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Pollution des Eaux

Projet Mer

APPLICATION OF LIAPOUNOV DIRECT METHOD

TO THE LOTKA-VOLTERRA EQUATIONS

by

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INTRODUCTION

Various studies have investigated the "stability" of the Volterra system of first order differential equations which control the population numbers of interacting species. Different definitions of the stability of an ecosystem, as well as different methods to study it, have been used. Among the more common approaches are computer simulation (e.g. Garfinkel, 1962, 1967), graphical analysis (e.g. Davis, 1960 ; Slobodkin, 1961) and neighbourhood stability analysis, i.e. the small perturbations' method (e.g. May, 1971, 1973 a, b, ; Strobeck, 1973). These methods however seem either too expensive or too restrictive for the simple, although non-linear, Volterra model.

May (1973 a) has pointed out that "a global stability analysis, in the analytical approach, is the construction of a "Liapounov function" throughout some domain". This technique allows to make statements on the stability of the equilibrium without any knowledge of the solutions of the differential equations. Moreover, the introduction of the concept of stability domains (in the phase space and in the parameter space) leads to a better characterisation of the intrinsic properties of a model.

In this paper, Liapounov direct method is applied to the Volterra model of interacting populations and it is shown that the equilibrium is either stable (conservative case) or asymptotically stable (dissipative case). In the stable case, a formula is presented for the prediction of the limits of the fluctuations for any given initial perturbation. The corresponding

bounds depend also on the various parameters of the system and the importance of each of them can then be studied. In the damped case a similar formula provides information on the rate of damping.

The definition and the theorems of stability used here can be found in Hahn (1963) or La Salle and Lefschetz (1961) among others.

1. VOLTERRA'S PREY-PREDATOR MODEL

The equations proposed by Volterra (1931) to describe the behavior in time of interacting biological species, having populations N_1, N_2, \dots, N_n treated as continuous variables, are

$$\frac{dN_i}{dt} = \epsilon_i N_i + \beta_i^{-1} \sum_{j=1}^n \alpha_{ji} N_j N_i \quad (1)$$

The first term describes partly (i.e. apart from saturation effect) the behaviour of species i if left to itself in the absence of other species in a given environment. Since ϵ_i is equal to the birth rate minus the death rate, this term may provide either an exponential (Malthusian) growth (if $\epsilon_i > 0$) or an exponential fall (if $\epsilon_i < 0$). The quadratic terms express the interaction of species i with all the other species and with itself (intra-species competition) when $\alpha_{ii} \neq 0$. The effect on the rate of change of N_i of the interaction with species j is assumed to be proportional to the number of possible binary encounters $N_i N_j$ between members of the two species. Since we consider prey-predator relationships, the interaction parameters α_{ji} and α_{ij} have opposite signs. The positive quantities $\beta_i^{-1} \alpha_{ji}$ are Volterra's "equivalent numbers". In the binary encounters of species i and j , the ratio of the number of i 's lost (or gained) per unit time to the number of j 's gained (or lost) per unit time is $\beta_i^{-1} / \beta_j^{-1}$. The interaction parameters are then anti-symmetric

$$\alpha_{ij} = -\alpha_{ji} \quad i \neq j \quad (2)$$

The steady states of the biological association are those for which all $\frac{dN_i}{dt}$ vanish. The steady state or equilibrium populations q_i are defined by

$$q_i \left(\epsilon_i + \beta_i^{-1} \sum_{j=1}^n \alpha_{ji} q_j \right) = 0 \quad (3)$$

Assuming none of the q 's vanish, we have

$$\epsilon_i \beta_i + \sum_{j=1}^n \alpha_{ji} q_j = 0 \quad (4)$$

Let us further assume that, unless otherwise stated, equations (4) have a unique solutions with all q_i positive. This hypothesis and its implications are discussed by Goel et al (1971, pp. 244-249). One more conclusion will be added to their statements concerning the case where one q_i is negative.

1.a. Conservative system

Volterra (1931) calls "conservative" a system described by (1), (2) and

$$\alpha_{ii} = 0 \quad i = 1, 2, \dots, n. \quad (5)$$

One of the most criticized aspects of this model (May, 1973 b), besides the antisymmetry of the "community matrix", is the fact that the number of interacting species has to be even in order to have finite equilibrium populations.

Rescigno and Richardson (1973) summarize some of Volterra's results for this case in the following theorem :

"If an association of an even number of species which has a stationary state starts from any state other than the stationary state, then the number of individuals of each species exhibits undamped oscillations between two positive limits".

1.b. Dissipative system

An example of a "dissipative" association is the system described by (1), (2) and

$$\alpha_{ii} < 0 \tag{6}$$

Volterra (1931) shows that, if a steady state exists, it is stable and all species will survive with finite variations. Moreover an intra-species competition term of the Verhulst-Pearl type, $\alpha_{ii} N_i^2$ with $\alpha_{ii} < 0$ will result in damping out fluctuations in the population of species i and only i around q_i . If (6) holds for all i , the equilibrium point is asymptotically stable.

All these results and a lesser requirement for asymptotic stability, namely the presence of at least one density dependent coefficient of autoincrease can be derived in a more straightforward way from Liapounov's theory.

2. HAMILTONIAN FORMALISM

Let us momentarily restrict ourselves to the conservative case. As shown by Volterra (1937) there is a constant of motion

$$\sum_{i=1}^n (\beta_i N_i - q_i \beta_i \log N_i) = C \quad (7)$$

In the same paper, by writing his equations in terms of the integrals of the population numbers, he introduces a variational principle and writes them in canonical form.

The most outstanding work along this line seems to be that of Kerner (1957, 1959, 1964, 1972) and it constitutes the basis of his statistical mechanical considerations. Defining the new dependent variables (Kerner, 1957; Nihoul, 1974)

$$v_i = \log \frac{N_i}{q_i} \quad (8)$$

one finds that the equations of "motion" may be written in the form

$$\beta_i \frac{dv_i}{dt} = \sum_{j=1}^n \alpha_{ji} q_j (e^{v_j} - 1) \quad (9)$$

Because of the antisymmetry of the α_{ji} they lead to the general integral

$$G = \sum_{i=1}^n \beta_i q_i (e^{v_i} - v_i) = \text{constant with respect to } t \quad (10)$$

The equations (9) may also be written in terms of G as

$$\frac{dv_i}{dt} = \left(\sum_{j=1}^n \gamma_{ji} \frac{\partial}{\partial v_j} \right) G \quad (11)$$

with

$$\gamma_{ji} = \frac{\alpha_{ji}}{\beta_j \beta_i} = -\gamma_{ij}$$

Kerner (1959) shows that equations (11) are comprehended under a variational principle similar to Hamilton's principle in mechanics

$$\delta \int_{t_1}^{t_2} \Lambda dt = 0$$

where

$$\mathcal{L} = \frac{1}{2} \sum_{i=2}^n \sum_{j=2}^n \Gamma_{ij} v_i \dot{v}_j \quad \dot{v}_j = \frac{dv_j}{dt}$$

with the Γ matrix identified as \tilde{f}^{-1}

Then introducing the "momentum" conjugate to the "coordinate" v_i

$$p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{v}_i} = \frac{1}{2} \sum_{j=2}^n \Gamma_{ji} v_j$$

he deduces the "Hamiltonian"

$$H = \sum_{i=2}^n p_i \dot{v}_i - \mathcal{L} = G$$

3. A LIAPOUNOV FUNCTION FOR VOLTERRA MODEL

3. a. Conservative system

a.1. Stability

A slightly modified form of the "Hamiltonian" can be used to prove the stability of the equilibrium point. This point is, by (8), the origin in the v-space. It is important to note that this transformation of state variables preserves stability properties (Rosen, 1970).

Since the function G is continuously differentiable, has an absolute minimum $G = \sum_{i=2}^n \beta_i q_i > 0$ for $v_i = 0 \quad i = 1, 2, \dots, n$ and is constant with respect to time, it is easy to define a Liapounov function for the conservative system by subtracting the value of this minimum from the function G

$$\begin{aligned} L(v_i) &= G - \sum_{i=2}^n \beta_i q_i \\ &= \sum_{i=2}^n \beta_i q_i (e^{v_i} - v_i - 1) \end{aligned} \quad (12)$$

The function L is positive definite and $\frac{dL}{dt} = 0$. So the origin is stable and the stability is not asymptotic. Since $L(v_i) \rightarrow \infty$ when $v_i \rightarrow 0$ or ∞ , v_i will remain finite and there will be neither extinction nor infinite growth. It can be shown that the long time

averages of the N_i 's are the q_i 's (e.g. Volterra (1931)) and that, in the two-species case, the motion is periodic (e.g. Goel et al (1971)).

a.2. Upper and lower bounds

The fact that L is a sum of individual terms, each relating to a separate species, allows the computation of the bounds of the variations as a function of the β_i 's, the q_i 's and the initial perturbation.

Each "component"

$$L_k(v_k) = \beta_k q_k (e^{v_k} - v_k - 1)$$

of L is zero at the equilibrium ($v_k=0$) and is continuously increasing with $|v_k|$.

Since

$$L[v_i(t)] = \sum_{i=1}^n L_i(v_i) = L[v_i(0)] \tag{13}$$

the upper and lower bounds of v_k are the roots of the equation

$$\begin{aligned} \beta_k q_k (e^{v_k} - v_k - 1) &= L[v_i(0)] \\ &= \sum_{i=1}^n \beta_i q_i (e^{v_i(0)} - v_i(0) - 1) \end{aligned} \tag{14}$$

For any given species, either one of these bounds can be reached if and only if all other species are at equilibrium. In the v-space, the larger the product q_k times β_k , the smaller the oscillations (according to Rescigno and Richardson the parameter β_k , to a first approximation, can be regarded as the average weight of a member of the species k).

Equation (14) has always a positive solution $v_{k \max}$ and a negative one $v_{k \min}$ ⁽¹⁾. The corresponding $N_{k \max}$ and $N_{k \min}$ are given by

$$\begin{aligned} N_{k \max} &= q_k e^{v_{k \max}} \\ &= q_k \left(1 + \log \frac{N_{k \max}}{q_k} \right) + \frac{L[v_i(0)]}{\beta_k} \end{aligned} \tag{15}$$

(1) The subscripts max and min denote respectively the upper and lower bounds of the v's or N's.

The usefulness of (15), in the case where more than two species are interacting, is enhanced by the fact that the fluctuations are generally not periodic, which means that numerical integration cannot provide this information.

Moreover, equation (15) can be used to study the variations of the amplitude of the fluctuations as a function of any of the parameters or the initial conditions of the system.

Defining

$$\Delta N_{\kappa \min} = q_{\kappa} - N_{\kappa \min} \quad (16)$$

and using a second-order approximation to the logarithm in (15), it is easy to deduce that

$$\Delta N_{\kappa \min} \approx \left(\frac{2 q_{\kappa} L [v_i(0)]}{\beta_{\kappa}} \right)^{1/2} \quad (17)$$

This formula is only valid for rather small perturbations but provides in that case an easier interpretation of (15).

It shows that the minimum value of N_{κ} will be further from the equilibrium the greater the size of the equilibrium population and the smaller the coefficient β_{κ} (Following Kerner (1972), "in a crude figure of speech the β 's can be thought of as biomasses of individuals". Aiken and Lapidus (1973) call its inverse β_{κ}^{-1} the "nutritional value" of the inter-species interaction). If we agree to measure the "amplitude" of the initial perturbation in terms of $L[v_i(0)]$, the distance (16) is proportional to the square root of this amplitude.

Similarly, if

$$\Delta N_{\kappa \max} = N_{\kappa \max} - q_{\kappa} \leq q_{\kappa}$$

$\Delta N_{\kappa \max}$ satisfies the approximation (17).

a.3. Case where one of the equilibrium populations is negative

Goel et al (1973) show that the function

$$L' = \sum_i \beta_i (N_i - q_i \log \frac{N_i}{19:1} - 19:1) = \sum_i L'_i (N_i)$$

whose derivative

$$\frac{dL'}{dt} = \sum_{i,j} \alpha_{ji} (N_i - q_i)(N_j - q_j)$$

equals zero because the antisymmetry of the α 's is a constant of motion.

They show that if only one of the q_i 's is negative, e.g. $q_K < 0$, the population N_K of the corresponding species disappears. Since

$$\lim_{N_K \rightarrow 0} L'_K = -\infty \quad \text{and} \quad L' = \sum_i L'_i [N_i(0)] = \text{Const.}$$

one can add that

- 1° N_K tends asymptotically towards zero but will never cross zero and be negative
- 2° At least one other L'_i , say L'_j , has to go to infinity (i.e. $N'_j \rightarrow 0$ or $N'_j \rightarrow \infty$) in order to keep L' constant. This could also have been deduced from the fact that one can not have an oscillating system with an odd number of species.

a.4. Remark

A similar Liapounov function for the two-species case has been deduced by Dubois (1973) using a different approach. It is interesting to note that the equivalent numbers' hypothesis is not restrictive (and then needless) for the one-prey-one-predator system

$$\frac{dN_1}{dt} = \epsilon_1 N_1 - \alpha N_1 N_2$$

$$\frac{dN_2}{dt} = -\epsilon_2 N_2 + \beta N_1 N_2$$

where ϵ_1 , ϵ_2 , α and β are all positive. The function

$$L = \beta q_2 (e^{v_2} - v_2 - 1) + \alpha q_1 (e^{v_1} - v_1 - 1)$$

satisfies the conditions required by Liapounov's theory to deduce the stability of the equilibrium point.

3.b. Completely damped system

Let us now consider the system defined by (1), (2) and (6) and assume that (6) holds for all i . The function G , or L , is no longer a constant of motion, but, as recently shown (Aiken and Lapidus (1973)), L can still be identified as a Liapounov function. It is positive definite, decrescent, and, using (9), (2) and (6), its time derivative

$$\begin{aligned} \frac{dL}{dt} &= \sum_{i=1}^n \beta_i q_i (e^{v_i} - 1) \frac{dv_i}{dt} \\ &= \sum_{i=1}^n \sum_{j=1}^n \alpha_{ji} q_i q_j (e^{v_i} - 1)(e^{v_j} - 1) \\ &= \sum_{i=1}^n \alpha_{ii} q_i^2 (e^{v_i} - 1)^2 \end{aligned} \tag{18}$$

is always and everywhere negative except at the origin where it vanishes. The origin is then uniformly asymptotically stable. Moreover, as L is radially unbounded (i.e. $\lim_{|v| \rightarrow \infty} L(v) = \infty$) the origin is asymptotically stable for trajectories starting from all points of phase space. The absolute upper and lower bounds in the v -space are still given by (14) but can only be reached at the moment of the initial perturbation. As functions of time the bounds of v_k are the two solutions of

$$e^{v_k} - v_k - 1 = \frac{1}{\beta_k q_k} L[v_i(t)] \tag{19}$$

which allows us to compare the rates of damping of the different species. The damping will be slower the larger the product $q_k \beta_k$. It is possible to prove (Aiken and Lapidus (1973)) that the stability is exponential-asymptotic near the steady state.

3.c. Mixed case

This is the case where only one or some of the equations have a damping term, say $\alpha_{ii} < 0$ for $i=1, \dots, k$ and $\alpha_{jj} = 0$, $j = k+1, \dots, n$. In this case, L is still positive definite and decrescent (i.e. $\lim_{|v| \rightarrow 0} L(v, t) = 0$ holds uniformly in t) but $\frac{dL}{dt}$ is no longer negative definite for not all v_i equal to zero as is claimed by Aiken and Lapidus (1973). Since we have now

$$\frac{dL}{dt} = \sum_{i=1}^k \alpha_{ii} q_i^2 (e^{v_i} - 1)^2 \tag{20}$$

it is clear that this expression will be zero for

$$v_i = 0 \quad i = 1, \dots, k \quad (21)$$

According to Liapounov's theorem all we can conclude is that the equilibrium is stable. However, it is still possible to prove that the stability is asymptotic by using the extension of Liapounov's theorem given in the appendix (another proof of that theorem can be found in Barbaskin (1970)). In order to apply that theorem we have to assume that the community matrix is such that there is a link or a succession of links leading from any undamped species to at least one of the density dependent species. This assumption is natural. It simply means that every species or group of species has to be coupled to the rest of the system. Then, whenever equations (21) are satisfied and therefore $\frac{dv_i}{dt} = 0$ while one or more

$v_j, j = k+1, \dots, n$ are not equal to zero, the equations of motion

$$\begin{aligned} \frac{dv_j}{dt} &= \frac{1}{\beta_l} \sum_{m=1}^n \alpha_{ml} q_m (e^{v_m} - 1) \\ &= \frac{1}{\beta_l} \left\{ \sum_{i=1}^k \alpha_{il} q_i (e^{v_i} - 1) + \sum_{j=k+1}^n \alpha_{jl} q_j (e^{v_j} - 1) \right\} \end{aligned}$$

indicate that the damped species will not all stay at equilibrium because the N_i are linearly independent.

In conclusion, the introduction of only one damping term in a system of "truly interacting" species is sufficient to lead to asymptotic stability for all species.

4. A numerical example

4.a. Conservative case

The system chosen to illustrate the theoretical results involves four interacting species. The "mass" flow graph and the conditions to have all q_i 's positive are discussed by Goel et al (1973 ; pp. 246-247, fig. 8 case i and equation (3.21)). The coefficients of equations (1) are

$$\alpha = \begin{pmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -4 \\ -1 & 0 & 0 & +5 \\ 0 & +4 & -5 & 0 \end{pmatrix} \quad \epsilon = \begin{pmatrix} 2 \\ -1 \\ -3 \\ -0.5 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 \\ 2.5 \\ 7. \\ 4. \end{pmatrix}$$

Only the first species feeds on the nutrient reservoir ($\epsilon_1 > 0$). As usual for that kind of simulation both the time scale and the absolute magnitude of the populations are arbitrary. Fig. 1. shows the oscillations of the four normalized populations ($x_i = \frac{N_i}{q_i}$) around their equilibrium values ($\bar{x}_i = 1$). The upper and lower bounds (dotted lines) have been computed by (15). The integration has been performed over a length of time (~ 150 units of time) such that the time averages of the x_i 's are approximately equal to 1. These x_i 's are given in table 1 as are the $x_i(0)$ and the computed and observed bounds. The initial conditions have been computed in such a way that at least one of the species ($i = 1$) reaches one of its bounds at some time t^* on the portion of trajectory which is plotted. It should be noted that all other species are at equilibrium for $t = t^*$. Lastly the amplitude of the normalized oscillations decreases as $\beta_i q_i$ increases ($\beta_4 q_4 < \beta_2 q_2 < \beta_3 q_3 < \beta_1 q_1$).

TABLE 1.

i	q_i	$x_i(0)$	\bar{x}_i	x_i Max computed	x_i Max observed	x_i min computed	x_i min observed
1	24.125	1.027	0.999	1.243	1.243	0.791	0.805
2	2.	1.180	1.005	1.582	1.461	0.582	0.680
3	2.	1.301	0.998	1.327	1.302	0.732	0.750
4	0.625	0.941	1.006	1.876	1.821	0.451	0.515

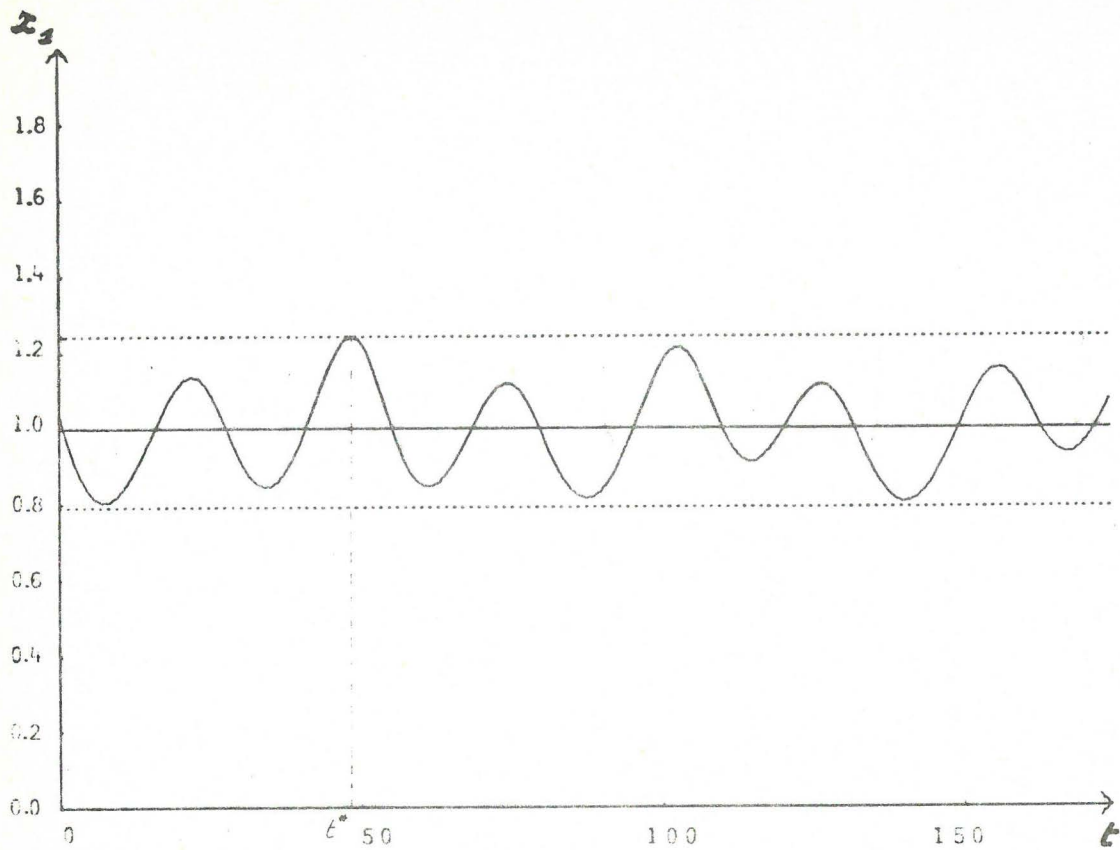


Fig. 1.a.

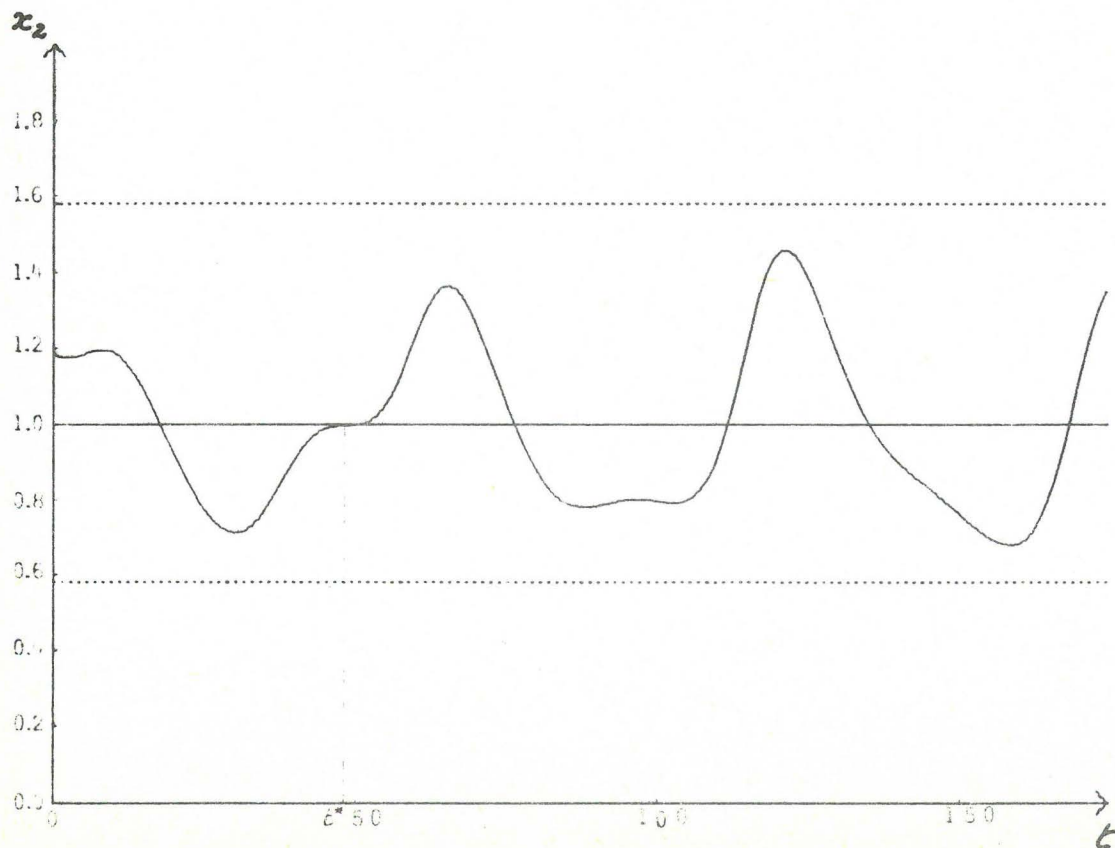


Fig. 1.b.

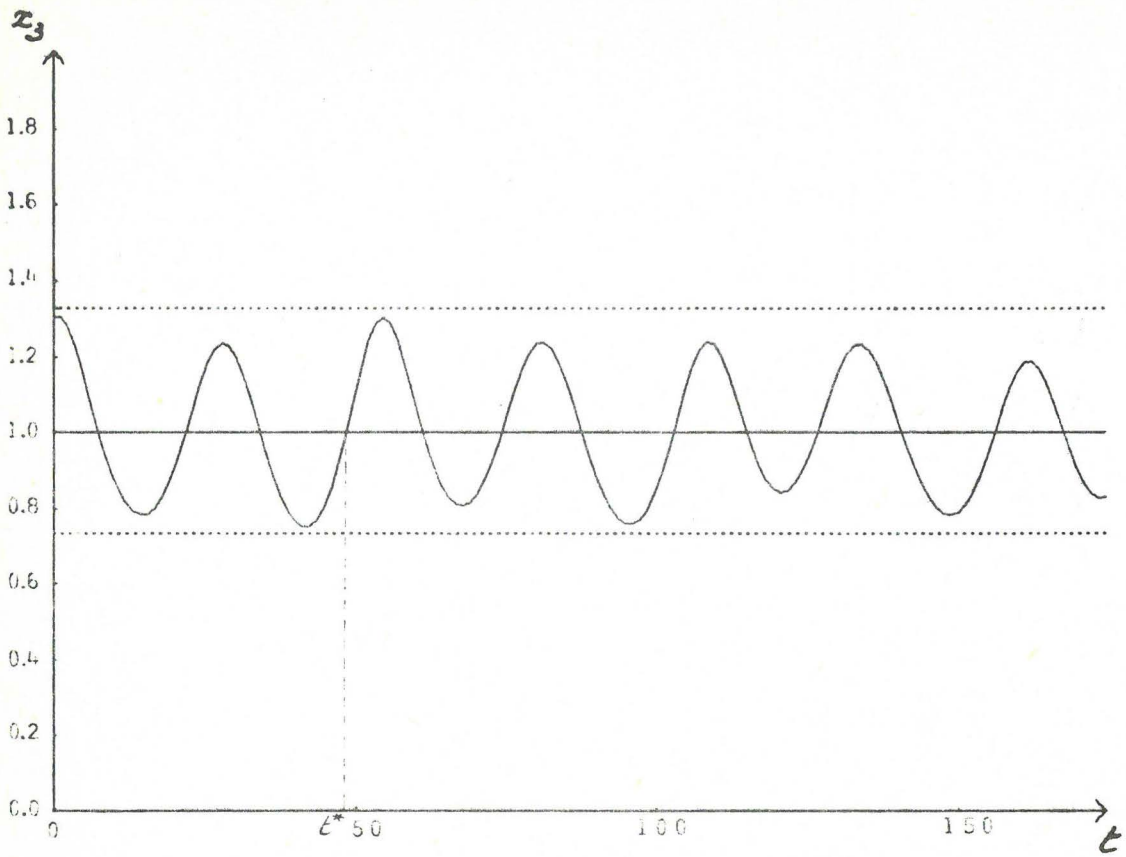


Fig. 1.c.

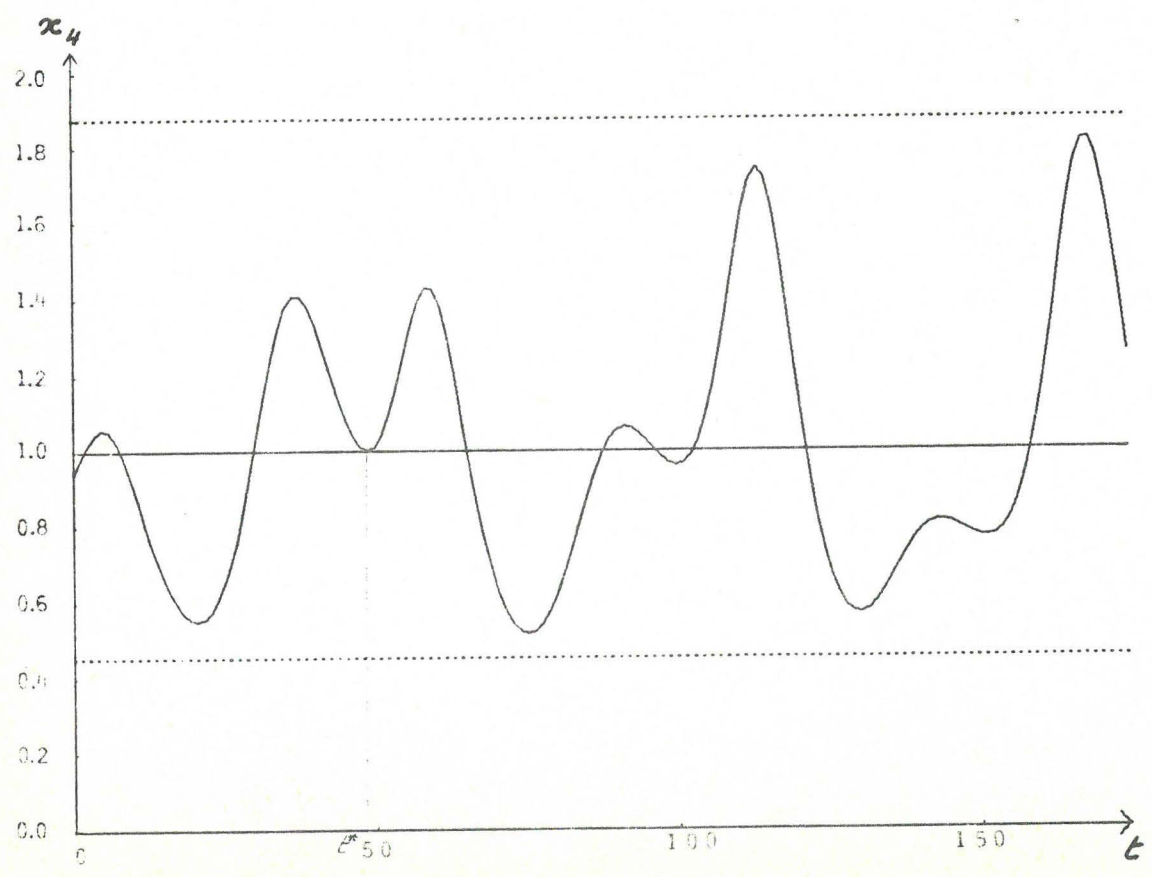


Fig. 1.d.

Some of the results of a study of the amplitude of fluctuations as a function of the initial conditions and the parameters of the system are exhibited on Fig. 2.

4.b. Dissipative case

Fig. 3 shows the damped oscillations of a four species system that has the same coefficients as in the conservative case except that $\alpha_{ii} = -3 \cdot 10^{-2}$ $i = 1, \dots$. The corresponding curves for the case where only one species has a damping term ($\alpha_{11} = -3 \cdot 10^{-2}$; $\alpha_{ii} = 0$ $i = 2, 3, 4$) (Fig. 4) have slightly slower damping rates but, as predicted, the equilibrium point is asymptotically stable. The dotted lines are the solutions of equation (19) and show that the damping rate is larger the smaller the product $\beta_i q_i$ ($\beta_4 q_4 < \beta_2 q_2 < \beta_3 q_3 < \beta_1 q_1$ in Fig. 3 and 4.). Fig. 5 shows the decrease of the Liapounov functions corresponding to those two cases. It should be noted that the initial populations were the same for the two systems and since the q_i 's are almost the same, the Liapounov functions had nearly the same initial values. However, the numerical results of the first 40 steps of the computation have not been plotted on Fig. 2.3.4 in order to have a better scale.

4.c. Case where one of the q_i 's is negative

Let us consider again the pattern of interactions described in 4.a. Goel et al (1973) show that for such a system one has always q_1, q_3 and q_4 positive and that the condition for q_2 to be also positive is that

$$E_1 \beta_1 \alpha_{34} > E_4 \beta_4 \alpha_{31} \tag{22}$$

Let us assume that for some reason (e.g. a disease or a disparition of nutrient that reduces the rate of increase of species 1 or an increase in the rate of death $|E_4|$) inequality (22) is no longer satisfied. Then species 2 will disappear because $q_2 < 0$.

At the same time the population of species 4, though it is preyed upon by species 2, will also disappear. Simple algebra shows that in this case ($E_1 \beta_1 \alpha_{34} < E_4 \beta_4 \alpha_{31}$ and $N_2 \approx 0$) $\frac{dN_4}{dt}$ is on the

average more often negative than it is positive. This is an interesting example of a situation where the presence of a predator is required for the survival of its prey.

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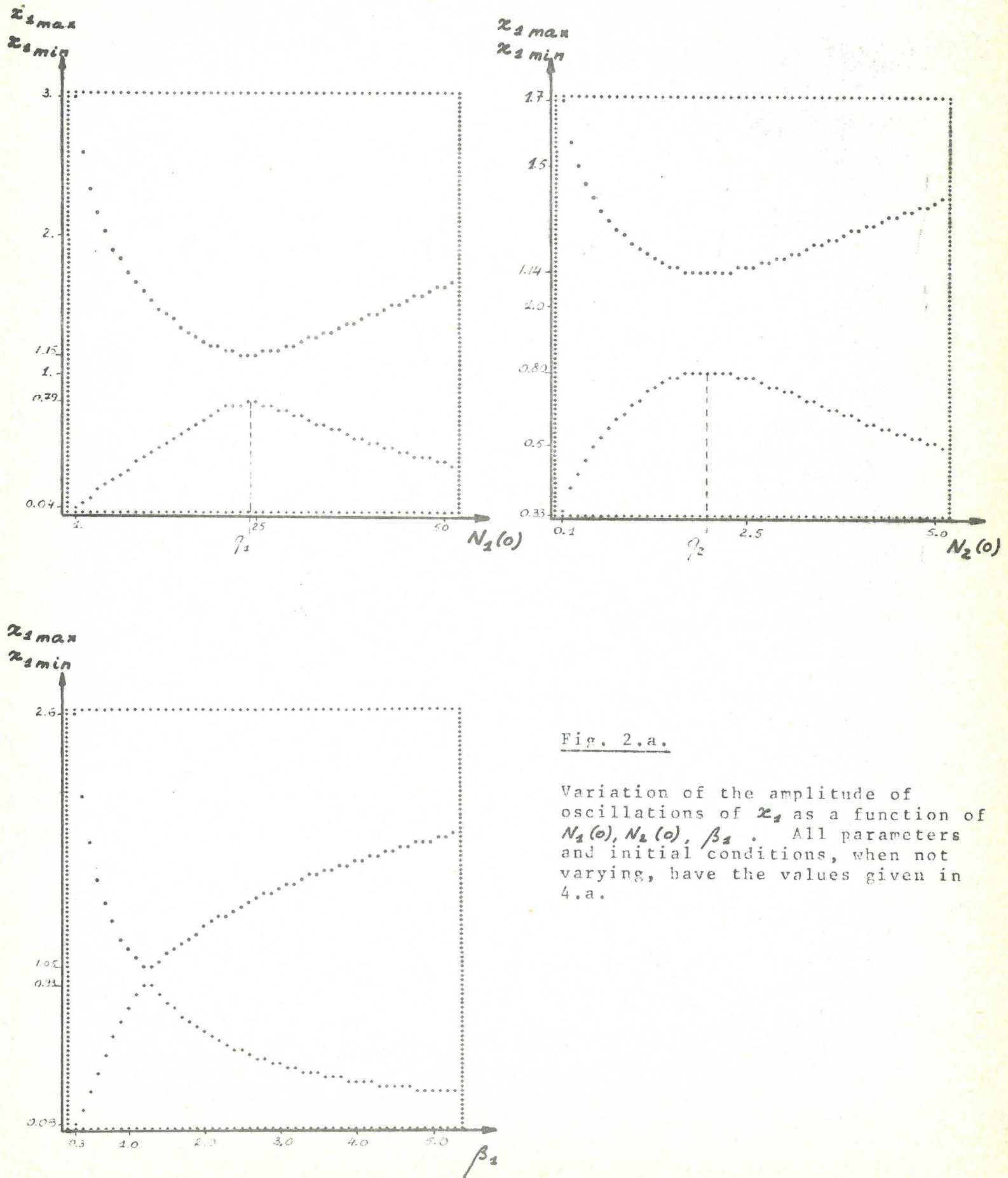


Fig. 2.a.

Variation of the amplitude of oscillations of x_1 as a function of $N_1(0)$, $N_2(0)$, β_1 . All parameters and initial conditions, when not varying, have the values given in 4.a.

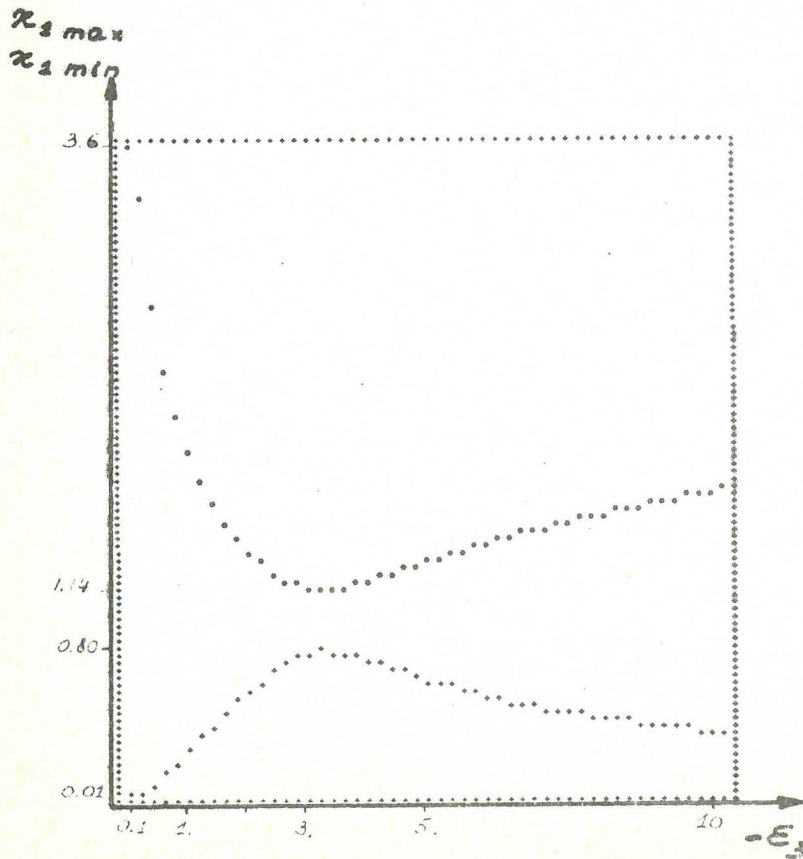
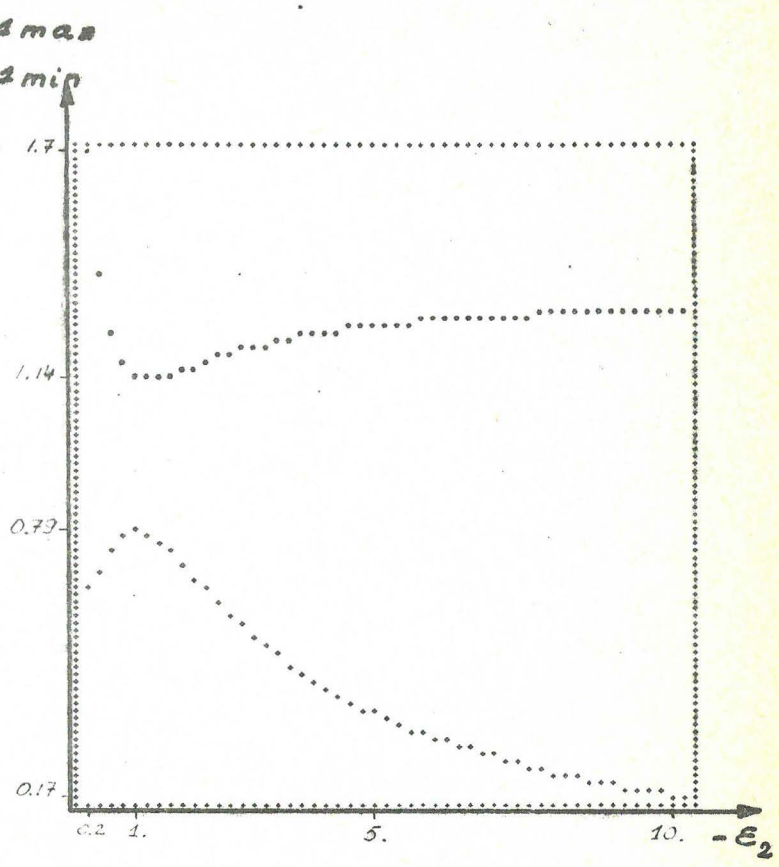
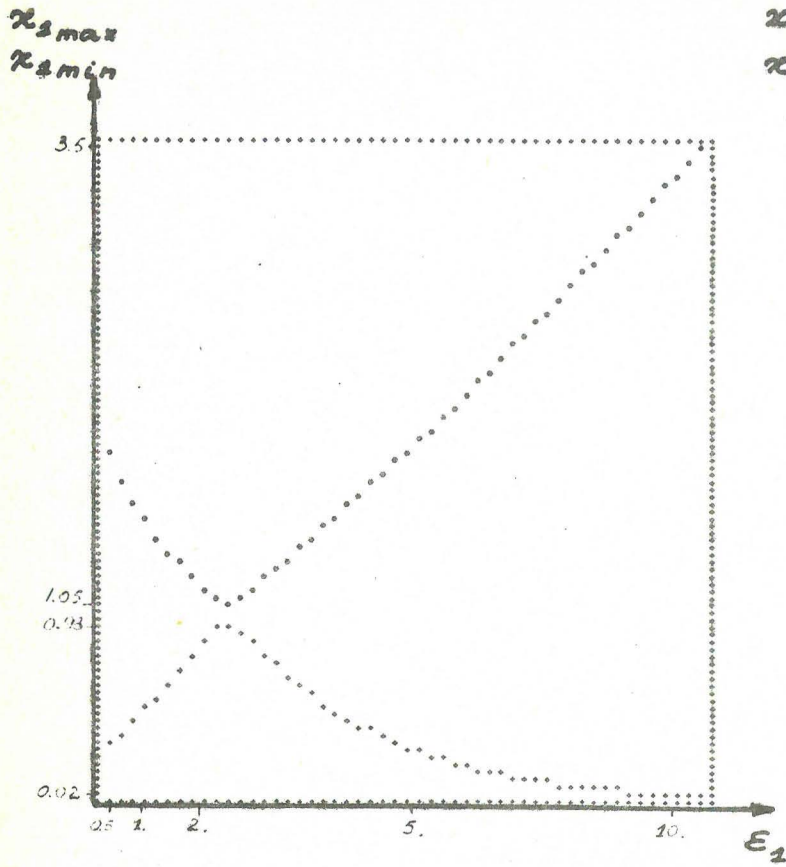


Fig. 2.b.

Variation of the amplitude of oscillations of x_2 as a function of $\epsilon_1, \epsilon_2, \epsilon_3$. All parameters and initial conditions, when not varying, have the values given in 4.a.

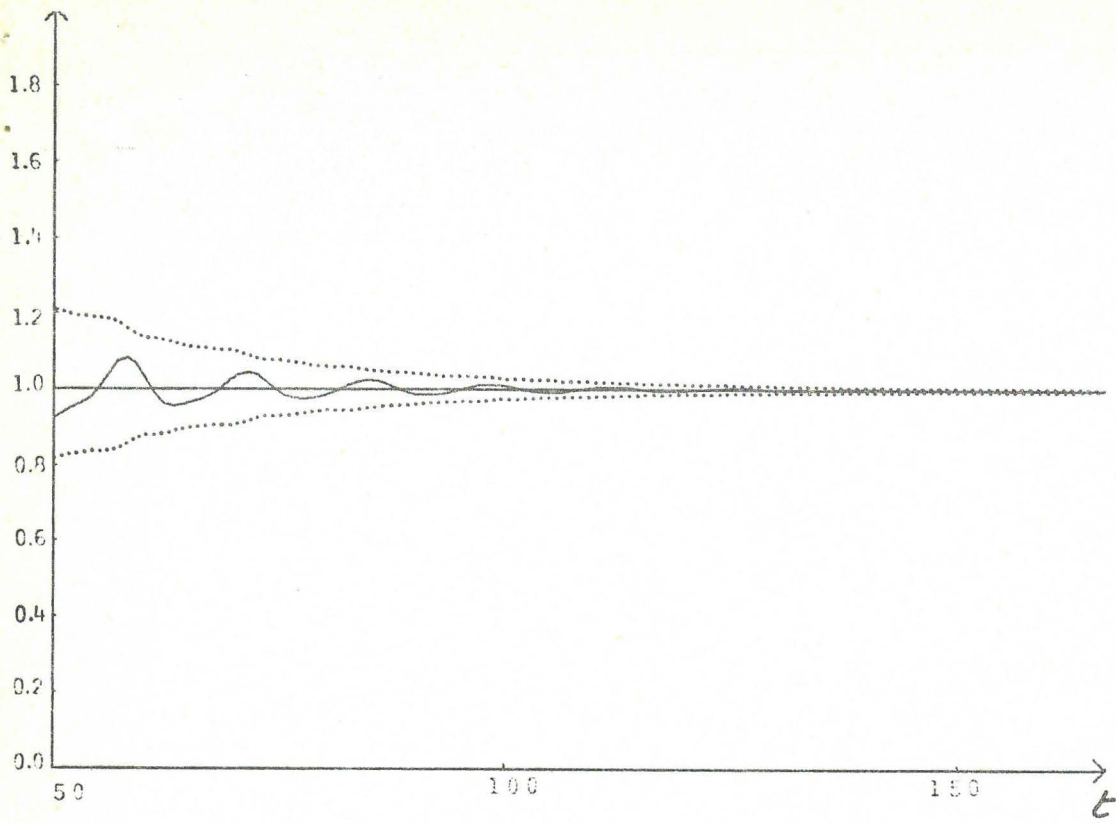


Fig. 3.a.

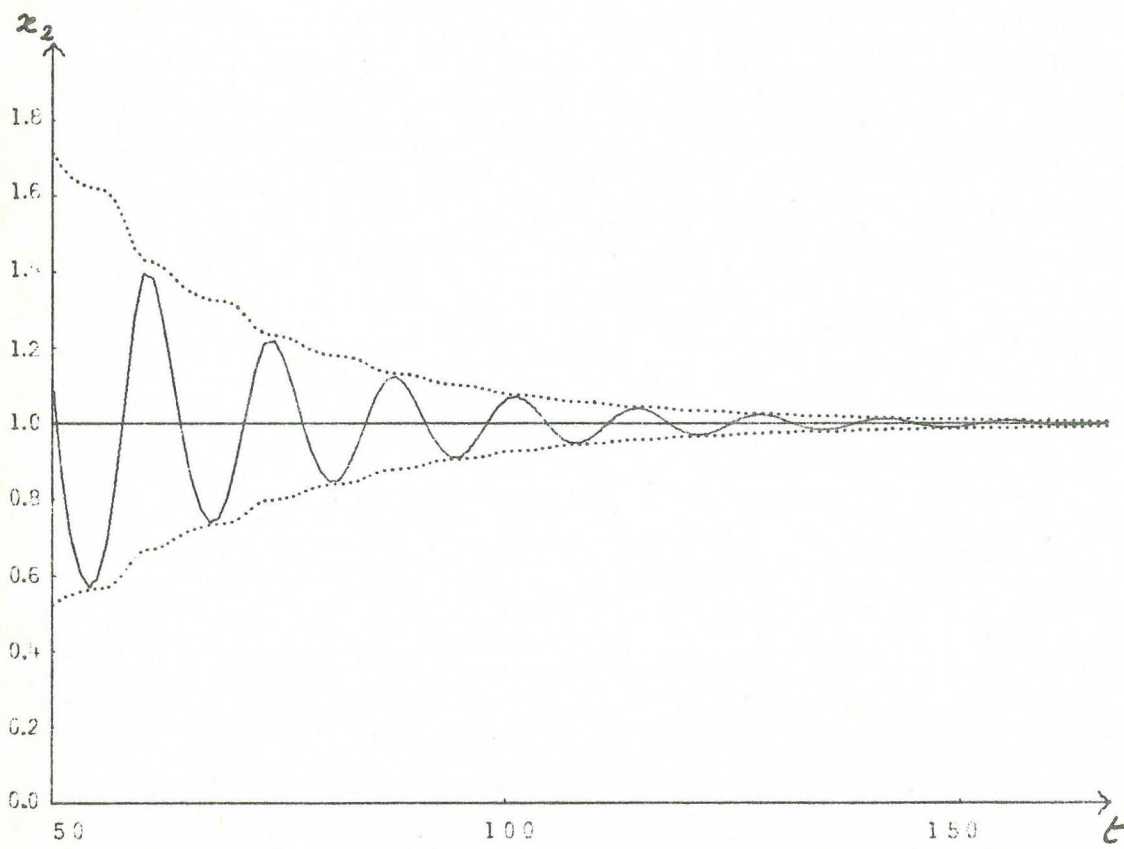


Fig. 3.b.

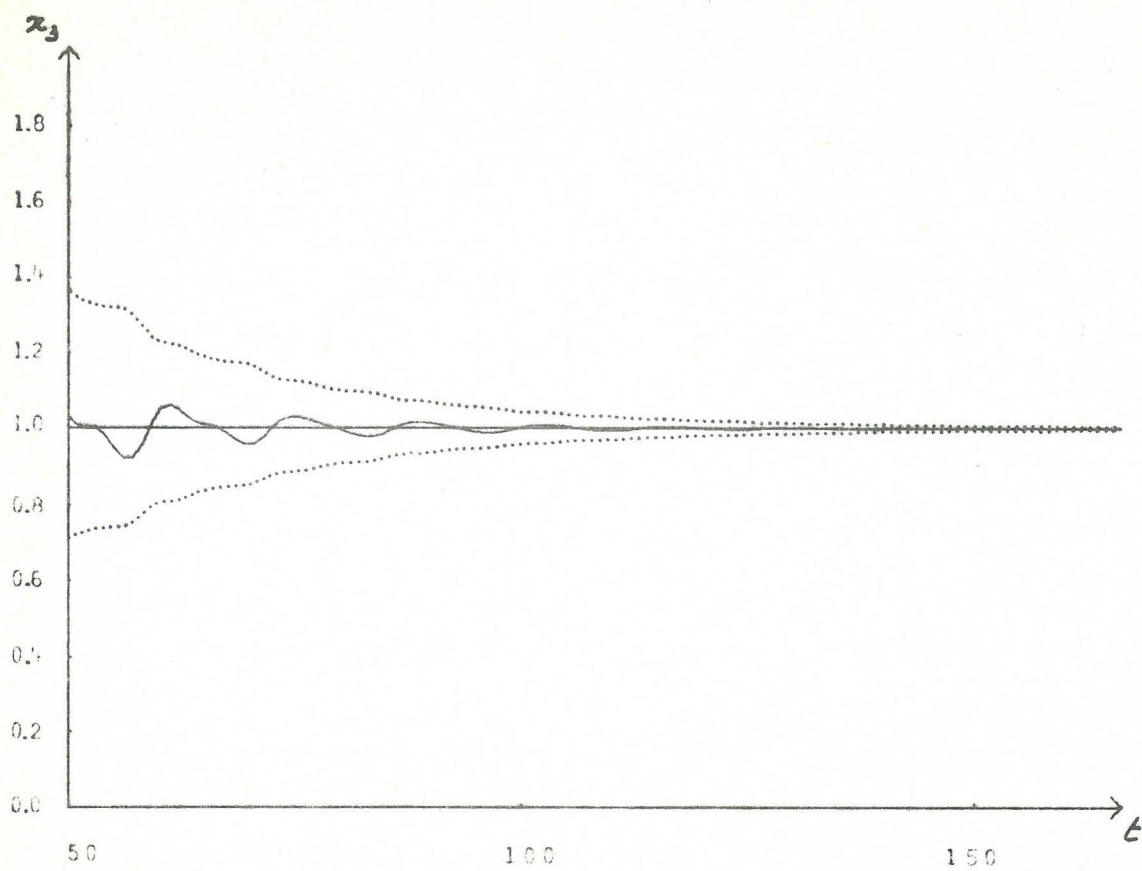


Fig. 3.e.

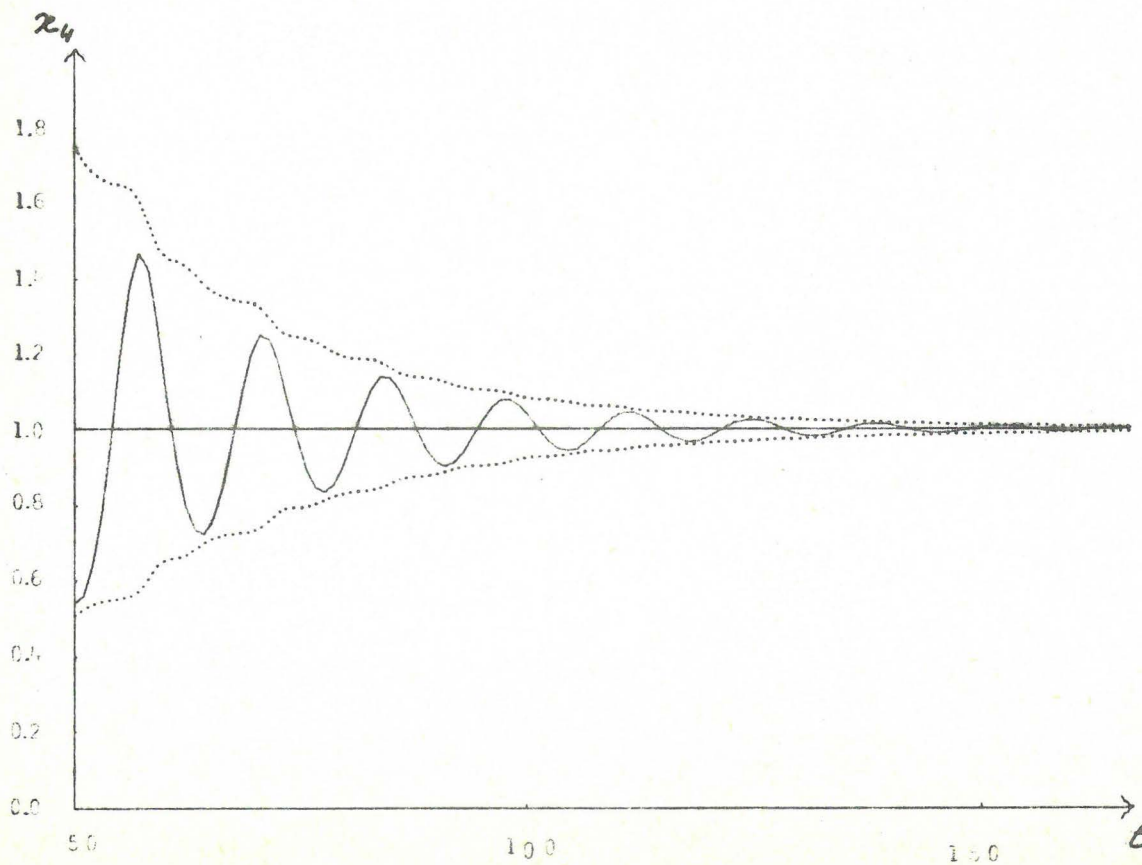


Fig. 3.d.

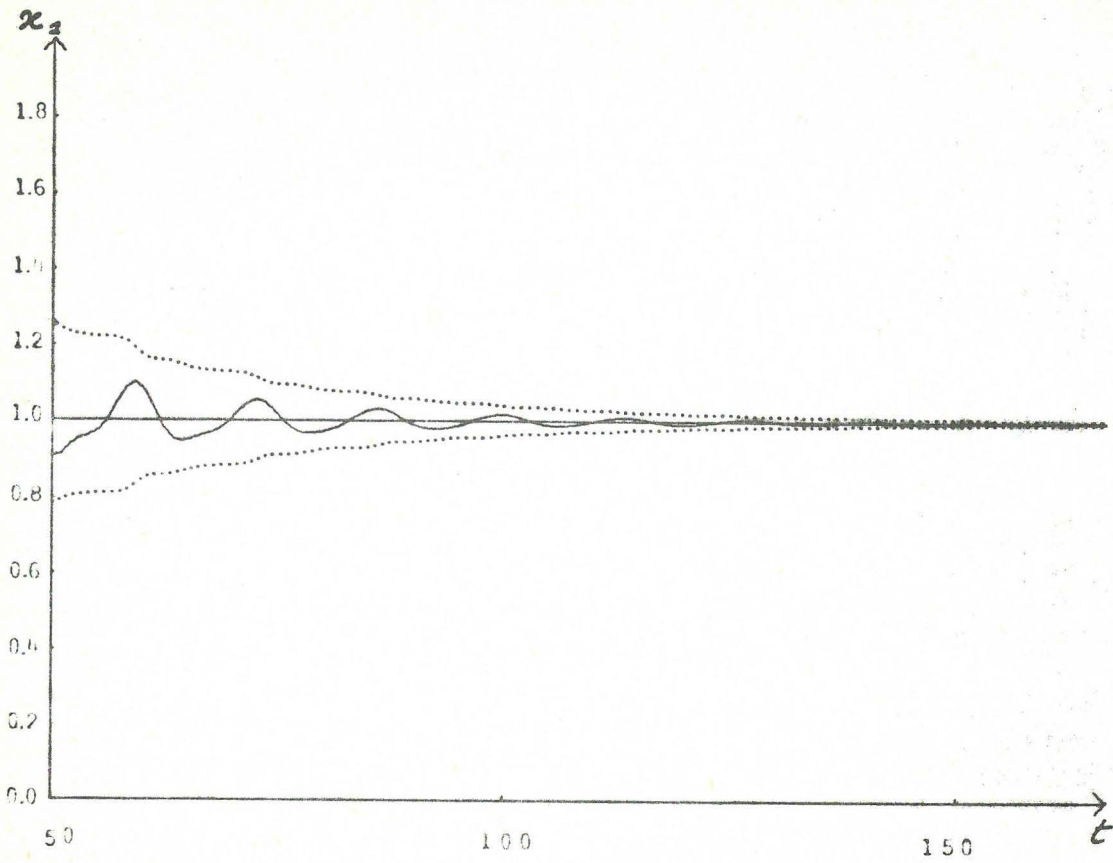


Fig. 4.a.

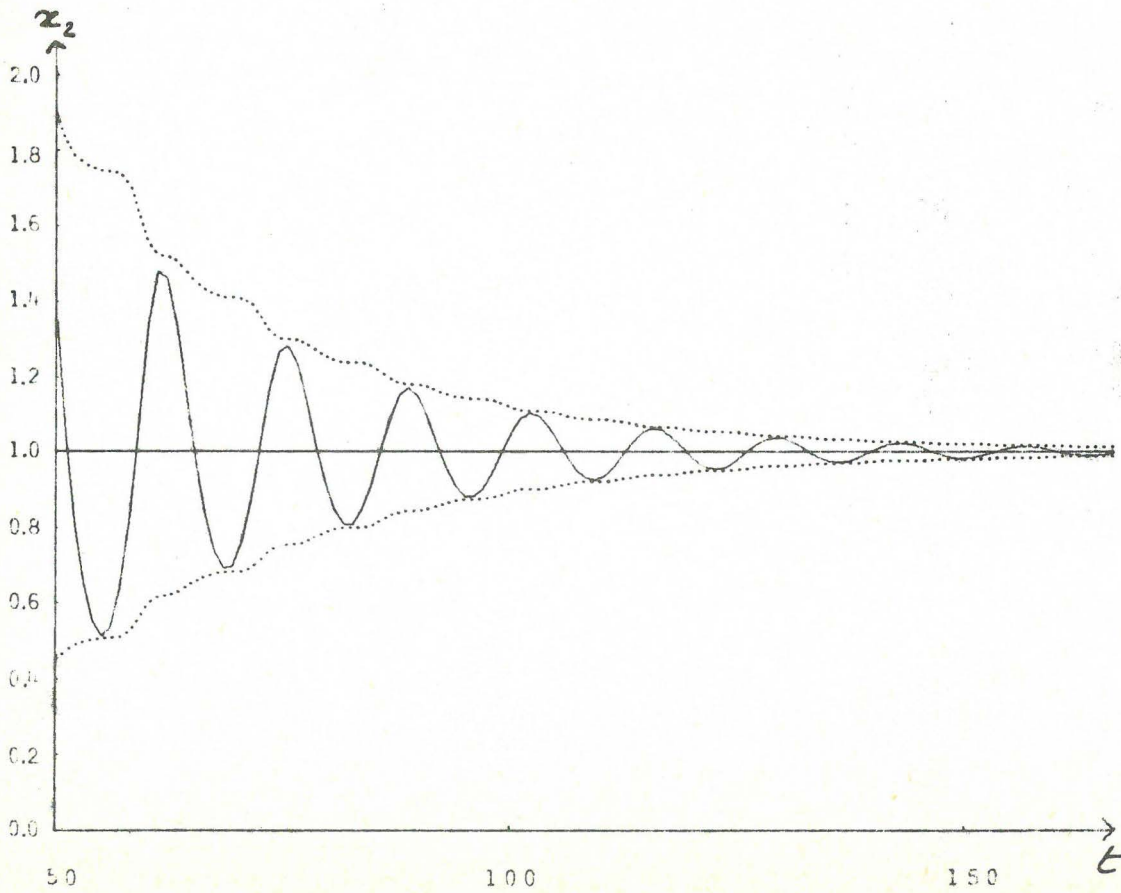


Fig. 4.b.

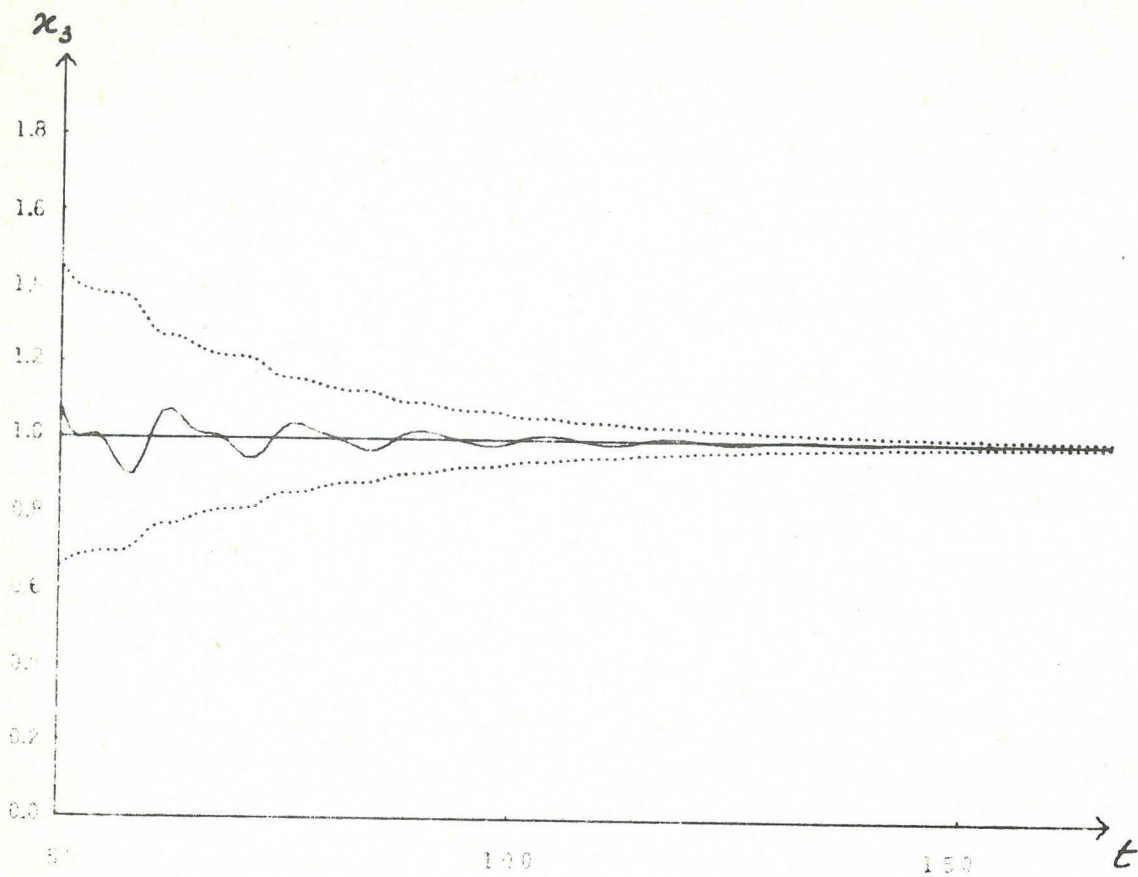


Fig. 4.c.

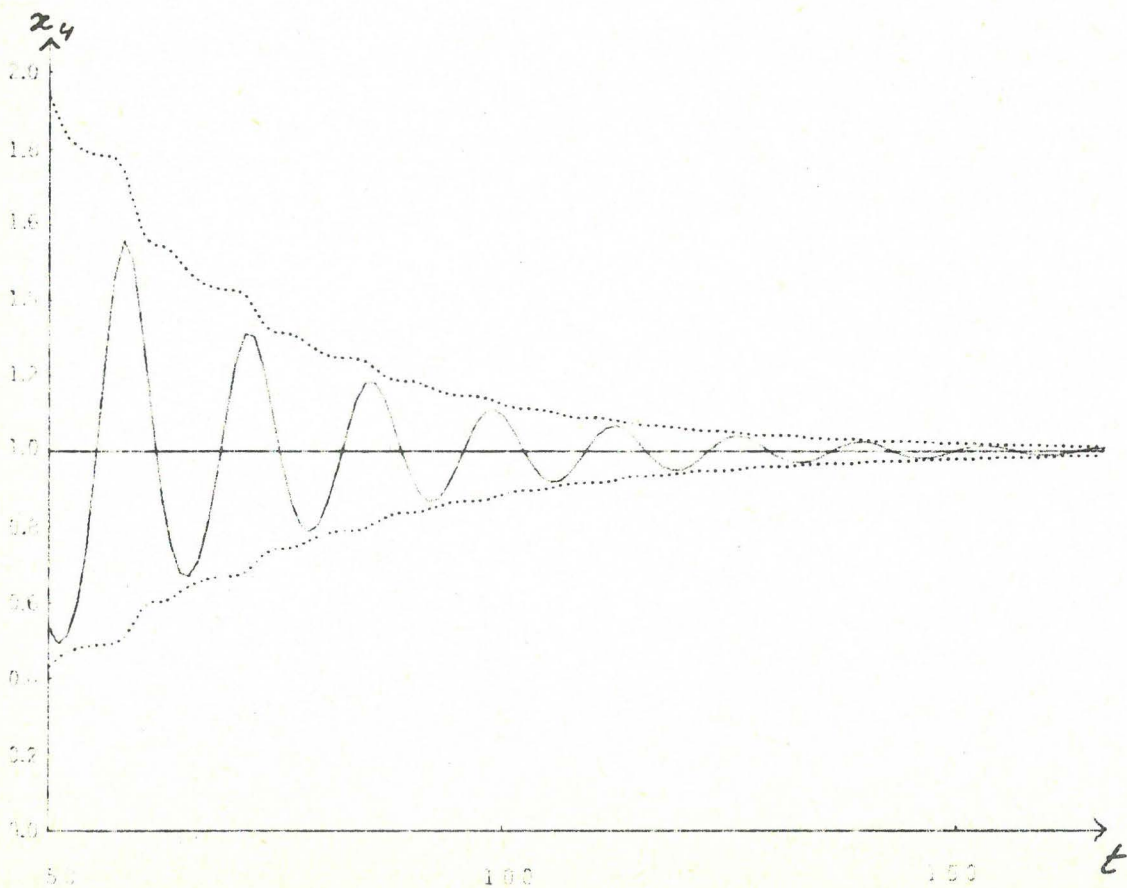


Fig. 4.d.

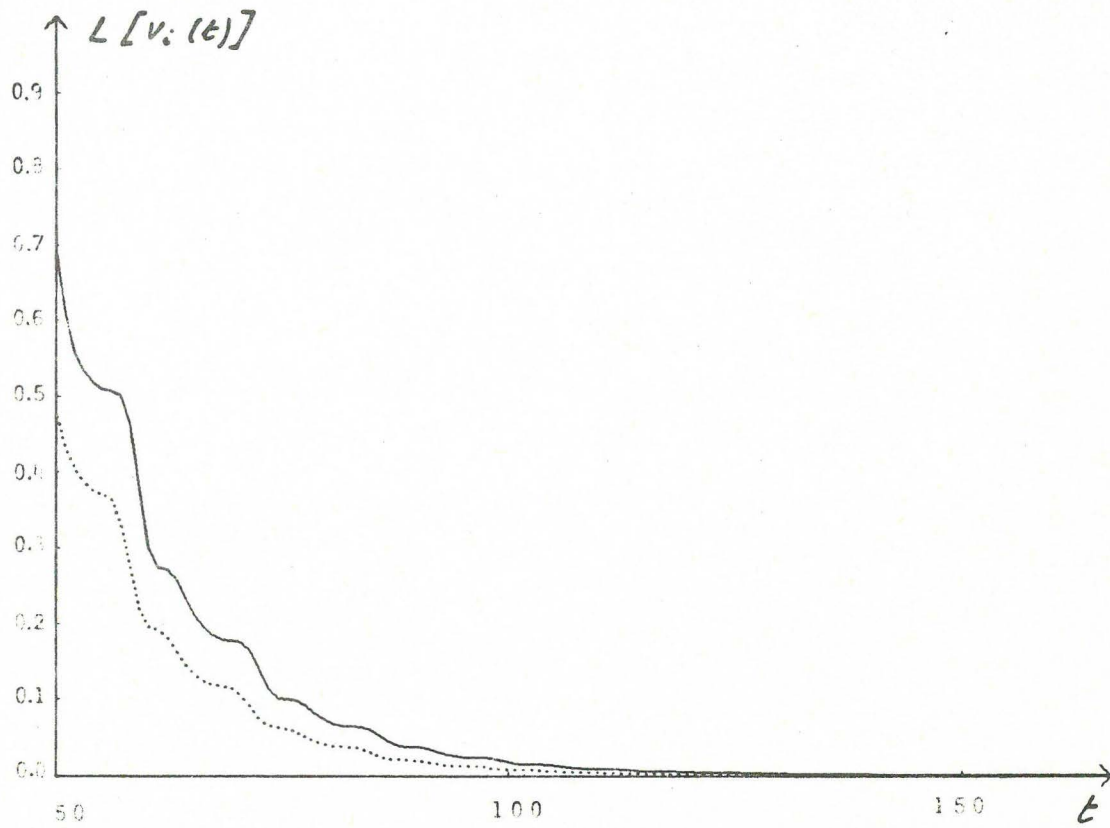


Fig. 5.

Lower curve : $\alpha_{ii} = -3 \cdot 10^{-2}$ $i = 1, 2, 3, 4$

Upper curve : $\alpha_{11} = -3 \cdot 10^{-2}$

$\alpha_{ii} = 0$ $i = 2, 3, 4$

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APPENDIX

A WEAKER ASSUMPTION FOR LIAPOUNOV'S THEOREM ON UNIFORM
ASYMPTOTIC STABILITY

Different ways of weakening the assumptions or strengthening the conclusions of Liapounov's original theorems have been given and are reviewed e.g. in Hahn (1965). The one presently proposed is aimed to allow the presence of a finite number of inflection points with horizontal tangent in the variation with time of a decreasing Liapounov function as well as the presence of a finite number of finite intervals during which the function is constant.

The notations and definitions that are used are those of Krasovskii (1963). The proof is closely parallel to that of Yoshizawa (1966).

The equations of perturbed motion are

$$\frac{dx_i}{dt} = X_i(x_1, \dots, x_n, t) \quad i = 1, \dots, n \quad (1)$$

Theorem

Suppose there exists a continuous function $v(x,t)$ defined for all $t > 0$ in some region $\|x\| < h$ that contains the unperturbed solution $x = 0$, which satisfies the following conditions :

- (i) $v(x,t)$ is definite along every trajectory of (1)
- (ii) $v(x,t)$ admits an upper bound which is infinitely small at the point $x = 0$

(iii) the total derivative $\left. \frac{dv(x,t)}{dt} \right|_{\dot{x}=X(x,t)} = 0$ for $x = 0$ and any t and, perhaps, for a countable set of points $(x^{(k)}, t_k)$ where $x^{(k)} \neq 0$ or a countable set of portions of trajectories. Anywhere else $\frac{dv}{dt}$ is of sign opposite to that of v .

Then every perturbed motion which is sufficiently close to the unperturbed trajectory $x = 0$ approaches the zero solution asymptotically.

Proof

The zero solution of (1) is uniformly stable with respect to t_0 since the assumptions of the theorem include those of the theorem on uniform stability (Krasovskii, p. 48). Hence there is a positive function $\delta = \delta(\varepsilon)$ of the positive argument $\varepsilon > 0$ such that for all t, t_0 with $t \geq t_0 > 0$, the relation $\|x(x_0, t_0, t)\| < \varepsilon$ holds for every initial value x_0 such that $\|x_0\| < \delta$. It will be shown that every solution $x(x_0, t_0, t)$ of (1) such that $t_0 > 0, \|x_0\| \leq \delta_0 = \delta(H)$, satisfies $\|x(x_0, t_0, t)\| < \delta(\varepsilon)$ at some t .

Without loss of generality, suppose $v(x, t) \geq w(\|x\|) > 0$ ($x \neq 0, 0 < t < \infty$), where $w(\|x\|)$ is a positive definite function. In this case assumption (ii) means that $v(x, t) \leq W(\|x\|)$, where $W(\|x\|)$ is continuous, $W(0) = 0$.

Suppose that, for some $\varepsilon, \delta(\varepsilon) \leq \|x(x_0, t_0, t)\| < H$ for all $t \geq t_0$. If that is true, we have $v(x, t) \geq w(\delta)$ for all $t \geq t_0$.

Let us consider

$$v(x, t) = v(x_0, t_0) + \int_{t_0}^t \left(\frac{dv}{dt}\right) dt \tag{2}$$

According to assumption (iii), there exists a $\alpha > 0$ such that $\frac{dv}{dt} < -\alpha$ along every trajectory of (1) except on some finite portions of the trajectory. At time t , let $(t_0, t_1), (t_2, t_3), \dots, (t_{2k}, t_{2k+1})$ with $t_i < t_{i+1} < t$ denote the intervals during which $\frac{dv}{dt} < -\alpha$. Outside these intervals,

$$\int_{t_{2i-1}}^{t_{2i}} \left(\frac{dv}{dt}\right) dt \text{ is certainly not positive.}$$

Consequently (2) leads to the inequality

$$v(x, t) < v(x_0, t_0) - \sum_{i=0}^k \alpha (t_{2i+1} - t_{2i}) - \alpha (t - t_{2k+1})$$

or

$$v(x, t) < v(x_0, t_0) - \alpha \left[t - t_0 - \sum_{i=2}^k (t_{2i} - t_{2i-1}) \right] \quad (3)$$

if $t > t_0 + \tau$, $\tau = \frac{1}{\alpha} [W(\delta_0) - w(\delta)] + \sum_{i=1}^k (t_{2i} - t_{2i+1})$

we have $v(x_0, t_0) - \alpha \left[t - t_0 - \sum_{i=1}^k (t_{2i}$

because $v(x_0, t_0) \leq W(\delta_0)$. This leads to $v(x, t) < w(\delta)$ which contradicts $v(x, t) \geq w(\delta)$. Thus, at some t_1 such that $t_0 \leq t_1 \leq t_0 + T$, we have $\|x(x_0, t_0, t_1)\| < \delta(\epsilon)$. Therefore, if $t \geq t_0 + T$, we have $\|x(x_0, t_0, t)\| < \epsilon$.

Remark

The theorem on asymptotic stability in the large can also be extended in a similar way.

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