1.7 \cdot 10^{-6} \quad , \quad A_2 = 19.9 \cdot 10^{-6} \tag{53}$$

By using $D_{v1} = 1.7.10^{-11}$ m^2/sec we find now from (47) and (13) that

$$p_1 = 0.036$$

with these values the yield stress at the top of the sediment bed which separates region 1 from region 2 follows from (48) as $\tau_y = 2.2$ N/m^2. This value corresponds nicely with the breaking point observed by Migniot (fig. 1). The value of p_2 can now be determined from the relation

$$U^* = p_1 \tau_y^{q_1} = p_2 \tau_y^{q_2}$$

which holds true at the breaking point where region 1 and 2 meet, as follows from (11). We obtain $p_2 = 0.033$.

The diffusion coefficient in region 2 follows now from (53), (47) and (13) as $D_{v2} = 370.10^{-11}$ which is 100 times larger than for series I. The equations for the erosion flux in series II are in M.K.S. units

$$J_1 = 1.7.10^{-6} (\tau - 0.34)\tau^{1/2} \qquad 0.34 \ N/m^2 \leq \tau \leq 1.48 \ N/m^2$$

$$J_2 = 19.9.10^{-6}(\tau^{0.5} - 1.12)\tau^{1/2} \qquad \tau \geq 1.48 \ N/m^2$$

These give a good description as shown in fig. 4.

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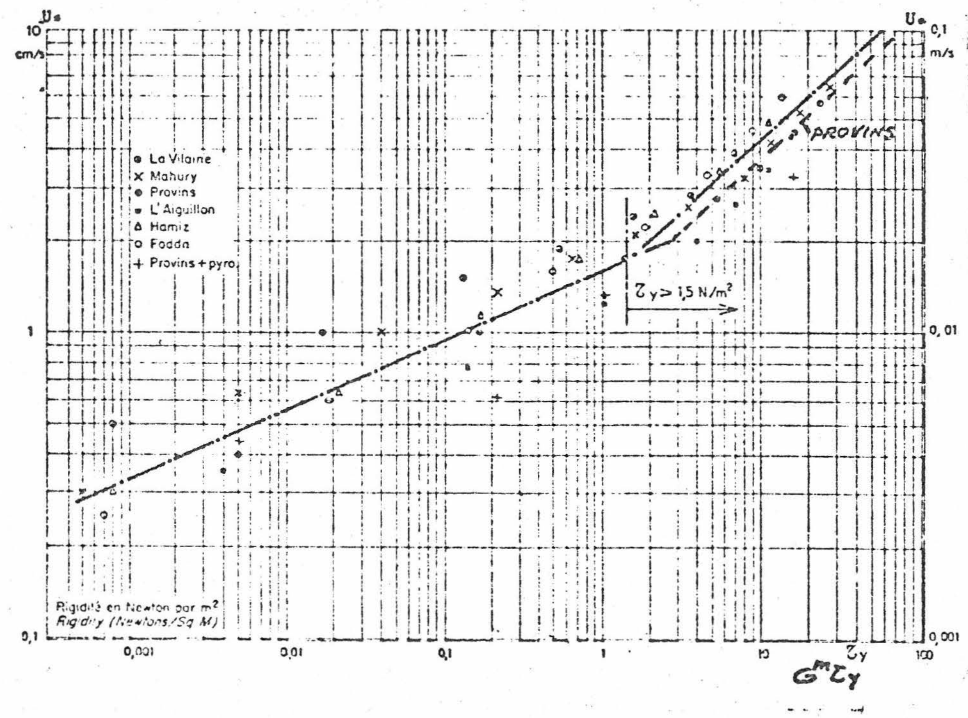

Fig.1. Critical friction velocity U_* versus yield stress, τ_y, in upper sediment bed region (after Migniot). The yield stress at the top of the sediment bed, $G^m \tau_y$, is also shown.

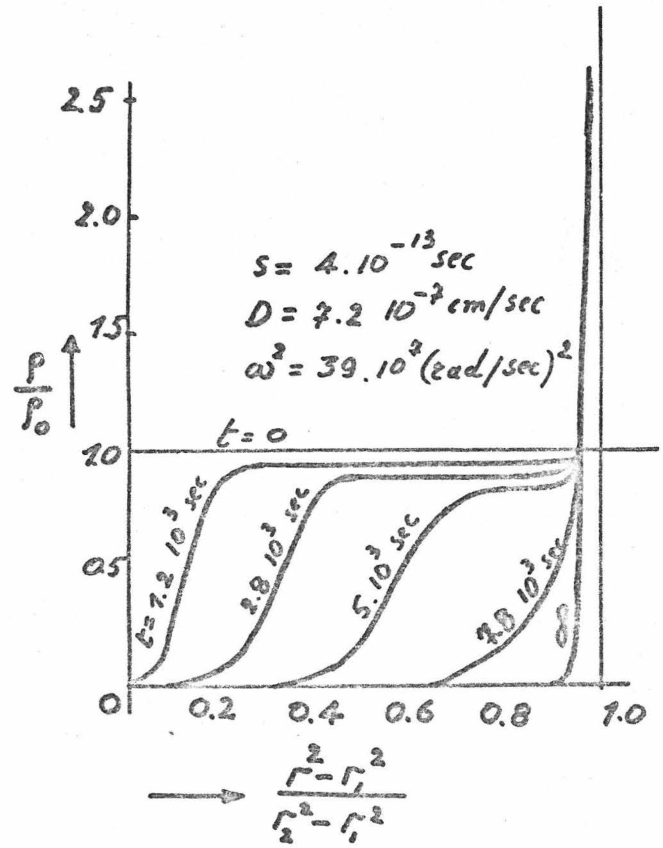

Fig. 2. Variation of density distribution in the ultra centrifuge. (after Fujita)

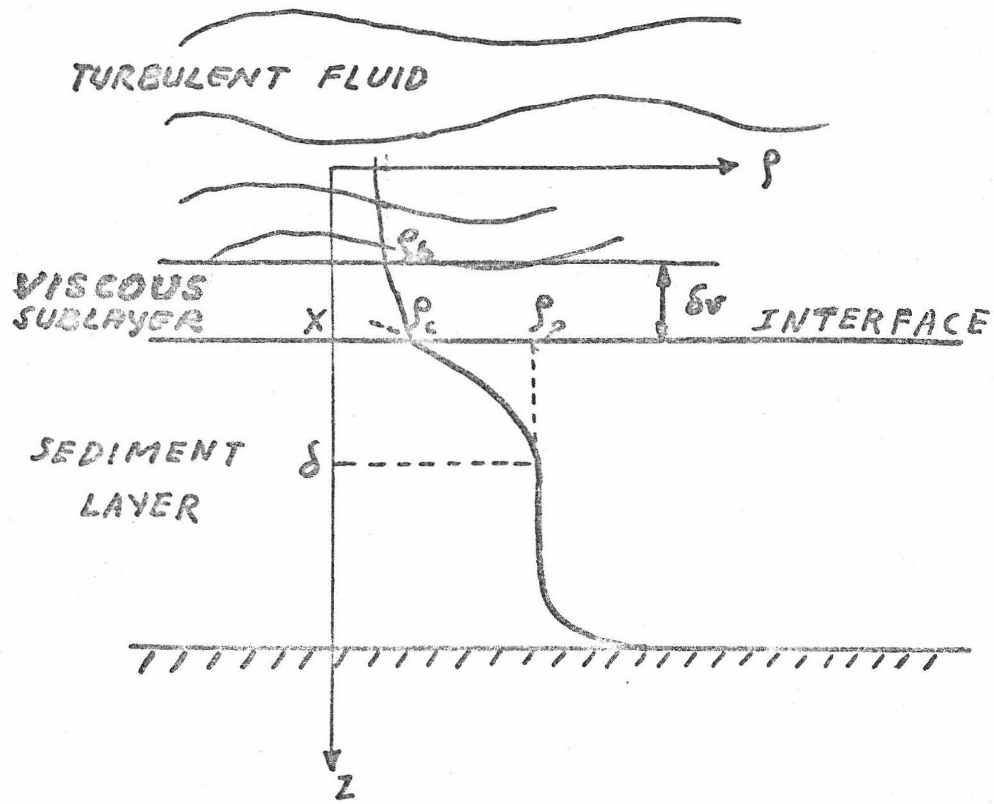

Fig. 3. Density distribution in the sediment bed.

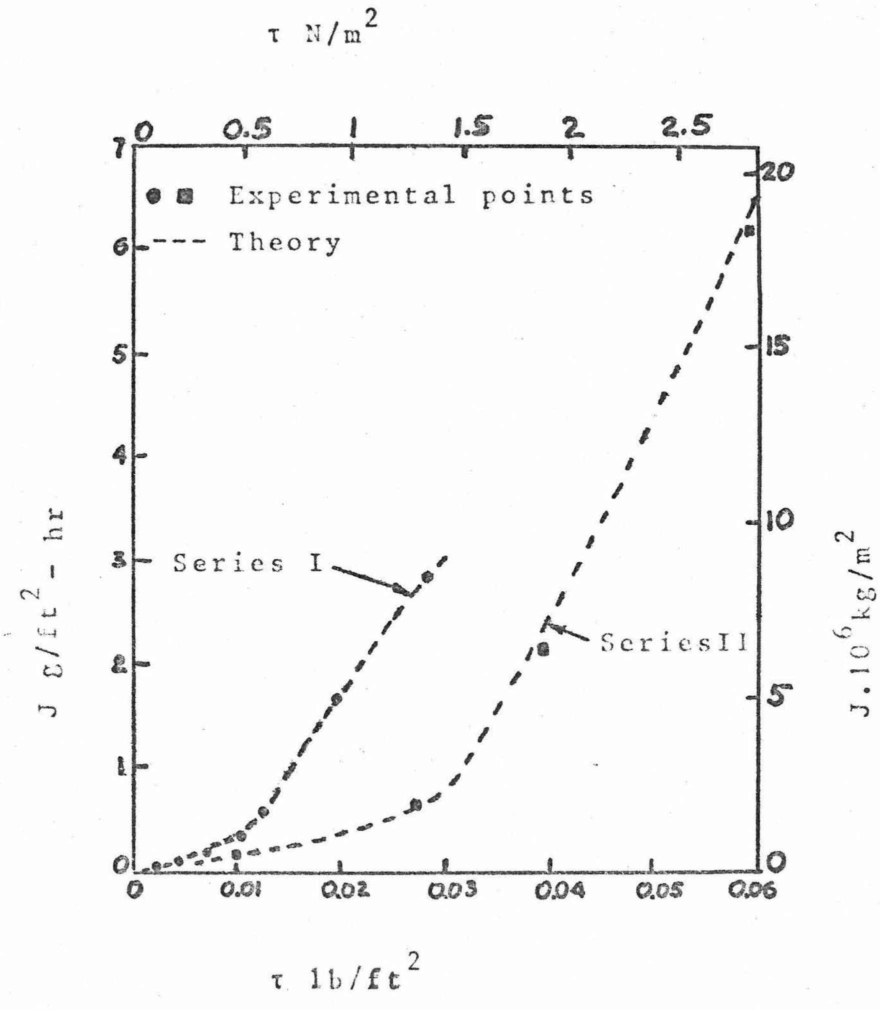

Fig.4. Variation of Erosion Rates, J, with Bottom Shear Stress, τ. (after Partheniades)