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Pollution des Eaux - Projet Mer

ON A NEW DERIVATION OF THE EROSION FLUX INDUCED BY A TURBULENT FLOW

by

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Introduction

Sedimentation means the transport of matter through a solution due to an external force field. Thereby sediment layers are built up at the bottom region of rivers, estuaries and oceans. Depending on the friction velocity or shear stress acting at the layer, deposition or erosion occurs.

In this paper we concentrate on the physical behaviour of the visco-plastic sediment layer and shall derive the expression for the erosion flux by considering the action of the turbulent flow.

As is evident the results provided by marine models are only truthworthy when correct boundary imputs are known.

The erosion flux induced by a two dimensional turbulent fluid flow

Consider a dynamically smooth and isotropic viscoplastic sediment layer (sea bed) in a gravitational field acted on by a two dimensional turbulent fluid flow.

For simplicity we consider a two component non charged chemically non reacting system, for example water and a single component fine grained sediment as clay.

The sediment mass flux with respect to the barycentric motion is defined by

$$\underline{J}^s = \rho^s(\underline{v}^s - \underline{v}) \quad . \tag{1}$$

The mass conservation equation for the sediment component reads

$$\frac{\partial \rho^s}{\partial t} = - \nabla \cdot (\rho^s \underline{v}^s) \tag{2}$$

Substituting of (1) into (2) gives

$$\frac{\partial \rho^s}{\partial t} = - \nabla \cdot \underline{J}^s - \nabla \cdot (\rho^s \underline{v}) \tag{3}$$

We shall neglect temperature gradients in the sediment layer and assume isotropy with constant diffusion, D^s , and sedimentation, S^s , coefficients. The sediment mass flux with respect to the barycentric velocity is in this case given by

$$\underline{J}^s = - D^s \nabla \rho^s + S^s \rho^s \underline{g} \tag{4}$$

Substituting (4) in (3) results in

$$\frac{\partial \rho^s}{\partial t} = D^s \nabla^2 \rho^s - S^s \underline{g} \cdot \nabla \rho^s - \nabla \cdot (\rho^s \underline{v}) \tag{5}$$

It is clear that the sediment density appearing in (5) may be replaced by the specific weight. Also ρ^s in (4) may be replaced by the specific weight if $\underline{J}^s$ appearing in it is interpreted as the weight flux per unit area and time with respect to the barycentric motion.

These equations are formally the same as for a fluid. However, the sediment layer differs from a fluid - the latter cannot by definition support a shear stress at equilibrium - in that it shows visco-plastic behaviour, i.e., possesses a yield stress. That is to say, under an applied shear stress less or equal to the yield shear stress, it behaves like an elastic solid while above this stress it shows a rate of deformation which is a linear function, close to equilibrium, of the difference between the applied and yield stress. The yield stress is a function of the solid component density which varies through the depth of the sediment layer.

Introduce a coordinate system wherein the z axis points downwards normal to the interface between the fluid and flat sediment layer. We consider a sediment layer in which the solid particle density depends only on z and t . The fluid flow above it is turbulent and two dimensional. Apart from a hydrostatic pressure the fluid exerts then only a shear stress to the top of the sea or river bed.

Consequently plastic shear deformation in the sediment layer is parallel to the x-y plane. A reasonable approximation inside the layer is then to assume that the barycentric velocity component in the z direction is negligible small while its other two components are functions of z and t only. Under these restrictions (5) reduces to

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial z^2} - S\, g_z \frac{\partial \rho}{\partial z} \qquad (6)$$

wherein we have omitted the superscript s referring to the solid contamitant.

The z component of (4) is

$$J_z = - D \frac{\partial \rho}{\partial z} + S\, \rho\, g_z \qquad (7)$$

The shear stress whereby erosion finds place is uniquely related to the yield stress. Thus for a given sediment layer, with solid particle density ρ_c at the top, erosion finds place when the shear stress exerted on it by the fluid reaches a certain critical value.

To make this statement explicit let us reproduce the experimental results found by Migniot (1968) .

The yield shear stress τ_y is for a great variety of muds experimentally found to be related to the solid density ρ by

$$\tau_y = n\, \rho^m \qquad (8)$$

where n and m are constants depending on the mud, m varies between 1 and 5.

The yield stress at the top of the sediment layer, where $\rho = \rho_c$, is thus

$$\tau_y = n\ \rho_c^m \qquad (9)$$

Further it is found experimentally that the critical friction velocity U_* acting at the top of the sediment layer, whereby erosion finds place is related to the yield stress by

$$\left.\begin{aligned} U_{*c}(cm/s) &= 0.5\ \tau_y^{1/2}(dynes/cm^2) \quad \text{for } \tau_y > 15\ dynes/cm^2) \\ U_{*c}(cm/s) &= \tau_y^{1/4}\ (dynes/cm^2) \quad \text{for } \tau_y < 15\ dynes/cm^2) \end{aligned}\right\} \qquad (10)$$

or

$$U_{*c} = p\ \tau_y^q \qquad (11)$$

where p and q depend(*) on τ_y

The critical friction velocity is related to the shear stress τ_c exerted by the fluid whereby erosion finds place by

$$U_{*c} = \sqrt{\frac{\tau_c}{\rho_v}} \qquad (12)$$

wherein ρ_v is the fluid density.

Combining (9) , (11) and (12) gives the shear stress at which erosion finds place as

$$\tau_c = E\ \rho_c^B \qquad (13)$$

wherein

$$E = \rho_v\ p^2\ n^{2q} \qquad B = 2\ mq$$

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* In M.K.S. units, one has :

$$U_{*c}(m/s) = 0{,}016\ \tau^{1/2}(N/m^2) \quad \text{for } \tau_y > 1{,}5\ N/m^2$$

$$U_{*c}(m/s) = \sqrt[4]{0{,}1}\ \ \tau^{1/4}(N/m^2) \quad \text{for } \tau_y < 1{,}5\ N/m^2$$

Consider now that after some time during which the sediment layer has been able to build up, the shear stress is rather suddenly raised, say, due to the occurence of a storm. It is clear from (14) that in a short time the sea bed will be eroded violently to the depth where the solid concentration corresponds with the critical density. During such sudden change one expects the formation of mud pubbels entering the fluid which have indeed been observed. Thereafter a strong discontinuity in solid concentration across the fluid-sea bed interface exists.

We proceed to calculate the erosion flux which occurs from there on; when the applied shear stress is either constant (steady state) or changes quasi-statically. Thereto (6) has to be solved under the appropriate initial and boundary conditions. Of course, the initial condition of the sediment layer depends strictly speaking on the whole previous history up to the formation of the ocean.

Nevertheless one can form a reasonable good idea of the solid particle density through the sediment layer from experiments in the ultracentrifuge where processes are speeded up enormously.
Figure 1 taken from Fujita (1962) shows how the solid particle density distribution ρ changes in a centrifuge with inner radius r_1 and outer radius r_2. The constant angular velocity is ω and initially the two component fluid is homogeneous with sediment density ρ_o.

The important thing to observe is that for a rather long time one observes a sediment density distribution having a "plateau" region and that the change in plateau density value varies only slowly with time. Migniot (1968) who performed experiments on the settling of solid particles in a gravitational field found also the distribution with a "plateau" region.

Guided by these results we assume a solid particle density distribution as shown in fig 2 and assume that the density ρ_p at the "plateau" $z = \delta$ remains constant during erosion. The density at the interface at $z = X$, is denoted by ρ_c. As long as erosion occurs its value is determined via (13) by the shear stress acting at the interface.

For $X \leq z \leq \delta$ a parabolic sediment density distribution of the following form may be assumed

$$\rho = \rho_p + (\rho_c - \rho_p)\left(1 - \frac{z-X}{\delta-X}\right)^2 \qquad X \leq z \leq \delta \tag{14}$$

which satisfies :

$$\begin{aligned} &\text{at } z = X \quad , \quad \rho = \rho_c \\ &\text{at } z = \delta \quad , \quad \rho = \rho_p \quad \text{and} \quad \frac{\partial \rho}{\partial z}(\delta, t) = 0 \end{aligned} \tag{15}$$

The introduced X and δ in (14) are, of course, unknown functions of the time. To determine them with (6) we need one additional boundary condition.

Thereto we set up the mass balance equation at the interface. To establish this relation, consider a control volume v bounded by a surface A containing the interface I, defined by $\rho_c(t)$ located at $z = X$ at time t (see fig. 3). This control volume moves together with the interface at the velocity $\underline{w}$.

Locally, the mass conservation law for the sediment is given by

$$\frac{\partial}{\partial t} \rho^s = - \operatorname{div} \rho^s \underline{v}^s$$

By integration over the moving control volume v and by using Gauss' theorem, one obtains

$$\int_v \frac{\partial}{\partial t} \rho^s dv = - \int_A \rho^s \underline{v}^s \cdot \underline{n} \, dA \tag{16}$$

According to Reynold's relation

$$\frac{d}{dt} \int_{v(t)} \rho^s dv = \int_{v(t)} \frac{\partial \rho^s}{\partial t} dv + \int_{A(t)} \rho^s \underline{w} \cdot \underline{n} \, dA \ , \tag{17}$$

expression (16) may be written as

$$\frac{d}{dt} \int_v \rho^s dv = - \int_A \rho^s (\underline{v}^s - \underline{w}) \cdot \underline{n} \, dA \ .$$

Let now the control volume go to zero. Denoting respectively by + and - the lower and the upper face adjacent to the interface, the above relation leads to

$$\lim_{v\to 0} \frac{d}{dt} \int_v \rho^s dv = - \int_I \left(\rho_+^s \left(v_+^s - \underline{w}\right) \cdot \underline{n}_+ + \rho_-^s \left(v_-^s - \underline{w}\right) \cdot \underline{n}_-\right) dA \tag{18}$$

The first term on the r.h.s. represents the z component J_z of the erosion flux through the moving surface as seen by an observer moving with the interface; the second term is equal to minus the z component of the solid particle flux which crosses the interface from the viscous sub-layer ; let us denote it $- J_{z,v}$. The l.h.s. represents the mass leaving the control volume per unit time due to erosion ; this quantity is clearly negative and given by

$$\lim_{v\to 0} \frac{d}{dt} \int_v \rho^s dv = - \rho_c^s \frac{dX}{dt}$$

where ρ_c^s is the density at the interface and $\frac{dX}{dt}$ the positive volume swept per unit time (for convenience the area of the interface has been taken equal to unity). According to these results, (18) may be written as follows :

$$- \rho_c^s \frac{dX}{dt} = - J_z + J_{z,v}$$

wherein we have omitted superscript s.

Combining this result with (7) and noting that $g_z = + g$ gives for the downward erosion flux across the interface

$$J_z = J_{z,v} + \rho_c \frac{dX}{dt} = \sigma \rho_c - D \frac{\partial \rho}{\partial z} (X,t) \qquad (19)$$

where we have put

$$\sigma = S g$$

To determine the flux $J_{z,v}$ which crosses from the fluid to the sediment layer we calculate the diffusion and sedimentation flux throug the viscous sublayer with thickness δ_v .
At the top of the viscous sublayer the particle density is ρ_b and at the interface its density is ρ_c .

Following Nihoul (1973) it is reasonable to approximate ρ_b by the depth averaged density over the fluid if the water column is rather well mixed as is observed in the Southern Bight of the North Sea.

The diffusion flux through the viscous sublayer, counted positive downwards, is

$$\hat{J}_{z,diff} = - D_v \frac{\rho_c - \rho_b}{\delta_v} = - L (\rho_c - \rho_b) \qquad (20)$$

wherein the transport coefficient

$$L = \frac{D_v}{\delta_v} > 0 \qquad (21)$$

is similar to the coefficient of heat transfer across a temperature dis continuity in heat conduction.

In addition to this diffusion flux there is a sediment flux through the viscous sublayer. Its average, counted positive downwards is

$$\hat{J}_{z,sed} = \frac{1}{\delta_v} \int_0^{\delta_v} J_{z,sed}\, dz = \frac{1}{\delta_v} \int_0^{\delta_v} \sigma_v\, g\, dz \quad . \qquad (22)$$

We shall assume that $\sigma_v = S_v g > 0$ is constant and that ρ varies linearly through the viscous sublayer(*), i.e.,

$$\rho_v = \frac{\rho_c - \rho_b}{\delta_v} z + \rho_b \tag{23}$$

Then (22) becomes

$$\hat{J}_{z,sed} = \frac{1}{2}\sigma_v(\rho_b + \rho_c) \tag{24}$$

The total flux through the sublayer, counted positive downwards, is found by adding (20) and (24) :

$$J_{z,v} = L(\rho_b - \rho_c) + \frac{1}{2}\sigma_v(\rho_b + \rho_c) \tag{25}$$

The viscous layer thickness appearing in (21) is (during erosion) related to the critical friction velocity U_{*c}, the fluid kinematic viscosity ν and the fluid density ρ_v by (Monin, 1965)

$$\delta_v = \alpha_v \frac{\nu}{U_{*c}} = \alpha_v \nu \sqrt{\frac{\rho_v}{\tau_c}} \tag{26}$$

where τ_c is the critical shear stress acting at the interface between fluid and sediment layer and α_v is a universal constant of order unity.

Hence

$$L = \frac{D_v U_{*c}}{\alpha_v \nu} = \frac{D_v}{\alpha_v \nu} \rho_v^{-1/2} \tau_c^{1/2} \tag{27}$$

The relation (19) with $J_{z,v}$ given by (25) and the L appearing herein by (27) determine the additional boundary condition to (15) With them we shall proceed to solve the differential equation (6) ; its solution is approximated by assuming the parabolic distribution given by (14) and yields the erosion flux.

(*) In the final approximate equation σ_v does not appear. Therefore the exact variation of solid particle density in the sublayer is in this approximation of no importance.

Integration of (6) between $z = X$ and $z = \delta$ yields

$$\int_{z=X(t)}^{z=\delta(t)} \frac{\partial \rho}{\partial t}\, dz = -\sigma \int_{X(t)}^{\delta(t)} \frac{\partial \rho}{\partial z}\, dz + D \int_{X(t)}^{\delta(t)} \frac{\partial^2 \rho}{\partial z^2}\, dz$$

$$= -\sigma\,(\rho_p - \rho_c) + \frac{2D}{\delta - X}\,(\rho_c - \rho_p) \qquad (28)$$

where we have made use of $\frac{\partial \rho}{\partial z}(\delta, t) = 0$ and

$$\frac{\partial \rho}{\partial z}(X,t) = -\frac{2}{\delta - X}\,(\rho_c - \rho_p) \qquad (29)$$

which follows from (14) .

Now by Leibnitz rule we have

$$\frac{d}{dt} \int_{z=X(t)}^{z=\delta(t)} \rho(z,t)\,dz = \int_X^{\delta} \frac{\partial \rho}{\partial z}\, dz + \rho(\delta,t)\frac{d\delta}{dt} - \rho(X,t)\frac{dX}{dt} \qquad (30)$$

Combining (28) and (30) and using (15) gives

$$\frac{d}{dt} \int_X^{\delta} \rho\, dz = \rho_p \frac{d\delta}{dt} - \rho_c \frac{dX}{dt} + \sigma\,(\rho_c - \rho_p) + \frac{2D}{\delta - X}\,(\rho_c - \rho_p) \qquad (31)$$

We proceed to evaluate the left hand side of this equation by means of (14) . We find

$$\int_X^{\delta} \rho\, dz = \frac{\delta - X}{3}\,(4\,\rho_p - \rho_c) \qquad (32)$$

We have pointed out that it is a good approximation to assume that ρ_p is constant, see fig. 1 . For a <u>steady</u> erosion flux which requires a constant applied shear stress at the top of the viscoplastic sediment layer one finds from (13) that ρ_c is constant. Hence

$$\frac{d}{dt} \int_X^{\delta} \rho\, dz = \left(\frac{d\delta}{dt} - \frac{dX}{dt}\right)\left(\frac{4\,\rho_p - \rho_c}{3}\right) \qquad (33)$$

Thus for this case (31) reduces to :

$$\frac{4\ \rho_p - \rho_c}{3}\left(\frac{d\delta}{dt} - \frac{dX}{dt}\right) = \rho_p \frac{d\delta}{dt} - \rho_c \frac{dX}{dt} + \sigma(\rho_c - \rho_p) + \frac{2D}{\delta - X}(\rho_c - \rho_p) \quad (34)$$

In addition to this relation we obtain a second one when (29) is substituted in (19)

$$J_z = J_{z,v} + \rho_c \frac{dX}{dt} \quad (35a)$$

$$= \sigma\ \rho_c + 2D\ \frac{\rho_c - \rho_p}{\delta - X} \quad (35b)$$

In the stationary state J_z and $J_{z,v}$ are constant. Hence it follows from (35a) that $\frac{dX}{dt}$ must be constant. It follows now from (35b) that $\delta - X$ must also be constant.

Thus

$$\frac{d\delta}{dt} = \frac{dX}{dt} \quad (36)$$

Whence for the stationary state we obtain from (34)

$$(\rho_c - \rho_p)\frac{dX}{dt} = \sigma(\rho_c - \rho_p) + \frac{2D}{\delta - X}(\rho_c - \rho_p) \quad (37)$$

Eliminating $\frac{2D}{\delta - X}(\rho_c - \rho_p)$ between (35b) and (37) gives

$$\frac{dX}{dt} = \sigma - \frac{J_{z,v}}{\rho_p} \quad (38)$$

Substituting this result in (35a) gives for the stationary erosion flux

$$J_z = \rho_c\ \sigma + \left(1 - \frac{\rho_c}{\rho_p}\right) J_{z,v} \quad (39)$$

The erosion flux, J, for the stationary state counted as <u>positive upwards</u> , i.e., $J = - J_z$, is finally found when (25) is substituted into (39) as

$$J = -\rho_c\sigma + \left(\frac{\rho_c}{\rho_p} - 1\right)\left\{L(\rho_b - \rho_c) + \frac{1}{2}\sigma_v(\rho_b + \rho_c)\right\} \quad (40)$$

wherein

$$\rho_c = \left(\frac{\tau_c}{E}\right)^{1/B} \tag{41}$$

as follows from (13) . L which is also a function of τ_c is given by (27).

The equation (40) has been derived under the assumption that erosion occurs. The auxiliary condition herefore is as follows : when the instantaneous sediment density at the top of the bed is ρ_c then erosion finds place if and only if the shear stress is larger or equal to the critical value determined by (41).

This loading condition can also be referred to the initial bed state. Consider a fluid flow such that no erosion finds place. (Either sediment deposition finds place or no deposition and no erosion). Denote the sediment density at the top of the sediment bed in this state by ρ_o . The minimum shear stress necessary for the onset of erosion follows from (13) as

$$\tau_o = E\,\rho_o^B \tag{42}$$

For a (positive) stress τ_{o-} which is an arbitrary small fraction less than τ_o no erosion occurs. Thus (40) gives the steady state erosion flux for monotonically increasing stresses τ_c such that

$$\tau_c > \tau_{o-} \tag{43}$$

Some authors (Nihoul 1973, Cormault...) have advanced a relation for the erosion flux of the following form

$$J = \alpha(\tau_c - \tau_{o-}) \quad \text{for} \quad \tau_c > \tau_{o-} \tag{44}$$

wherein α is a positive coefficient.

To arrive at this relation assume that

$$L >> \sigma_v \quad , \quad L >> \sigma \tag{45}$$

and the much more restrictive condition

$$\frac{\rho_c}{\rho_p} << 1 \tag{46}$$

Then (40) reduces to, when use is made of (41)

$$J = L\left\{\left(\frac{\tau_c}{E}\right)^{1/B} - \rho_b\right\} \quad \text{for} \quad \tau_c > \tau_{o-} \qquad (47)$$

As $J = 0$ for $\tau_c = \tau_{o-}$ we have

$$\rho_{b_o} = \left(\frac{\tau_{o-}}{E}\right)^{1/B} \qquad (48)$$

and thus if ρ_b is constant,

$$J = \frac{L}{E^{1/B}}\left[\tau_c^{1/B} - \tau_{o-}^{1/B}\right] \quad \text{for} \quad \tau_c > \tau_{o-} \qquad (49)$$

When $B = 1$ (say $q = \frac{1}{4}$ and $m = 2$) this relation reduces to

$$J = \frac{L}{E}\left(\tau_c - \tau_{o-}\right) \text{ for } \tau_c > \tau_{o-} \qquad (50)$$

which is formally the same as (44). However L is not a constant. Substituting (27) into (50) yields

$$J = \frac{D_v}{\alpha_v\ \nu}\ \frac{1}{\rho_v^{1/2}\ E}\left(\tau_c - \tau_o\right)\tau_c^{1/2} \quad \text{for} \quad \tau > \tau_{o-} \qquad (51)$$

This relation has been derived earlier by Bagnold (1956) on entirely different grounds(*). The quantity $\frac{D_v}{\alpha_v\ \nu}\ \frac{1}{\rho_v^{1/2}\ E}$ in (51) which varies for different sediments is then not determined from theory but taken from an experimental graph where it is plotted against solid particle diameter.

The relation has been verified experimentally, at least for a sand bed with excellent fit. However in Bagnold's derivation peculiar energies, as available energy and useful work, are used which have no meaning in thermodynamics. On these grounds his derivation has been criticized by Yalin (1972).

Of course the equations we have derived should apply equally to the flow of air over a dust layer, at least when the latter has similar properties as the sedimentation layer we considered above.

(*) Bagnold calls it the bed-load which is actually the ill defined sediment flux taken along by the fluid in the neighbourhood of the sediment layer.

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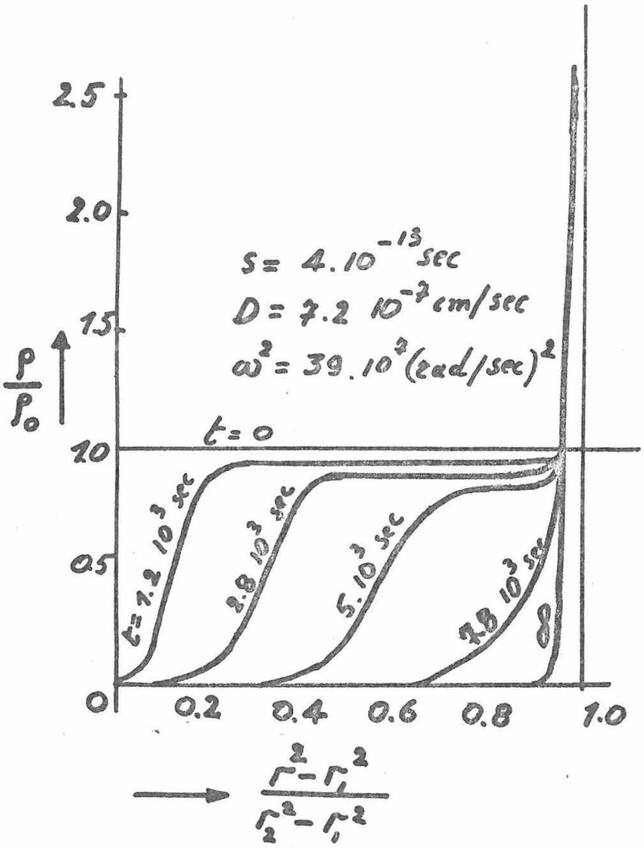

2.5
2.0
1.5
1.0
0.5
0
0.2
0.4
0.6
0.8
1.0
$s = 4.10^{-13}$ sec
$D = 7.2\ 10^{-7}$ cm/sec
$\omega^2 = 39.10^7 (rad/sec)^2$
$\frac{\rho}{\rho_0}$
t = 0
$t = 1.2\ 10^3$ sec
$2.8\ 10^3$ sec
5.10^3 sec
$7.8\ 10^3$ sec
∞
$\frac{r^2 - r_1^2}{r_2^2 - r_1^2}$

FIG. 1

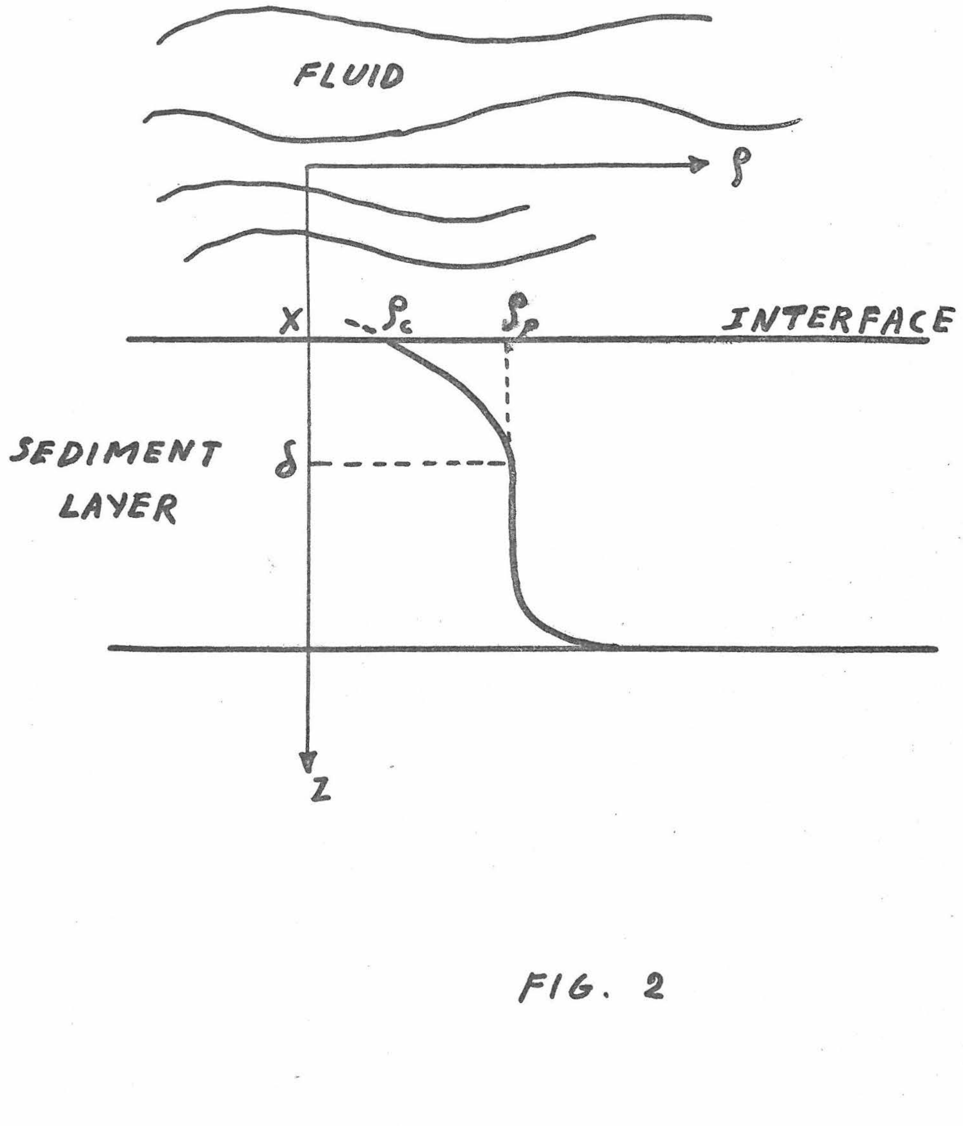

FIG. 2

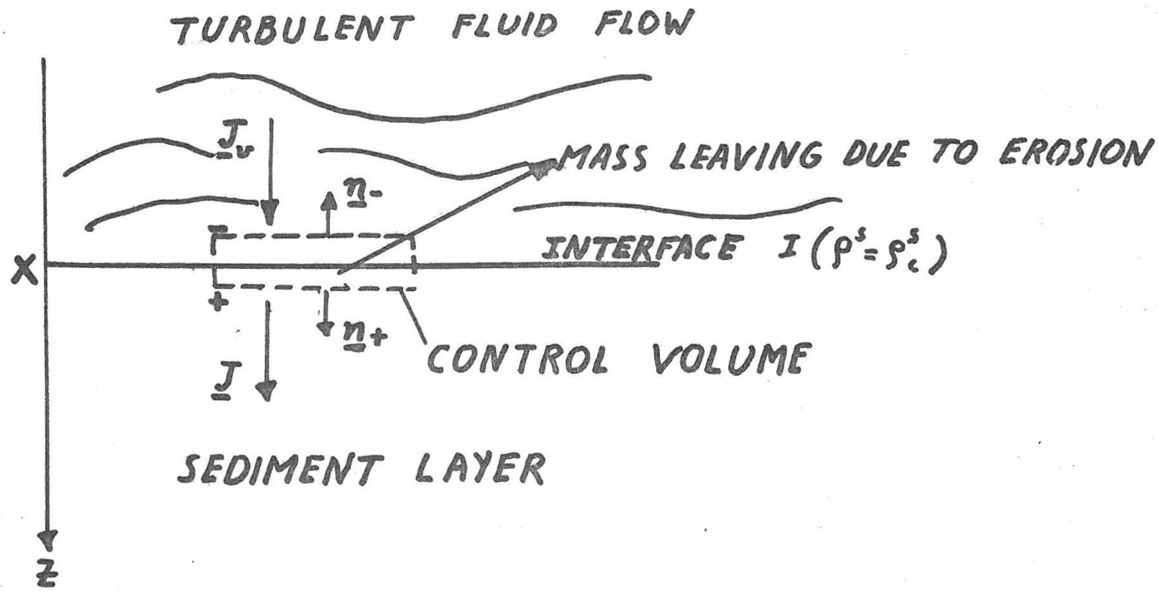
TURBULENT FLUID FLOW
J_v
n-
MASS LEAVING DUE TO EROSION
-
INTERFACE I (ρ^s = ρ^s_c)
X
+
n+
CONTROL VOLUME
J
SEDIMENT LAYER
Z

FIG. 3