ON THE

MATHEMATICAL INVESTIGATION OF OCEAN CURRENTS,

BY

J. W. SANDSTRÖM & B. HELLAND-HANSEN.

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NOTE.—This Paper, by Dr. J. W. Sandström and Mr. B. Helland-Hansen, was first published under the title "Ueber Die Berechnung von Meeresströmungen," in the Report on Norwegian Fishery and Marine Investigations, Vol. II., No. 4, Bergen, 1903. The great importance of the subject, and the fact that the original Paper was unknown to, and beyond the reach of, English Physicists, seemed to me to justify its reproduction, and I have accordingly made the following translation, sanctioned and revised by the Authors.—D.W.T.

ON THE MATHEMATICAL INVESTIGATION OF OCEAN CURRENTS,

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J. W. SANDSTRÖM & B. HELLAND-HANSEN.

In a discussion of the laws that govern the movements of the sea, we must continually hark back to the fundamental principles of mechanics. But we have so far a choice of ways, that we may either apply these laws directly in their original and simple form, or transmute them first into new and more developed forms, designed for the end in view.

Hitherto Hydrographers have worked throughout according to the former plan; and Mohn's great work on the Dynamics of the North Atlantic may be cited as an apt example. He calculated the distribution of pressure by help of the actual observed weight of the stratum of water; then from the distribution of pressure he deduced the current-movements, taking into account the influence of the wind and of the earth's rotation.

The practical application of the method, as Mohn's work shows, involves very laborious numerical calculations, and we may readily suppose that the ends may be more easily attained by introducing the necessary dynamical principles in more highly developed forms.

The next step, then, which suggests itself, is to attempt to apply the general laws of Helmholtz on the vortex motion of fluids*, or the similar laws which Lord Kelvin has given for the circulation of a curve of fluid particles†.

But for immediate application to the current-movements of the sea, these laws are too specialised. In their treatment of the problem, Helmholtz and Lord Kelvin have introduced certain conditional limitations, the former assuming that the density remains constant at all points, the latter, that the density alters only with the pressure; furthermore, it is taken for granted by both that the fluid be frictionless, and both have left the rotation of the earth wholly out of consideration. Their results, accordingly, cannot, without further elaboration, be applied to the case of the ocean moving on the surface of the rotating earth, since the water is not exempt from friction and since the specific gravity of the water varies and is dependent not only upon pressure, but also on its own temperature and salinity.

Now, however, Bjerknes has generalised these laws of vortices and of circulation, so that they are independent of any limitations on the score of density.‡ By the help of Bjerknes' generalisation, we may calculate the circulation as Lord Kelvin has done, no matter how the density may change. The influence of the earth's rotation and of friction is also taken into account by Bjerknes. An essential basis of simplification and of general applicability results from the fact that the *same unit* is introduced for the influence of density, of terrestrial rotation and of friction, with the result that a few small and handy Tables suffice for the calculation of the various elements.

Lord Kelvin has, as we have already mentioned, studied the movement of a series of water-particles which form a closed curve. From the velocity of the individual particles, he obtains the tangential components along the curve, whose sum he calls the *circulation of the curve.* If then we denominate the tangential components by v_t and a longitudinal element of the curve by ds, the circulation, C_a, is expressed by the integral—

$$C_a = \int v_t ds. \quad (1)$$

In the integration, we follow the whole curve once round. The practical calculation of this integral is most simply performed by a graphic method, representing the length s of the curve by abscissae and the tangential velocity v_t by ordinates in a rectangular system of co-ordinates, and so measuring the area which the integral (1) represents. Alternatively, we may represent the mean tangential velocity of the particles and multiply this by the length of the curve.

Lord Kelvin, in defining his conception of circulation, has only taken absolute movements into consideration. Accordingly, by Lord Kelvin's definition, circulation is that which one discovers by observing the velocities from a fixed point in space. This circulation we shall call in the sequel the *absolute circulation.*

In the meantime, we are not observing absolute movements, but movements relative to the rotating earth. By methods precisely analogous to these which deal with absolute movements, we may also calculate from the relative movements another circulation that we shall call the *relative circulation*, and denominate by C_r. If u be the velocity relative to the earth, we have, in analogy with (1) :—

$$C_r = \int u_t ds. \quad (2)$$

Bjerknes has deduced the following simple relation§ :—

$$C_a = C_r + 2\omega S, \quad (3)$$

where ω represents the angular velocity of the earth and S the area of the closed curve (as projected) on the equatorial plane. By help of this formula, we can very easily calculate the absolute circulation when we know the relative circulation, and *vice versâ.*

* H. Helmholtz, Wissenschäftliche Abhandlungen, T. I, p. 101.

† Sir W. Thomson: On Vortex Motion. Trans. Roy. Soc., Edinb., 1869, p. 217.

‡ V. Bjerknes: Ueber einen hydrodynamischen Fundamentalsatz, und seine Anwendung besonders auf die Mechanik der Atmosphäre und des Weltmeeres. Kongl. Sv. Vet. Akad. Handlingar, Vol. 31, No. 4. Stockholm, 1898.

§ V. Bjerknes: Cirkulation relativ zu der Erde. Oefversigt af Kongl. Vet. Akad. Handl., No. 10. Stockholm, 1901.

Bjerknes has calculated the change of the relative circulation with the time $\left(\frac{dC_r}{dt}\right)$, on the assumption that the movement of the water is influenced only by gravity, by the distribution of density and of pressure, by terrestrial rotation and by friction.* This value is equal to the sum of the tangential components of the acceleration of the particles upon the curve. If we denominate these tangential components by $\dot{u}_t$, then :—

$$\frac{dC_r}{dt}=\int\dot{u}_t ds. \tag{4}$$

When the tangential acceleration $\dot{u}_t$ is known in a sufficient number of particles of the curve, then $\frac{dC_r}{dt}$ may be calculated from the tangential velocity in the same way as the value C.

In figure (1), SS represents a portion of the closed curve, and TT the tangent to the same at the point m. The water-particle (m) is acted upon by several forces, the several accelerations due to which are indicated by arrows in the

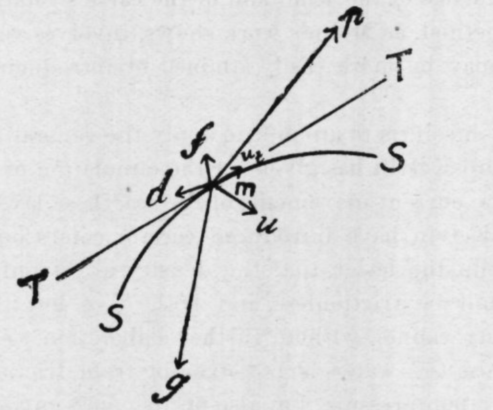

FIG. 1.

figure. The arrow mg gives the direction and magnitude of the acceleration due to gravity, and in like manner the other arrows mp, md, and mf, indicate the accelerations due to pressure, terrestrial rotation and friction. By compounding these accelerations according to the parallelogram of forces, we obtain the resultant acceleration $\dot{u}$ of the particle. The projection $\dot{u}_t$ of the vector $\dot{u}$ upon this line, is then the tangential acceleration of the particle m upon the closed curve. If this be integrated along the whole curve, we obtain as our result the value $\frac{dC_r}{dt}$ (4). The same value $\dot{u}_t$ may also be found by drawing and then summing the tangential components of the various vectors. If the index (t) indicate the tangential components, then :—

$$\dot{u}_t=g_t+p_t+d_t+f_t. \tag{5}$$

If this be incorporated with (4) we have :—

$$\frac{dC_r}{dt}=\int g_t ds+\int p_t ds+\int d_t ds+\int f_t ds. \tag{6}$$

Each one of these integrals may be calculated graphically in the same way as the circulation was calculated from formula (1), if only we are acquainted with the values g_t, p_t, d_t, and f_t at a sufficient number of points of the closed curve. The integrals, however, may be translated into other forms better adapted for calculation in actual practice.

The value g_t in the first integral in equation (6) is the component of the acceleration due to gravity along the linear element ds. Consequently the value $g_t ds$ is the work which we must perform in displacing a unit of mass along the linear element (ds) in order to overcome the force of gravity. The integral $g_t ds$ is therefore the work which must be done to overcome gravity in displacing the unit of mass along the entire closed curve s. This work is, however, nil, because our unit of mass is restored to its original position, that is to say, gravity has no influence upon the circulation of a closed curve of water-particles. We have, therefore :—

$$\int g_t ds=0. \tag{7}$$

The value p_t in the second integral of (6) is the component along the linear element ds of the acceleration induced by the distribution of pressure. This is proportional to the change of pressure $\frac{dp}{ds}$ along the linear element ds, and inversely proportional to ρ, the density of the water. We have accordingly :—

$$p_t=-\frac{1}{\rho}\cdot\frac{dp}{ds}.$$

* V. Bjerknes : Circulation relativ zu der Erde, *loc. cit.*

The negative sign indicates that p_t acts in the direction of diminishing pressure. If, instead of the density ρ, we make use of the specific volume $v=\frac{1}{\rho}$, we obtain :—

$$p_t=-v\frac{dp}{ds}.$$

If this be introduced in the second integral in equation (6), we obtain :—

$$\int p_t ds=-\int v dp. \qquad (8)$$

The vector m*d* in figure (1) is the so-called *Coriolian* force. If we integrate its tangential component d_t along the closed curve, we get the influence of the earth's rotation on the circulation of the curve. However, this influence is found much more easily by recourse to (3). By differentiating this equation with regard to time, we get :—

$$\frac{dC_a}{dt}=\frac{dC_r}{dt}+2\omega\frac{dS}{dt}. \qquad (9)$$

The change of the circulation relatively to the earth $\left(\frac{dC_r}{dt}\right)$, is given by formula (6). The effect of gravity, density, pressure-distribution, and friction, is precisely as in the case of the relative circulation ; but the *coriolian* force has only to do with the relative movement and is accordingly equal to 0 in the case of the absolute movement. The change of the absolute circulation with the time amounts therefore to :—

$$\frac{dC_a}{dt}=\int g_t ds+\int p_t ds+\int f_t ds. \qquad (10)$$

A comparison of formulæ (6) and (10) gives us :—

$$\frac{dC_a}{dt}=\frac{dC_r}{dt}-\int d_t ds,$$

and a comparison of this formula with (9) gives :—

$$\int d_t ds=-2\omega\frac{dS}{dt}. \qquad (11)$$

The last integral ($\int f_t ds$) in formula (6) represents the influence of friction. The direct calculation of this value entails considerable difficulties, and the only practicable method is to calculate it from formula (6), in cases where the other values are known. We will denominate this integral, for the sake of brevity, by —R, the minus sign indicating that the friction acts in a direction contrary to the circulation. We have then :—

$$\int f_t ds=-R. \qquad (12)$$

By introducing into (6) the equivalents shown in the formulæ (7), (8), (11), and (12), we obtain :—

$$\frac{dC_r}{dt}=-\int v dp-2\omega\frac{dS}{dt}-R. \qquad (13)$$

The dimension of the magnitude C_r is, according to formula (2), $\frac{\text{length}^2}{\text{time}}$; the value $\frac{dC_r}{dt}$ has therefore the dimension $\frac{\text{length}^2}{\text{time}^2}$. The other elements in formula (13) are of the same dimension. If we express all the values in c.g.s. units, the various elements are expressed in $\frac{cm^2}{sec^2}$ units.

The first element on the right-hand side of equation (13) is of special importance, because it contains the primary cause of the movements in the sea and in the atmosphere. Terrestrial rotation and friction cannot set a mass of water in motion from a state of rest, but can only deform an already existing motion. The two last elements in equation (13) are significant, accordingly, only in regard to the further course of a movement already set up by the distributions of pressure and of density. On this account Professor Bjerknes has made a rigorous investigation of the first element. The results achieved may be best illustrated by an example. We want to calculate the element for a closed curve composed of water-particles in a meridian plane. In order to do this, we must know the pressure and the specific volume in the meridian plane. The distribution of these values is best represented by lines of equal pressure (isobars) and by lines of equal specific volume (isosteres) in the meridian plane. The isobars run more or less parallel to the earth's surface; the isosteres are inclined, in general, downwards towards the equator. Figure 2 may be taken to represent these systems of lines. The first integral on the right of equation (13) must be calculated for the closed

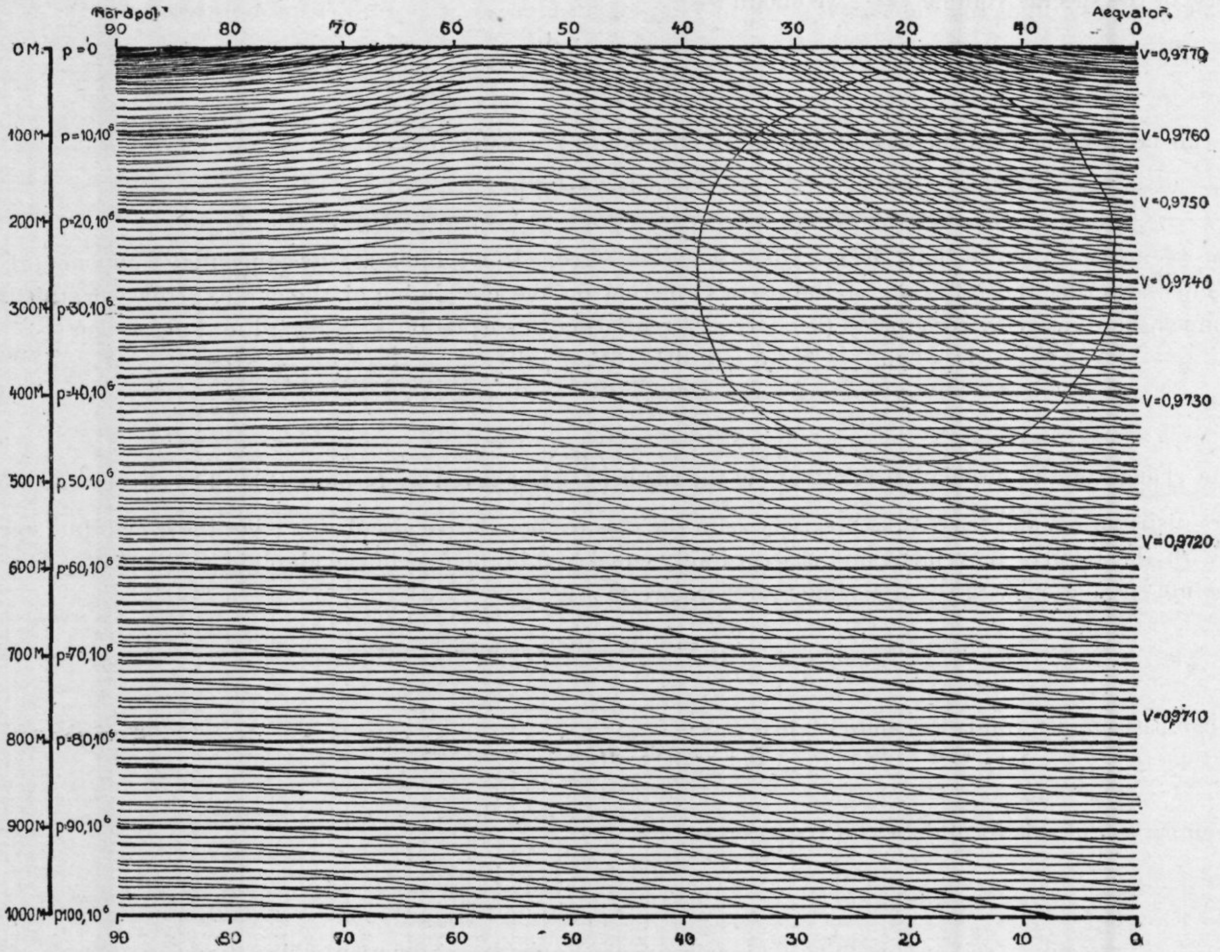

Fig. 2.

curve represented in Figure (2). This can be achieved by plotting the pressure in the form of ordinates and the specific gravity in the form of abscissæ, in a rectangular plane system of co-ordinates, and by drawing upon this plane the closed curves which possesses the same corresponding values of p and v as the curve in Figure 2.

Figure (3) shows the system of co-ordinates together with the curve which is depicted upon Figure (2). According to the theory of integrals, the integral vdp is equal to the area enclosed by the curve in Figure (3). This area is, however, proportional to the number of squares in Figure (3) enclosed by the curve, because each of these squares contains a unit of area. If the horizontal lines in Figure (3) be looked upon as isobars and the vertical lines as isosteres, then each of the squares in Figure (3) corresponds to one of the parallelogram-like figures in Figure (2); and therefore the closed curve in Figure (2) contains precisely the same number of parallelogram-like figures as the closed curve in Figure (3) contains squares. The integral $-\int vdp$ for the closed curve in Figure 2, expressed in c.g.s. units, is, therefore, equal to the number of the parallelogram-shaped figures enclosed by the curve, multiplied by a proportional factor.

In order to find the value for this factor, one must work out the integral $-\int vdp$ for one of the parallelogram-like figures in Figure (2). We note that each of these figures is bounded by two isobars $p=p_0$ and $p=p_1$ and by two isobars $v=v_0$ and $v=v_1$. The working out of the integration around such a curve gives :—

$$-\int vdp=(v_1-v_0)\,(p_1-p_0).$$

For each of the parallelograms in Figure (2), we now have :—

$$(v_1-v_0)=0{\cdot}0001\frac{(cm)^3}{g}, \text{ and}$$

$$(p_1-p_0)=10^4\frac{g}{cm.sec.^2}.$$

that is to say :—

$$-\int vdp = 0{\cdot}0001 \times 10^6 = 100 \frac{cm^2}{sec^2}.$$

By enumeration, one finds that the closed curve in Figure (2) comprises 495 parallelogram-like figures, therefore for this closed curve :—

$$-\int vdp = 495 \times 100 = 49500 \frac{cm^2}{sec^2}.$$

If we now consider, not only the relations in a meridian section, but those in the sea as a whole, we have to deal with isobaric and isosteric surfaces instead of isobaric and isosteric lines. As the two systems of surfaces cut one another,

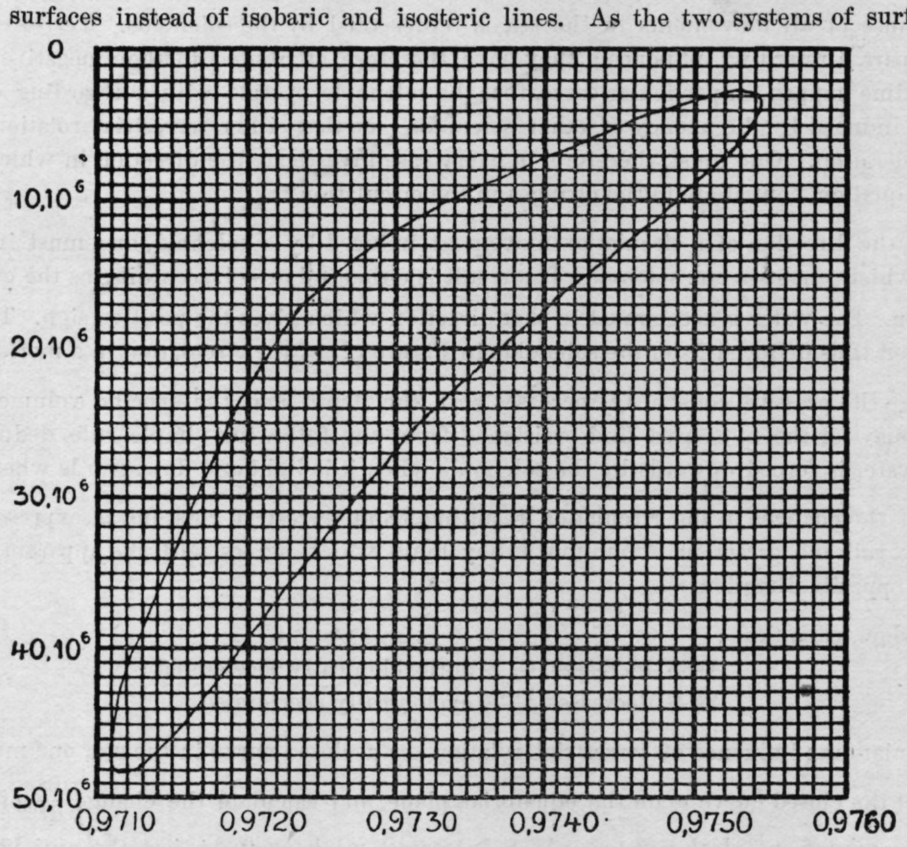

FIG. 3.

there is produced a system of tubes, and the parallelogram-like figures in Figure (2) are the cross sections of these tubes in the meridian plane. Each of these tubes is represented by a corresponding square in Figure (3). The integral $-\int vdp$ is accordingly equal, for any closed curve in the sea, to the number of isobar-isostere tubes enclosed within the curve; these tubes are called *solenoids* by Bjerknes. The closed curve in Figure (2) encloses therefore 49,500 c.g.s. solenoids.

When a closed curve in the sea encloses a number A of solenoids, then, for this curve,

$$-\int vdp = A. \tag{14}$$

If this expression be introduced into (13), we have

$$\frac{dC_r}{dt} = A - 2\omega \frac{dS}{dt} - R. \tag{15}$$

and this is the equation for the circulation deduced by Bjerknes.

The value $\frac{dC_r}{dt}$ is the change of the circulation of any closed curve in the sea, as viewed from the rotating earth, with reference to a time unit. To investigate the meaning of the quantity $\frac{dC_r}{dt}$ we return to formula (4). If we denote by $\dot{u}_{tm}$ the average amount of the tangential components of the acceleration, and by s the length of the curve, then, according to (4),

$$\frac{dC_r}{dt} = \int \dot{u}_t ds = \dot{u}_{tm} s,$$

therefore,

$$\dot{u}_{tm} = \frac{1}{s} \frac{dC_r}{dt}. \tag{16}$$

Here s is known, and the quantity $\frac{dC_r}{dt}$ is calculable from formula (15). Therefore also, by (16) the average tangential acceleration along the curve may be calculated.

The three expressions on the right-hand side of (15) supply the values for the change in the circulation which results from the distribution of pressure and of density, from the terrestrial rotation and from friction. A is the number of isobar-isostere solenoids enclosed by the curve, ω the angular velocity of the earth and S the area of the projection of the closed curve on the equatorial plane. R is the influence of friction on the circulation of the curve.

The primary cause of all movements in the sea is represented by the solenoids. Terrestrial rotation and friction offer, for the most part, a retarding influence ; they have, therefore, attributed to them negative signs in equation (15). It may at the same time happen, in particular cases, that the solenoids operate with a retarding effect upon an already existing circulation induced by the agency of other solenoids : or that either terrestrial rotation or friction may act in the direction of acceleration. One must, therefore, in every case investigate the direction in which the various elements, on the right of the equation, contribute to the changes in the circulation.

In order to find the direction of a circulation-movement induced by a solenoid, one must integrate the expression $-\int vdp$ for a curve which encloses one solenoid. The result is either $+1$ or -1, according as the curve be followed in one or the other direction. The water is accelerated in that direction which gives the positive sign. The arrows in Figure (5) represent the direction thus deduced from the solenoids in Figure (2) of the circulation in a meridial plane.

In the sea, where the pressure constantly increases with the depth, and the specific volume constantly diminishes with the depth, we may for the most part find assistance from the following simple rule, deduced from the process of integration :—The water is forced upwards by the solenoids when it is light and downwards when it is heavy.

The direction of the changes in the circulation resulting from terrestrial rotation, is expressed by Bjerknes in the following rule. We call the direction of motion anticyclonic which agrees with the apparent motion of the sun, and cyclonic that in the opposite direction.

Bjerknes' Law then, runs thus :

As S increases, C_r is anticyclonically accelerated ;
As S decreases, C_r is cyclonically accelerated.

In order to calculate the influence of terrestrial rotation on a closed curve in the sea, one must therefore proceed as follows. We project the closed curve upon the equatorial plane, and calculate the change $\frac{dS}{dt}$ of the area of projection consequent upon the movement of the water. As S increases, we have to project the anticyclonic direction in the projected curve upon the curve in the sea ; as S diminishes, we project the cyclonic direction in order to obtain the sense of the circulation acceleration. The amount of the influence of terrestrial rotation is, according to (15), equal to $2\omega\frac{dS}{dt}$.

The significance of the acceleration due to friction is, as a general rule, opposed in direction to the actual circulation, since friction tends to equalise the relative motions. Yet it may sometimes happen that friction operates in the direction of acceleration, for example, when a current runs along, and not far from, the coast. When all the particles of the current have, to begin with, the same velocity, then the circulation of all closed curves in the current is equal to zero. Owing to friction against the coast, the water in its neighbourhood is somewhat retarded, so that its velocity is there less than at a greater distance from land. Accordingly, the circulation of the closed curves is no longer *nil*, or in other words, the circulation-movement has been increased as the result of friction.

The influence of friction may be calculated provided that we know the coefficient of friction and the relative movements of an infinitesimal order in the neighbourhood of the closed curve. It is naturally impossible to effect observations on this relative movement ; and therefore in the calculation of R, we may best proceed from equation (15) in those cases where the other elements of the equation are known. The stationary movement is well adapted to this investigation, because in this case the circulation C_r is constant and therefore $\frac{dC_r}{dt} = 0$. According to (15) then :—

$$R = A - 2\omega\frac{dS}{dt}. \quad (17)$$

In calculating R according to this method, we obtain also the significance of the frictional acceleration in regard to the circulation. If A be less than $2\omega\frac{dS}{dt}$ the acceleration due to friction is in the same direction as the acceleration of the solenoids, and so conversely.

If we assume the friction to be proportional to the velocity, then the influence of friction upon the circulation is proportional to the circulation itself, and therefore

$$R=\mu C_r, \tag{18}$$

where μ denotes the coefficient of friction according to Mohn. When R and C_r are both known for a closed curve the coefficient μ can then be calculated.

II.—THE NUMERICAL CALCULATION OF THE CIRCULATION ACCELERATION.

(1.) Estimation of the Number of Solenoids.

The number of the solenoids is obtained very easily for such curves as are constituted of two verticals a and b and two isobars $p=p_0$ and $p=p_1$. Such special curves, moreover, suffice in the majority of cases for the calculation of the movements of the sea. The essential cause of simplification in this case is, that upon the portions of the curve represented by the isobars, $dp=o$, so that these have no effect upon the integral $-\int vdp$ for the curve as a whole, and consequently one has only to perform the integration along the verticals a and b. In place of (14) we may therefore write—

$$A=\left(\int_{p_0}^{p_1} vdp\right)_a-\left(\int_{p_0}^{p_1} vdp\right)_b \tag{19}$$

The signs are necessarily different, since in the process of integration around the curve, we proceed up the one vertical line and down the other. If we next calculate for different verticals, the integral

$$E=\int_{p_0}^{p_1} vdp, \tag{20}$$

we have, according to (19), only to take the differences in order to obtain the number (A) of the solenoids. It is, on this account, desirable to calculate the value of E for our hydrographic stations. We shall see in the sequel that this calculation may be very easily and quickly performed. Let us take note, to begin with, that the pressure increases in the sea by one atmosphere or every 10 metres depth, and that one atmosphere corresponds to about 10^6 c.g.s. units; the pressure, in other words, ncreases by about 10^5 c.g.s. units for each metre. It will accordingly be very convenient to choose 10^5 c.g.s. as our unit of pressure, since then the pressure will increase by one unit per metre depth; and then the values of E and A obtained by integration have simply to be multiplied by 10^5 to be expressed in c.g.s. units. Instead of this, we may multiply all the specific volumes by 10^5, which is convenient, because the specific volume varies very little. In the following discussion we shall throughout express pressure in 10^5 and specific volume in 10^{-5} c.g.s. units. The values of E and A, as got by integration, are then found directly in c.g.s. units.

TABLE 1.

Depth of the Isobars in Latitude 45°.

Pressure in 10^5 c.g.s. Units.	Fresh Water 0°C. Depth in m.	Sea Water 0°C. Depth in m.
0	0	0
10	$10_{.19}$	$9_{.91}$
20	$20_{.38}$	$19_{.82}$
50	$50_{.95}$	$49_{.55}$
100	$101_{.9}$	$99_{.09}$
200	$203_{.7}$	$198_{.1}$
500	$509_{.0}$	$495_{.0}$
1000	1017	989
1500	1523	1482
2000	2029	1974
3000	3036	2955
4000	4039	3931
5000	5038	4904

Table 1 shows the depths of the different isobars in fresh and salt water (salinity 35°/$_{oo}$) at 0° C. and 45° latitude. The isobar 1000 lies, according to this Table, at a depth of 989 metres in sea water and 1,017 metres in fresh water; at most, therefore, the position of this isobar can only be at a distance of 17 metres on one side or other of the 1000-metre line, or 1·7 %, in this case, of the depth. In no case, among the other isobars, does the discrepancy exceed 2 % of the whole depth. Owing to obliquity of the sounding-line and other causes, our hydrographic soundings do not attain to this degree of exactitude, and we may therefore assume that the values found at any particular depth for salinity and temperature may stand for the corresponding isobars; and therefore the specific volumes calculated from these results hold good also for the isobars. By this coincidence of the isobaric zones with zones of equal depth, the numerical working-out of the integration $\int_{p_0}^{p_1} v dp$ is very greatly simplified.

The specific volume may, with the help of modern methods, be calculated nearly to five places. If we desire to calculate out the above integral directly, we have accordingly to deal with numbers of five figures. It is, however, easy to see that to do so is superfluous. We may, that is to say, diminish each of the two elements on the right of (19) by an optional amount E_0, and the difference between the two expressions will still remain equal to A. If we denote the two elements by E_a and E_b, respectively, we may write, instead of (19)

$$A = E_a - E_b. \qquad (21)$$

This equation may also be written

$$A = (E_a - E_0) - (E_b - E_0). \qquad (22)$$

By making suitable choice of a value for E_0, we may get small numbers for the numerical values of the differences $(E_a - E_0)$ and $(E_b - E_0)$. For example, we may choose our value for E_0 in the following way: Calculate the value E according to (20) for a vertical in the sea, where only water of 0° C. and 35°/$_{oo}$ occurs, and then take this result as the value of E_0. It is, accordingly,

$$E_0 = \int_{p_0}^{p_1} v_{35°/_{oo}, 0°C.} dp. \qquad (23)$$

Table 2 contains in the first place $\rho_{35°/_{oo}, 0°C.}$ This value is calculated from Tait's compressibility experiments and Martin Knudsen's Hydrographic Tables.* The Table also contains the values obtained from the former ones by inversion and multiplication by 10^5. By mechanical integration of the value of $v_{35°/_{oo}, 0°C.}$, we obtain according to (23) the value of E_0; this also is shown in Table 2. The integration is performed from sea-level downwards.

TABLE 2.

Density, specific Volume and Value of E_0 for a Perpendicular Line in the Sea where only Water of 35°/$_{oo}$ and 0°C. occurs.

Depth in Metres	$\rho_{35°/_{oo}, 0°C.}$	$V_{35°/_{oo}, 0°C.}$	E_0
0	1.02812_6	$97264._3$	0
10	1.02817_3	$97259._9$	972 621
20	1.02822_0	$97255._4$	1 945 200
30	1.02826_7	$97251._0$	2 917 730
40	1.02831_4	$97246._6$	3 890 220
50	1.02836_1	$97242._1$	4 862 660
100	1.02859_5	$97220._0$	9 724 210
150	1.02882_8	$97198._0$	14 584 700
200	1.02906_2	$97175._9$	19 444 000
250	1.02929_5	$97153._9$	24 302 300
300	1.02952_8	$97131._9$	29 159 400
350	1.02976_0	$97110._0$	34 015 400
400	1.02999_2	$97088._1$	38 870 400
450	1.03022_4	$97066._3$	43 724 300
500	1.03045_6	$97044._4$	48 577 000

* Martin Knudsen: Hydrographical Tables. Copenhagen, 1901.

P. G. Tait. Report on some of the Physical Properties of Fresh Water and of Sea Water. The Voyage of H.M.S. Challenger. Vol. II. London, 1889.

TABLE 2—*continued.*

Depth in Metres.	$\rho_{35^0/_{00},\,0^0C.}$	$V_{35^0/_{00},\,0^0C.}$	E_0
600	1.03091_8	$97000._9$	58 279 300
700	1.03138_0	$96957._5$	67 977 200
800	1.03184_0	$96914._3$	77 670 800
900	1.03229_9	$96871._2$	87 360 100
1000	1.03275_7	$96828._2$	97 045 000
1200	1.03367_0	$96742._7$	116 402 000
1500	1.03503_1	$96615._5$	145 406 000
2000	1.03727_8	$96406._2$	193 661 000
2500	1.03949_8	$96200._3$	241 813 000
3000	1.04169_3	$95997._6$	289 862 000
3500	1.04386_1	$95798._2$	337 811 000
4000	1.04600_4	$95601._9$	385 661 000
4500	1.04812_4	$95408._6$	433 413 000
5000	1.05021_7	$95218._4$	481 070 000

In order now to calculate the value of $E-E_0$ from our hydrographical observations, we write, according to (20) and (23),

$$E-E_0=\int_{p_0}^{p_1} v\,dp-\int_{p_0}^{p_1} v_{35^0/_{00},\,0^0C.}\,dp,$$

or

$$E-E_0=\int_{p_0}^{p_1}(v-v_{35^0/_{00},\,0^0})\,dp, \qquad (24)$$

that is to say, the coefficient $v-v_{35^0/_{00},\,0^0}$ must be integrated along the vertical line. The value of this expression changes very little with the compression, because both v and $v_{35^0/_{00},\,0^0}$ are influenced by the pressure in nearly the same degree. We may accordingly allow for the influence of the pressure by applying a correction. (*See* Table 4.)

Table 3 contains $v-v_{35^0/_{00},\,0^0}$ as a function of salinity and temperature under a pressure of one atmosphere.

TABLE 3.

$v-v_{35^0/_{00},\,0^0C.}$ at a Pressure of one Atmosphere.

Salinity. Temp. °C.	$33._5$ °/oo	$33._6$	$33._7$	$33._8$	$33._9$	$34._0$ °/oo	$34._1$	$34._2$	$34._3$	$34._4$	$34._5$ °/oo	$34._6$	$34._7$	$34._8$	$34._9$	$35._0$ °/oo	$35._1$	$35._2$	$35._3$	$35._4$	$35._5$ °/oo	Temp. °C.
$-2._0$	$108._0$	100	93	85	77	$69._6$	62	54	47	39	$31._2$	24	16	8	0	$-7._2$	−15	−23	−30	−38	$-45._7$	$-2._0$
$-1._9$	$108._2$	101	93	85	78	70	62	54	47	39	$31._4$	24	16	8	1	−7	−15	−22	−30	−38	$-45._6$	$-1._9$
$-1._8$	$108._3$	101	93	85	78	70	62	55	47	39	$31._6$	24	16	9	1	−7	−15	−22	−30	−38	$-45._4$	$-1._8$
$-1._7$	$108._6$	101	93	86	78	70	63	55	47	39	$31._8$	24	16	9	1	−7	−14	−22	−30	−37	$-45._1$	$-1._7$
$-1._6$	$108._8$	101	93	86	78	70	63	55	47	40	$32._1$	24	17	9	1	−6	−14	−22	−30	−37	$-44._8$	$-1._6$
$-1._5$	$109._1$	101	94	86	78	71	63	55	48	40	$32._3$	25	17	9	2	−6	−14	−21	−29	−37	$-44._5$	$-1._5$
$-1._4$	$109._3$	102	94	86	79	71	65	56	48	40	$32._6$	25	17	10	2	−6	−13	−21	−29	−37	$-44._2$	$-1._4$
$-1._3$	$109._6$	102	94	87	79	71	64	56	48	41	$32._9$	25	18	10	2	−6	−13	−21	−29	−36	$-43._9$	$-1._3$
$-1._2$	$109._9$	102	95	87	79	72	64	56	49	41	$33._2$	26	18	10	3	−5	−13	−20	−28	−36	$-43._5$	$-1._2$
$-1._1$	$110._2$	103	95	87	80	72	64	57	49	41	$33._5$	26	18	11	3	−5	−12	−20	−28	−35	$-43._1$	$-1._1$

TABLE 3—*continued.*

$v - v_{35‰, 0°C}$ at a Pressure of one Atmosphere.

Salinity. Temp.°C.	33.5‰	33.6	33.7	33.8	33.9	34.0‰	34.1	34.2	34.3	34.4	34.5‰	34.6	34.7	34.8	34.9	35.0‰	35.1	35.2	35.3	35.4	35.5‰	Temp.°C.
−1.0	110.5	103	95	88	80	72.2	65	57	49	42	33.9	26	19	11	3	−4.1	−12	−20	−27	−35	−42.7	−1.0
−0.9	110.8	103	96	88	80	73	65	57	50	42	34.3	27	19	11	4	−4	−12	−19	−27	−35	−42.3	−0.9
−0.8	111.2	104	96	88	81	73	65	58	50	42	34.7	27	19	12	4	−4	−11	−19	−27	−34	−41.9	−0.8
−0.7	111.6	104	96	89	81	73	66	58	50	43	35.1	27	20	12	4	−3	−11	−19	−26	−34	−41.5	−0.7
−0.6	111.9	104	97	89	81	74	66	58	51	43	35.5	28	20	13	5	−3	−10	−18	−26	−33	−41.1	−0.6
−0.5	112.3	105	97	89	82	74	66	59	51	44	35.9	28	21	13	5	−2	−10	−18	−25	−33	−40.6	−0.5
−0.4	112.7	105	97	90	82	75	67	59	52	44	36.3	29	21	13	6	−2	−10	−17	−25	−33	−40.2	−0.4
−0.3	113.1	105	98	90	83	75	67	60	52	44	36.7	29	21	14	6	−1	− 9	−17	−24	−32	−39.7	−0.3
−0.2	113.5	106	98	91	83	75	68	60	52	45	37.2	30	22	14	7	−1	− 9	−16	−24	−32	−39.2	−0.2
−0.1	114.0	106	99	91	83	76	68	61	53	45	37.7	30	22	15	7	0	− 8	−16	−23	−31	−38.7	−0.1
0.0	114.5	107	99	92	84	76.1	69	61	53	46	38.2	31	23	15	8	0.0	− 8	−15	−23	−31	−38.2	0.0
0.1	115.0	107	100	92	84	77	69	62	54	46	38.7	31	23	16	8	0	− 7	−15	−22	−30	−37.7	0.1
0.2	115.5	108	100	93	85	77	70	62	54	47	39.2	32	24	16	9	1	− 7	−14	−22	−30	−37.2	0.2
0.3	116.0	108	101	93	85	78	70	63	55	47	39.7	32	24	17	9	1	− 6	−14	−21	−29	−36.6	0.3
0.4	116.5	109	101	94	86	78	71	63	55	48	40.2	33	25	17	10	2	− 6	−13	−21	−28	−36.1	0.4
0.5	117.0	109	102	94	86	79	71	64	56	48	40.8	33	25	18	10	2	− 5	−13	−20	−28	−35.5	0.5
0.6	117.5	110	102	95	87	79	72	64	57	49	41.3	34	26	18	11	3	− 4	−12	−20	−27	−34.9	0.6
0.7	118.0	110	103	95	88	80	72	65	57	50	41.9	34	27	19	11	4	− 4	−11	−19	−27	−34.3	0.7
0.8	118.6	111	103	96	88	81	73	65	58	50	42.5	35	27	20	12	4	− 3	−11	−18	−26	−33.6	0.8
0.9	119.2	112	104	96	89	81	74	66	58	51	43.1	35	28	20	13	5	− 3	−10	−18	−25	−33.0	0.9
1.0	119.8	112	105	97	89	81.8	74	67	59	51	43.7	36	28	21	13	5.7	− 2	−10	−17	−25	−32.4	1.0
1.1	120.4	113	105	98	90	82	75	67	60	52	44.3	37	29	22	14	6	− 1	− 9	−17	−24	−31.7	1.1
1.2	121.0	113	106	98	91	83	75	68	60	53	44.9	37	30	22	14	7	− 1	− 8	−16	−24	−31.1	1.2
1.3	121.5	114	106	99	91	84	76	68	61	53	45.6	38	30	23	15	8	0	− 8	−15	−23	−30.4	1.3
1.4	122.1	115	107	99	92	84	77	69	61	54	46.2	39	31	23	16	8	1	− 7	−15	−22	−29.7	1.4
1.5	122.8	115	108	100	92	85	77	70	62	54	46.9	39	32	24	17	9	1	− 6	−14	−21	−29.0	1.5
1.6	123.4	116	108	101	93	85	78	70	63	55	47.6	40	32	25	17	10	2	− 6	−13	−21	−28.3	1.6
1.7	124.1	117	109	101	94	86	79	71	63	56	48.3	41	33	26	18	10	3	− 5	−12	−20	−27.5	1.7
1.8	124.8	117	110	102	94	87	79	72	64	57	49.0	41	34	26	19	11	4	− 4	−12	−19	−26.8	1.8
1.9	125.5	118	110	103	95	88	80	73	65	57	49.8	42	35	27	19	12	4	− 3	−11	−18	−26.0	1.9
2.0	126.2	119	111	103	96	88.4	81	73	66	58	50.5	43	35	28	20	12.6	5	− 3	−10	−18	−25.3	2.0
2.1	126.9	119	112	104	97	89	82	74	66	59	51.3	44	36	29	21	13	6	− 2	− 9	−17	−24.6	2.1
2.2	127.6	120	112	105	97	90	82	75	67	60	52.0	44	37	29	22	14	7	− 1	− 9	−16	−23.8	2.2
2.3	128.4	121	113	106	98	91	83	75	68	60	52.8	45	38	30	22	15	7	0	− 8	−15	−23.0	2.3
2.4	129.1	122	114	106	99	91	84	76	69	61	53.6	46	38	31	23	16	8	1	− 7	−15	−22.2	2.4

Table 3—*continued.*

$v - v_{35‰, 0°C.}$ at a Pressure of one Atmosphere.

Salinity. Temp. °C.	33.5‰	33.6	33.7	33.8	33.9	34.0‰	34.1	34.2	34.3	34.4	34.5‰	34.6	34.7	34.8	34.9	35.0‰	35.1	35.2	35.3	35.4	35.5‰	Temp. °C.
2.5	129.9	122	115	107	100	92	85	77	70	62	54.4	47	39	32	24	17	9	1	—6	—14	—21.4	2.5
2.6	130.7	123	116	108	101	93	85	78	70	63	55.2	48	40	32	25	17	10	2	—5	—13	—20.5	2.6
2.7	131.5	124	116	109	101	94	86	79	71	64	56.0	48	41	33	26	18	11	3	—5	—12	—19.7	2.7
2.8	132.3	125	117	110	102	95	87	79	72	64	56.8	49	42	34	27	19	11	4	—4	—11	—18.8	2.8
2.9	133.1	126	118	110	103	95	88	80	73	65	57.6	50	43	35	27	20	12	5	—3	—10	—17.9	2.9
3.0	134.0	126	119	111	104	96.3	89	81	74	66	58.5	51	43	36	28	20.8	13	6	—2	—9	—17.0	3.0
3.1	134.8	127	120	112	105	97	90	82	74	67	59.4	52	44	37	29	22	14	7	—1	—9	—16.1	3.1
3.2	135.7	128	121	113	106	98	90	83	75	68	60.3	53	45	38	30	23	15	7	0	—8	—15.2	3.2
3.3	136.6	129	122	114	106	99	91	84	76	69	61.2	54	46	39	31	24	16	8	1	—7	—14.2	3.3
3.4	137.5	130	122	115	107	100	92	85	77	70	62.1	55	47	39	32	24	17	9	2	—6	—13.3	3.4
3.5	138.5	131	123	116	108	101	93	86	78	71	63.0	55	48	40	33	25	18	10	3	—5	—12.4	3.5
3.6	139.4	132	124	117	109	102	94	87	79	71	63.9	56	49	41	34	26	19	11	4	—4	—11.5	3.6
3.7	140.3	133	125	118	110	103	95	88	80	72	64.9	57	50	42	35	27	20	12	5	—3	—10.5	3.7
3.8	141.3	134	126	119	111	104	96	88	81	73	65.8	58	51	43	36	28	21	13	6	—2	— 9.5	3.8
3.9	142.3	135	127	120	112	105	97	89	82	74	66.8	59	52	44	37	29	22	14	7	—1	— 8.5	3.9
4.0	143.2	136	128	121	113	105.5	98	90	83	75	67.8	60	53	45	38	30.2	23	15	8	0	— 7.5	4.0
4.1	144.1	137	129	122	114	106	99	91	84	76	68.8	61	54	46	39	31	24	16	9	1	— 6.5	4.1
4.2	145.1	138	130	123	115	107	100	92	85	77	69.8	62	55	47	40	32	25	17	10	2	— 5.5	4.2
4.3	146.1	139	131	124	116	108	101	93	86	78	70.8	63	56	48	41	33	26	18	11	3	— 4.4	4.3
4.4	147.0	139	132	124	117	109	102	94	87	79	71.8	64	57	49	42	34	27	19	12	4	— 3.3	4.4
4.5	148.0	140	133	125	118	110	103	95	88	80	72.8	65	58	50	43	35	28	20	13	5	— 2.3	4.5
4.6	149.0	141	134	126	119	111	104	96	89	81	73.9	66	59	51	44	36	29	21	14	6	— 1.2	4.6
4.7	150.0	142	135	127	120	112	105	97	90	82	74.9	67	60	52	45	37	30	22	15	7	— 0.1	4.7
4.8	151.1	144	136	129	121	114	106	99	91	84	76.0	69	61	54	46	39	31	24	16	9	1.0	4.8
4.9	152.1	145	137	130	122	115	107	100	92	85	77.1	70	62	55	47	40	32	25	17	10	2.1	4.9
5.0	153.2	146	138	131	123	115.7	108	101	93	86	78.2	71	63	56	48	40.7	33	26	18	11	3.2	5.0
5.1	154.3	147	139	132	124	117	109	102	94	87	79.3	72	64	57	49	42	34	27	19	12	4.3	5.1
5.2	155.4	148	140	133	125	118	110	103	95	88	80.4	73	65	58	50	43	35	28	20	13	5.4	5.2
5.3	156.5	149	142	134	127	119	112	104	97	89	81.6	74	67	59	52	44	37	29	22	14	6.6	5.3
5.4	157.6	150	143	135	128	120	113	105	98	90	82.7	75	68	60	53	45	38	30	23	15	7.7	5.4
5.5	158.7	151	144	136	129	121	114	106	99	91	83.8	76	69	61	54	46	39	31	24	16	8.9	5.5
5.6	159.9	152	145	137	130	122	115	107	100	92	84.9	77	70	62	55	47	40	33	25	18	10.1	5.6
5.7	161.0	154	146	139	131	124	116	109	101	94	86.1	79	71	64	56	49	41	34	26	19	11.3	5.7
5.8	162.1	155	147	140	132	125	117	110	102	95	87.3	80	72	65	57	50	42	35	27	20	12.5	5.8
5.9	163.3	156	148	141	133	126	118	111	103	96	88.5	81	74	66	59	51	44	36	29	21	13.7	5.9

TABLE 3—*continued.*

$v - v_{35‰, 0°C.}$ at a Pressure of one Atmosphere.

Salinity.	33.5 ‰	33.6	33.7	33.8	33.9	34.0 ‰	34.1	34.2	34.3	34.4	34.5 ‰	34.6	34.7	34.8	34.9	35.0 ‰	35.1	35.2	35.3	35.4	35.5 ‰	
Temp. °C.																						Temp. °C.
6.0	164.5	157	150	142	135	127.1	120	112	105	97	89.7	82	75	67	60	52.3	45	37	30	22	14.9	6.0
6.1	165.7	158	151	143	136	128	121	113	106	98	90.9	83	76	68	61	54	46	39	31	24	16.1	6.1
6.2	166.9	159	152	144	137	130	122	115	107	100	92.2	85	77	70	62	55	47	40	32	25	17.4	6.2
6.3	168.1	161	153	146	138	131	123	116	108	101	93.4	86	78	71	63	56	49	41	34	26	18.6	6.3
6.4	169.3	162	154	147	139	132	124	117	110	102	94.6	87	80	72	65	57	50	42	35	27	19.9	6.4
6.5	170.5	163	156	148	141	133	126	118	111	103	95.9	88	81	73	66	59	51	44	36	29	21.1	6.5
6.6	171.7	164	157	149	142	134	127	120	112	105	97.2	90	82	75	67	60	52	45	37	30	22.4	6.6
6.7	173.0	166	158	151	143	136	128	121	113	106	98.4	91	83	76	69	61	54	46	39	31	23.7	6.7
6.8	174.3	167	159	152	144	137	130	122	115	107	99.7	92	85	77	70	62	55	47	40	32	25.0	6.8
6.9	175.6	168	161	153	146	138	131	123	116	108	101.0	94	86	79	71	64	56	49	41	34	26.4	6.9
7.0	176.9	169	162	155	147	139.6	132	125	117	110	102.3	95	87	80	72	65.0	58	50	43	35	27.7	7.0
7.1	178.2	171	163	156	148	141	133	126	119	111	103.6	96	89	81	74	66	59	51	44	37	29.1	7.1
7.2	179.5	172	165	157	150	142	135	127	120	112	104.9	97	90	83	75	68	60	53	45	38	30.4	7.2
7.3	180.8	173	166	158	151	144	136	129	121	114	106.3	99	91	84	77	69	62	54	47	39	31.8	7.3
7.4	182.1	175	167	160	152	145	137	130	123	115	107.7	100	93	85	78	70	63	56	48	41	33.2	7.4
7.5	183.5	176	169	161	154	146	139	131	124	117	109.1	102	94	87	79	72	64	57	50	42	34.6	7.5
7.6	184.9	177	170	163	155	148	140	133	125	118	110.5	103	96	88	81	73	66	58	51	43	36.0	7.6
7.7	186.2	179	171	164	156	149	142	134	127	119	111.9	104	97	90	82	75	67	60	52	45	37.4	7.7
7.8	187.6	180	173	165	158	150	143	136	128	121	113.3	106	98	91	84	76	69	61	54	46	38.8	7.8
7.9	189.0	182	174	167	159	152	144	137	130	122	114.7	107	100	92	85	77	70	63	55	48	40.2	7.9
8.0	190.4	183	176	168	161	153.3	146	138	131	124	116.1	109	101	94	86	78.8	71	64	57	49	41.6	8.0
8.1	191.8	184	177	170	162	155	147	140	132	125	117.5	110	103	95	88	80	73	65	58	51	43.1	8.1
8.2	193.2	186	178	171	163	156	149	141	134	126	118.9	111	104	97	89	82	74	67	59	52	44.5	8.2
8.3	194.6	187	180	172	165	157	150	143	135	128	120.3	113	105	98	91	83	76	68	61	53	46.0	8.3
8.4	196.1	189	181	174	166	159	152	144	137	129	121.8	114	107	100	92	85	77	70	62	55	47.5	8.4
8.5	197.5	190	183	175	168	160	153	146	138	131	123.3	116	108	101	94	86	79	71	64	56	49.0	8.5
8.6	198.9	191	184	177	169	162	154	147	140	132	124.8	117	110	103	95	88	80	73	65	58	50.5	8.6
8.7	200.4	193	186	178	171	163	156	149	141	134	126.3	119	111	104	97	89	82	74	67	59	52.0	8.7
8.8	201.8	194	187	180	172	165	157	150	143	135	127.8	120	113	106	98	91	83	76	68	61	53.5	8.8
8.9	203.3	196	189	181	174	166	159	152	144	137	129.3	122	114	107	100	92	85	77	70	62	55.0	8.9
9.0	204.8	197	190	183	175	167.9	160	153	146	138	130.8	123	116	109	101	93.7	86	79	71	64	56.5	9.0
9.1	206.4	199	192	184	177	169	162	155	147	140	132.3	125	117	110	103	95	88	80	73	65	58.0	9.1
9.2	207.9	200	193	186	178	171	163	156	149	141	133.8	126	119	112	104	97	89	82	74	67	59.6	9.2
9.3	209.4	202	195	187	180	172	165	158	150	143	135.3	128	120	113	106	98	91	83	76	69	61.2	9.3
9.4	210.9	203	196	189	181	174	166	159	152	144	136.8	129	122	115	107	100	92	85	78	70	62.8	9.4

TABLE 3—*continued.*

$v - v_{35‰, 0°C.}$ at a Pressure of one Atmosphere.

Salinity.	33.5‰	33.6	33.7	33.8	33.9	34.0‰	34.1	34.2	34.3	34.4	34.5‰	34.6	34.7	34.8	34.9	35.0‰	35.1	35.2	35.3	35.4	35.5‰	
Temp. °C.																						Temp. °C.
9.5	212.4	205	198	190	183	175	168	161	153	146	138.4	131	124	116	109	101	94	87	79	72	64.4	9.5
9.6	214.0	207	199	192	184	177	170	162	155	147	140.0	133	125	118	110	103	96	88	81	73	66.0	9.6
9.7	215.5	208	201	193	186	179	171	164	156	149	141.6	134	127	119	112	105	97	90	82	75	67.6	9.7
9.8	217.1	210	202	195	188	180	173	165	158	151	143.2	136	128	121	114	106	99	91	84	77	69.2	9.8
9.9	218.7	211	204	197	189	182	174	167	160	152	144.8	137	130	123	115	108	100	93	86	78	70.8	9.9
10.0	220.3	213	206	198	191	183.4	176	169	161	154	146.4	139	132	124	117	109.4	102	95	87	80	72.4	10.0
10.1	221.9	215	207	200	192	185	178	170	163	155	148.0	141	133	126	118	111	104	96	89	81	74.0	10.1
10.2	223.5	216	209	201	194	187	179	172	164	157	149.7	142	135	127	120	113	105	98	90	83	75.6	10.2
10.3	225.1	218	210	203	196	188	181	173	166	159	151.3	144	137	129	122	114	107	100	92	85	77.3	10.3
10.4	226.8	219	212	205	197	190	183	175	168	160	153.0	146	138	131	123	116	109	101	94	86	79.0	10.4
10.5	228.4	221	214	206	199	192	184	177	169	162	154.6	147	140	132	125	118	110	103	95	88	80.6	10.5
10.6	230.1	223	215	208	201	193	186	178	171	164	156.2	149	141	134	127	119	112	104	97	90	82.3	10.6
10.7	231.8	224	217	210	202	195	187	180	173	165	157.9	151	143	136	128	121	114	106	99	91	84.0	10.7
10.8	233.4	226	219	211	204	196	189	182	174	167	159.5	152	145	137	130	123	115	108	101	93	85.8	10.8
10.9	235.1	228	220	213	206	198	191	183	176	169	161.2	154	146	139	132	124	117	110	102	95	87.5	10.9
11.0	236.8	229	222	215	207	200.0	192	185	178	170	162.9	156	148	141	133	126.1	119	111	104	97	89.2	11.0
11.1	238.5	231	224	216	209	202	194	187	179	172	164.6	157	150	143	135	128	120	113	106	98	91.0	11.1
11.2	240.2	233	225	218	211	203	196	189	181	174	166.4	159	152	144	137	130	122	115	108	100	92.8	11.2
11.3	241.9	235	227	220	212	205	198	190	183	176	168.2	161	153	146	139	131	124	117	109	102	94.5	11.3
11.4	243.7	236	229	222	214	207	199	192	185	177	169.9	163	155	148	140	133	126	118	111	104	96.3	11.4
11.5	245.4	238	231	223	216	209	201	194	186	179	171.7	164	157	150	142	135	127	120	113	105	98.0	11.5
11.6	247.2	240	232	225	218	210	203	196	188	181	173.5	166	159	151	144	137	129	122	115	107	99.8	11.6
11.7	249.0	242	234	227	220	212	205	197	190	183	175.3	168	161	153	146	138	131	124	116	109	101.6	11.7
11.8	250.7	243	236	229	221	214	207	199	192	184	177.1	170	162	155	148	140	133	126	118	111	103.5	11.8
11.9	252.5	245	238	230	223	216	208	201	194	186	178.9	172	164	157	149	142	135	127	120	113	105.3	11.9
12.0	254.3	247	240	232	225	217.6	210	203	195	188	180.7	173	166	159	151	144.0	137	129	122	114	107.1	12.0
12.1	256.1	249	241	234	227	219	212	205	197	190	182.5	175	168	160	153	146	138	131	124	116	108.9	12.1
12.2	257.9	251	243	236	228	221	214	206	199	192	184.3	177	170	162	155	148	140	133	126	118	110.8	12.2
12.3	259.7	252	245	238	230	223	216	208	201	194	186.2	179	171	164	157	149	142	135	127	120	112.6	12.3
12.4	261.5	254	247	239	232	225	217	210	203	195	188.0	181	173	166	159	151	144	137	129	122	114.5	12.4
12.5	263.3	256	249	241	234	227	219	212	205	197	189.8	182	175	168	160	153	146	138	131	124	116.3	12.5
12.6	265.1	258	250	243	236	228	221	214	206	199	191.6	184	177	170	162	155	148	140	133	126	118.2	12.6
12.7	266.9	260	252	245	238	230	223	216	208	201	193.5	186	179	171	164	157	149	142	135	127	120.1	12.7
12.8	268.8	261	254	247	239	232	225	217	210	203	195.4	188	181	173	166	159	151	144	137	129	121.9	12.8
12.9	270.7	263	256	249	241	234	227	219	212	205	197.3	190	183	175	168	161	153	146	139	131	123.8	12.9

TABLE 3—*continued.*

$v - v_{35‰, 0°C}$ at a Pressure of one Atmosphere.

Salinity.	33.5‰	33.6	33.7	33.8	33.9	34.0‰	34.1	34.2	34.3	34.4	34.5‰	34.6	34.7	34.8	34.9	35.0‰	35.1	35.2	35.3	35.4	35.5‰	
Temp.°C.																						Temp.°C.
13.0	272.6	265	258	251	243	235.9	229	221	214	207	199.2	192	185	177	170	162.5	155	148	140	133	125.7	13.0
13.1	274.5	267	260	252	245	238	230	223	216	208	201.1	194	186	179	172	164	157	150	142	135	127.6	13.1
13.2	276.4	269	262	254	247	240	232	225	218	210	203.0	196	188	181	174	166	159	152	144	137	129.6	13.2
13.3	278.3	271	264	256	249	242	234	227	220	212	204.9	198	190	183	176	168	161	154	146	139	131.5	13.3
13.4	280.2	273	266	258	251	244	236	229	222	214	206.9	200	192	185	178	170	163	156	148	141	133.5	13.4
13.5	282.2	275	268	260	253	246	238	231	224	216	208.9	202	194	187	180	172	165	158	150	143	135.5	13.5
13.6	284.1	277	269	262	255	248	240	233	226	218	210.9	204	196	189	182	174	167	160	152	145	137.5	13.6
13.7	286.1	279	271	264	257	250	242	235	228	220	212.9	206	198	191	184	176	169	162	154	147	139.5	13.7
13.8	288.1	281	273	266	259	251	244	237	229	222	214.8	207	200	193	185	178	171	163	156	149	141.5	13.8
13.9	290.0	283	275	268	261	253	246	239	231	224	216.8	209	202	195	187	180	173	165	158	151	143.5	13.9
14.0	292.0	285	277	270	263	255.4	248	241	233	226	218.8	211	204	197	189	182.1	175	167	160	153	145.5	14.0
14.1	294.0	287	279	272	265	258	250	243	235	228	220.8	213	206	199	191	184	177	169	162	155	147.5	14.1
14.2	296.0	289	281	274	267	260	252	245	237	230	222.8	215	208	201	193	186	179	171	164	157	149.5	14.2
14.3	298.0	291	283	276	269	262	254	247	239	232	224.8	217	210	203	195	188	181	173	166	159	151.5	14.3
14.4	300.0	293	285	278	271	264	256	249	241	234	226.8	219	212	205	197	190	183	175	168	161	153.5	14.4
14.5	302.0	295	287	280	273	266	258	251	243	236	228.8	221	214	207	200	192	185	178	170	163	155.6	14.5
14.6	304.0	297	289	282	275	268	260	253	246	238	230.9	224	216	209	202	194	187	180	172	165	157.6	14.6
14.7	306.0	299	291	284	277	270	262	255	248	240	232.9	226	218	211	204	196	189	182	174	167	159.7	14.7
14.8	308.1	301	293	286	279	272	264	257	250	242	235.0	228	220	213	206	198	191	184	176	169	161.7	14.8
14.9	310.1	303	295	288	281	274	266	259	252	244	237.0	230	222	215	208	200	193	186	178	171	163.8	14.9
15.0	312.2	305	298	290	283	275.7	268	261	254	246	239.1	232	224	217	210	202.5	195	188	181	173	165.9	15.0

Table 4 shows the correction for compression as a function of temperature and depth. The relation between salinity and compression was not fully investigated by Tait. It is very much to be desired that Tait's researches should be repeated and extended, and as soon as this be done, Table 4 will require to be recalculated. Our Table includes, indeed, what we at present know concerning the compressibility of sea-water; but it is very far from being comparable to Knudsen's Tables in its degree of exactitude. Table 3 on the other hand, which is directly derived from Knudsen's Tables and formulæ, may be regarded as final.

TABLE 4.

Correction for Compression, to be applied to the values in Table 3.

Depth (Metres).	Temperature, Centigrade.																	
	−2	−1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
100	−1	0	0	0	1	1	1	1	2	2	2	2	2	3	3	3	3	3
200	−1	−1	0	1	1	2	2	3	3	4	4	5	5	5	6	6	6	7
300	−2	−1	0	1	2	3	3	4	5	6	6	7	7	8	9	9	10	10
400	−2	−1	0	1	2	3	4	5	6	7	8	9	10	11	11	12	13	13
500	−3	−2	0	1	3	4	5	7	8	9	10	11	12	13	14	15	16	17
600	−4	−2	0	2	3	5	7	8	9	11	12	13	15	16	17	18	19	20
700	−4	−2	0	2	4	6	8	9	11	13	14	16	17	18	20	21	22	23
800	−5	−2	0	2	5	7	9	11	13	14	16	18	19	21	22	24	25	26
900	−5	−3	0	3	5	7	10	12	14	16	18	20	22	24	25	27	28	30
1000	−6	−3	0	3	6	8	11	13	16	18	20	22						
1100	−6	−3	0	3	6	9	12	14	17	20	22							
1200	−7	−4	0	3	7	10	13	16	19	21								
1300	−8	−4	0	4	7	11	14	17	20									
1400	−8	−4	0	4	8	11	15	18										
1500	−9	−4	0	4	8	12	16	20										
1600	−9	−5	0	4	9	13	17											
1700	−10	−5	0	5	10	14	18											
1800	−11	−5	0	5	10	15												
1900	−11	−6	0	5	10	15												
2000	−12	−6	0	6	11	16												
2500	−15	−7	0	7	14	20												
3000	−17	−9	0	8	16	24												

By the help of Tables 3 and 4 we may calculate $v - v_{35^0/_{00}, 0^0}$ for the water of the ocean. In *inland seas*, where the salinity is subject to great variations, Tables 5, 6, and 7 must be employed. Table 5 gives $v - v_{35^0/_{00}, 0^0}$ at 0°C. and one atmosphere of pressure, as a function of the salinity, and Tables 6 and 7 contain the corrections that have to be added, respectively, for temperature and for compression.

In *lakes*, the specific volume depends only on temperature and pressure; and the specific volumes need not, in this case, be referred to $v_{35^0/_{00}, 0^0}$, but only to v_{4^0}. Instead of formulæ (23) and (24), we now write—

$$E_1 = \int_{p_0}^{p_1} v_{4^0} dp. \tag{25}$$

$$E - E_1 = \int_{p_0}^{p_1} (v - v_{4^0}) dp. \tag{26}$$

TABLE 5.

$v - v_{35^0/_{00},\,0^0C.}$ at 0°C. and one Atmosphere Pressure.

S°/₀₀	Tenths per Mille.									
	0	1	2	3	4	5	6	7	8	9
2	2582_5	2574	2566	2558	2550	2542	2534	2526	2518	2510
3	2501_5	2493	2485	2477	2469	2461	2453	2445	2437	2429
4	2420_8	2413	2405	2397	2389	2381	2372	2364	2356	2348
5	2340_3	2332	2324	2316	2308	2300	2292	2284	2276	2268
6	2259_9	2252	2244	2236	2228	2220	2212	2204	2196	2188
7	2179_8	2172	2164	2156	2148	2140	2132	2124	2116	2108
8	2099_9	2092	2084	2076	2068	2060	2052	2044	2036	2028
9	2020_2	2012	2004	1996	1988	1980	1972	1965	1957	1949
10	1940_7	1933	1925	1917	1909	1901	1893	1885	1877	1869
11	1861_3	1853	1845	1838	1830	1822	1814	1806	1798	1790
12	1782_1	1774	1766	1758	1751	1743	1735	1727	1719	1711
13	1703_1	1695	1687	1679	1672	1664	1656	1648	1640	1632
14	1624_3	1616	1609	1601	1593	1585	1577	1569	1561	1553
15	1545_6	1538	1530	1522	1514	1506	1499	1491	1483	1475
16	1467_1	1459	1451	1444	1436	1428	1420	1412	1404	1397
17	1388_7	1381	1373	1365	1357	1350	1342	1334	1326	1318
18	1310_5	1303	1295	1287	1279	1272	1264	1256	1248	1240
19	1232_5	1225	1217	1209	1201	1194	1186	1178	1170	1162
20	1154_6	1147	1139	1131	1123	1116	1108	1100	1092	1085
21	1076_3	1069	1061	1053	1046	1038	1030	1022	1015	1007
22	999_1	991	984	976	968	960	953	945	937	929
23	921_6	914	906	898	891	883	875	867	860	852
24	844_2	836	829	821	813	806	798	790	782	775
25	766_9	759	751	744	736	728	721	713	705	697
26	689_8	682	674	667	659	651	644	636	628	620
27	612_8	605	597	590	582	574	567	559	551	544
28	535_8	528	520	513	505	497	490	482	474	467
29	459_0	451	444	436	428	421	413	405	398	390
30	382_3	375	367	359	352	344	336	329	321	313
31	305_6	298	290	283	275	267	260	252	244	237
32	229_1	221	214	206	198	191	183	176	168	160
33	152_6	145	137	130	122	114	107	99	92	84
34	76_3	69	61	53	46	38	31	23	15	8
35	0	−8	−15	−23	−31	−38	−46	−53	−61	−69

TABLE 6.

Correction for Temperature, to be applied to the Values in Table 5.

S.°/∞	Temperature, Centigrade.																	
	−2	−1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	16	7	0	−5	−9	−10	−10	−9	−6	−1	5	12	20	31	42	54	68	83
3	15	7	0	−5	−8	−9	−9	−7	−4	1	7	15	24	34	45	58	72	87
4	14	6	0	−4	−7	−8	−7	−5	−2	3	10	18	27	37	49	62	76	91
5	13	6	0	−4	−6	−7	−6	−4	0	6	12	20	30	40	52	66	80	95
6	12	5	0	−4	−6	−6	−5	−2	2	8	15	23	33	44	56	69	84	99
7	12	5	0	−3	−5	−5	−3	0	4	10	17	26	36	47	59	73	87	103
8	11	4	0	−3	−4	−4	−2	1	6	12	19	28	39	50	62	76	91	107
9	10	4	0	−3	−3	−3	−1	3	8	14	22	31	42	53	65	80	95	111
10	9	4	0	−2	−3	−2	1	5	10	17	24	34	44	56	69	83	99	115
11	9	3	0	−2	−2	−1	2	6	12	19	27	36	47	59	72	87	102	119
12	8	3	0	−2	−1	0	3	8	14	21	29	39	50	62	75	90	106	123
13	7	3	0	−1	−1	1	5	9	16	23	32	42	53	65	79	94	110	126
14	6	2	0	−1	0	2	6	11	17	25	34	44	56	68	82	97	113	130
15	6	2	0	−1	1	3	8	13	19	27	37	47	59	71	85	101	117	134
16	5	2	0	0	1	4	9	14	21	29	39	50	61	74	89	104	120	138
17	4	1	0	0	2	5	10	16	23	31	41	52	64	77	92	107	124	141
18	4	1	0	1	3	6	11	17	25	33	43	55	67	80	95	111	127	145
19	3	1	0	1	3	7	12	19	26	35	46	57	69	83	98	114	131	149
20	2	0	0	1	4	8	13	20	28	37	48	60	72	86	101	117	134	152
21	1	0	0	2	5	9	15	22	30	39	50	62	75	89	104	120	138	156
22	1	0	0	2	5	10	16	23	31	41	52	64	77	92	107	123	141	159
23	0	−1	0	2	6	11	17	24	33	43	54	67	80	94	110	127	144	163
24	−1	−1	0	2	6	12	18	26	35	45	56	69	83	97	113	130	147	166
25	−1	−1	0	3	7	12	19	27	37	47	59	71	85	100	116	133	151	170
26	−2	−2	0	3	7	13	20	29	38	49	61	74	88	103	119	136	154	173
27	−2	−2	0	3	8	14	21	30	40	51	63	76	90	105	121	139	157	176
28	−3	−2	0	4	9	15	23	31	41	52	65	78	93	108	124	142	160	180
29	−4	−3	0	4	9	16	24	33	43	54	67	81	95	111	127	145	164	183
30	−4	−3	0	4	10	17	25	34	45	56	69	83	98	113	130	148	167	186
31	−5	−3	0	5	10	17	26	35	46	58	71	85	100	116	133	151	170	190
32	−6	−4	0	5	11	18	27	37	48	60	73	87	102	118	136	154	173	193
33	−6	−4	0	5	11	19	28	38	49	61	75	89	105	121	138	157	176	196
34	−7	−4	0	5	12	20	29	39	51	63	77	92	107	123	141	160	179	199
35	−7	−4	0	6	13	21	30	41	52	65	79	91	109	126	144	163	182	203

TABLE 7.

Correction for Compression, to be applied to the Values in Table 5.

Depth (Metres).	Temperature, Centigrade.																	
	− 2	− 1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
100	− 1	0	0	0	1	1	1	2	2	2	2	3	3	3	3	4	4	4
200	− 1	− 1	0	1	1	2	3	3	4	4	5	5	6	6	7	7	7	8
300	− 2	− 1	0	1	2	3	4	5	5	6	7	8	8	9	10	10	11	12
400	− 3	− 1	0	1	3	4	5	6	7	8	9	10	11	12	13	14	15	16
500	− 3	− 2	0	2	3	5	6	8	9	10	11	13	14	15	16	17	19	20
600	− 4	− 2	0	2	4	6	8	9	11	12	14	15	16	18	19	21	22	24
700	− 5	− 2	0	2	4	7	9	11	12	14	16	17	19	21	23	24	26	
800	− 5	− 3	0	3	5	8	10	12	14	16	18	20	22	24	25	27		
900	− 6	− 3	0	3	6	9	11	14	16	18	20	22	25	27	29			
1000	− 7	− 3	0	3	6	9	13	15	18	20	22	25	27	30				
1200	− 8	− 4	0	4	8	11	15	18	21	24	27	30	33					
1500	−10	− 5	0	5	9	14	19	22	26	30	33	37						
2000	−13	− 6	0	6	12	19	25	30	34	39	44							

Table 8 contains ρ_{4°, v_{4° and E_1 along a vertical line entirely occupied by fresh-water of 4° C.; Table 9 gives $v-v_{4^\circ}$ at a pressure of one atmosphere as a function of the temperature; and Table 10 gives the correction for pressure to be applied to $v-v_{4^\circ}$.

TABLE 8.

Density, Specific Volume, and Value of E, along a Vertical where only Fresh Water of 4°C. is present.

Depth (Metres).	ρ_4^0	V_4^0	E_1
0	$1.000\,00_0$	$100\,000._0$	0
10	$1.000\,04_9$	$99\,995._1$	999 976
20	$1.000\,09_7$	$99\,990._3$	1 999 900
30	$1.000\,14_6$	$99\,985._4$	2 999 780
40	$1.000\,19_4$	$99\,980._6$	3 999 610
50	$1.000\,24_3$	$99\,975._7$	4 999 390
60	$1.000\,29_1$	$99\,970._9$	5 999 120
80	$1.000\,38_8$	$99\,961._2$	7 998 450
100	$1.000\,48_5$	$99\,951._5$	9 997 570
120	$1.000\,58_2$	$99\,941._8$	11 996 500
150	$1.000\,72_8$	$99\,927._3$	14 994 500
200	$1.000\,96_9$	$99\,903._2$	19 990 300
250	$1.001\,21_0$	$99\,879._1$	24 984 900
300	$1.001\,45_1$	$99\,855._1$	29 978 200
350	$1.001\,69_3$	$99\,831._0$	34 970 400
400	$1.001\,93_4$	$99\,807._0$	39 961 300
450	$1.002\,17_4$	$99\,783._1$	44 951 100
500	$1.002\,41_4$	$99\,759._2$	49 939 600
600	$1.002\,89_2$	$99\,711._6$	59 913 200
700	$1.003\,37_0$	$99\,664._1$	69 882 000
800	$1.003\,84_7$	$99\,616._8$	79 846 000
900	$1.004\,32_3$	$99\,569._6$	89 805 300
1000	$1.004\,79_7$	$99\,522._6$	99 759 900
1200	$1.005\,74_2$	$99\,429._1$	119 655 000
1500	$1.007\,15_0$	$99\,290._1$	149 463 000
2000	$1.009\,47_4$	$99\,061._5$	199 051 000

TABLE 9.

$v - v_{4°C.}$ at a Pressure of one Atmosphere.

Temperature (Centigrade).	Tenths of a Degree Centigrade.									
	0	1	2	3	4	5	6	7	8	9
0	12.9	13	12	11	11	10	10	9	8	8
1	7.2	7	6	6	6	5	5	4	4	4
2	3.1	3	3	2	2	2	2	2	1	1
3	0.9	1	1	1	1	0	0	0	0	0
4	0.0	0	0	0	0	0	1	1	1	1
5	1.0	1	1	2	2	2	2	2	3	3
6	3.0	3	4	4	4	5	5	6	6	6
7	6.7	7	8	8	9	9	10	10	10	11
8	11.4	12	13	13	14	14	15	16	16	17
9	17.8	18	19	20	21	21	22	23	24	25
10	25.3	26	27	28	29	30	31	32	33	34
11	34.5	36	37	38	39	40	41	42	43	44
12	45.1	46	47	49	50	51	52	53	55	56
13	57.0	58	60	61	62	64	65	66	67	69
14	70.1	71	73	74	76	77	78	80	81	83
15	84.1	86	87	89	90	92	94	95	97	98
16	99.9	102	103	105	106	108	110	111	113	114
17	116.0	118	120	122	124	125	127	129	131	133
18	134.8	137	139	141	143	144	146	148	150	152
19	154.2	156	158	160	162	164	166	168	170	172
20	174.4	177	179	181	183	185	187	189	191	194
21	195.7	198	200	202	204	207	209	211	213	215
22	217.7	220	222	225	227	229	231	234	236	238
23	240.5	243	245	248	250	252	255	257	259	262
24	264.1	267	269	272	274	276	279	281	284	286
25	288.8	291	294	296	299	302	304	307	309	312
26	314.4	317	320	322	325	328	330	333	336	338
27	340.8	344	346	349	352	354	357	360	363	365
28	368.2	371	374	377	380	382	385	388	391	394
29	396.5	399	402	405	408	411	414	417	420	423

TABLE 10.

Correction for Compression, to be applied to the Values in Table 9.

Temp. (Cent.).	Depth in Metres.													
	0	100	200	300	400	500	600	700	800	900	1000	1200	1500	2000
0	0	−1	−3	−4	−6	−7	−8	−10	−11	−13	−14	−17	−21	−28
1	0	−1	−2	−3	−4	−5	−6	−7	−8	−9	−11	−13	−16	−21
2	0	−1	−1	−2	−3	−3	−4	−5	−6	−6	−7	−8	−10	−14
3	0	0	−1	−1	−1	−2	−2	−2	−3	−3	−3	−4	−5	−7
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	1	1	1	2	2	2	2	3	3	4	5	6
6	0	1	1	2	2	3	4	4	5	6	6	7	9	12
7	0	1	2	3	4	5	6	7	7	8	9	11	14	18
8	0	1	3	4	5	6	8	9	10	11	12	15	18	24
9	0	2	3	5	6	8	10	11	13	14	16	19	23	30
10	0	2	4	6	8	10	12	14	15	17	19	23		
11	0	2	5	7	9	11	14	16	18	20				
12	0	3	5	8	11	13	16	18						
13	0	3	6	9	12	15	18							
14	0	3	6	10	13	16								
15	0	4	7	11	15									
16	0	4	8	11										
17	0	4	8											
18	0	4	9											
19	0	5	9											
20	0	5												
21	0	5												
22	0	5												
23	0	6												
24	0	6												
25	0	6												
26	0	6												
27	0	7												
28	0	7												
29	0	7												

With the aid of these Tables, we may calculate the values of $v—v_{35‰, 0°}$ or $v—v_{4°}$ in the great majority of cases, and in extreme cases, to which the Tables do not extend, we may obtain assistance by means of small exterpolations. After these values have been calculated for a hydrographic station, they must then be integrated according to formulæ (24) or (26). This may be most simply done by taking successive intervals, for which one assumes mean values of $v—v_{35‰, 0°}$, or $v—v_{4°}$, and then multiplies these by the pressure difference. If we indicate the mean values by $(v—v_{35‰, 0°})_m$, and $(v—v_{4°})_m$, then we have, in the case of salt-water, according to (24) :—

$$E—E_0 = (v—v_{35‰, 0°})_m \, (p_1—p_0) \qquad (27)$$

nd, in the case of fresh-water, according to (26) :—

$$E—E_1 = (v—v_{4°})_m \, (p_1—p_0) \qquad (28)$$

It follows from the inferences already drawn, that the pressure difference $p_1 - p_0$ is simply equal to the corresponding difference of depth. In calculating the values of $(v - v_{35‰, 0°})_m$, or $(v - v_{4°})_m$, it is, in general, sufficient to employ arithmetic means, because $(v - v_{35‰, 0°})$ and $(v - v_{4°})$ generally vary but little with the depth. The integration is, accordingly, not a troublesome one.

As an illustration of the method of calculating the value of $E - E_0$, we may choose the hydrographic observations of the "Heimdal" on 9th May, 1901, at station No. 10, latitude 64° 08′ N., longitude 4° 52′ W. Table 11 contains, in the first place, the observed depths, salinities and temperatures; then the values given by Tables 3 and 4, and the values for $v - v_{35‰, 0°}$ obtained by adding these together; the seventh column contains $(v - v_{35‰, 0°})_m$, and these numbers are, as we perceive, simple arithmetical means between the successive values of $(v - v_{35‰, 0°})$. If we multiply $(v - v_{35‰, 0°})_m$ by the corresponding differences of depth in metres, we obtain $(v - v_{35‰, 0°})_m \times (p_1 - p_0)$, that is to say, $E - E_0$ for each interval; and this value finds its place in the last column but one. Finally, by summing up these values, we obtain the values of $E - E_0$ from the surface to the various depths; and these values are shown in the last column.

TABLE 11.

Dynamical treatment of the observations made on board S.S. "Heimdal" at Hydrographical Station No. 10, latitude 64° 08′ N., longitude 4° 52′ W., May, 1901.

Depth.	Salinity.	Temperature.	Values from Table 3.	Values from Table 4.	$V - V_{35‰, 0°}$	$(V - V_{35‰, 0°})_m$	$(V - V_{35‰, 0°})_m \cdot (p - p_0)$	$E - E_0$
0	35.08	7.0	59	0	59			0
						55.5	555	
10	35.07	6.32	52	0	52			555
						43.5	652	
25	34.98	4.29	35	0	35			1 207
						31	775	
50	34.96	3.47	28	0	28			1 982
						28	700	
75	34.93	3.06	27	+1	28			2 682
						27.5	688	
100	34.92	2.95	26	+1	27			3 370
						26.5	662	
125	34.93	2.94	25	+1	26			4 032
						26	650	
150	34.93	2.90	25	+1	26			4 682
						26	1 300	
200	34.93	2.87	24	+2	26			5 982
						25	2 500	
300	34.89	2.13	22	+2	24			8 482
						21.5	2 150	
400	34.90	1.62	17	+2	19			10 632

The values of $E - E_0$ and $v - v_{35‰, 0°}$, in conformity with the above Table are transferred to sections, in the manner shown in Figure 4, in which figure the above observations are shown, together with those at other neighbouring stations.

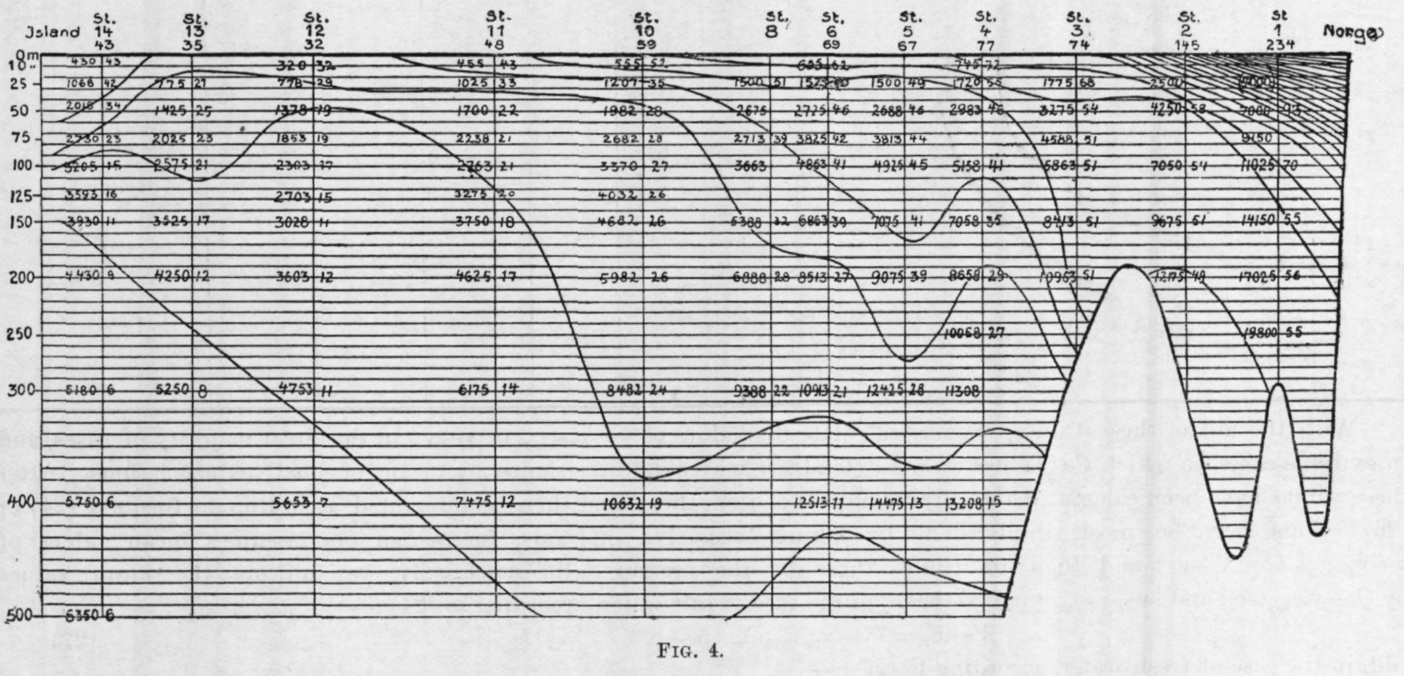

FIG. 4.

In order to find the number of solenoids that are enclosed by a curve composed of two horizontal and two vertical lines in this section, we must take, according to equation (22), the difference of the values of $E-E_0$ for the two verticals. This operation is especially easy in the case of those curves in which one horizontal boundary coincides with the surface of the sea, owing to the fact that all the values of $E-E_0$ at the surface in Figure 4, are equal to nil. Let us consider for example the closed curve whose two horizontal boundaries are at the surface and at 400 metres depth, and whose two vertical boundaries are at stations 4 and 14. For the vertical at station 4, $E-E_0$ amounts, according to the data given in Figure 4, to 13,208 c.g.s. units, and for the vertical at station 14, to 5,750. The difference of the two numbers is 7,458, and there are therefore 7,458 c.g.s. solenoids within the closed curve that we are considering. To see how the number of solenoids may be discovered when both horizontal boundaries lie beneath the surface, let us examine the closed curve which is built up of horizontals at 200 and at 400 metres depth and of verticals at stations 10 and 12. For the vertical at station 10, $E-E_0$ equals $10632-5982=4650$, and for the vertical at station 12, $E-E_0=5653-3603=2050$. The difference of these two values of $E-E_0$ amounts to $4650-2050=2600$, and therefore, within this curve, there are 2,600 c.g.s. solenoids. In other words, the number 2,600 is derived from the numbers at the four corners of the curve by the following process :—

$$(10632-5982)-(5653-3603)=2600.$$

These numbers may, if we prefer it, be grouped as follows :—

$$(10632+3603)-(5653+5982)=2600,$$

that is to say, we find the number of the solenoids by first adding the diagonally opposite numbers and then taking the difference of their sums.

The significance of the circulation-acceleration resulting from the solenoids may be deduced from the values of $v-v_{35^0/_{00},\,0^0\,C.}$ That is to say, if we compare with one another these values for points at the same level, we have the simple rule, that the water is driven upwards where $v-v_{35^0/_{00},\,0^0\,C.}$ is large, and downwards where it is small. This result may be very clearly seen by drawing lines upon our section connecting points where the values of $v-v_{35^0/_{00},\,0^0\,C.}$ are identical. The curved lines in Figure 4 are such lines as these. The solenoids tend to move the water in such a way as to bring these lines horizontal. Therefore, where these lines lie deepest, the water is being forced most powerfully upwards, and where they stand highest, the water is being driven most powerfully downwards.

The above explanations have made it clear that in calculating the number of solenoids, we may deal just as well with the quantity $v-v_{35^0/_{00},\,0^0C.}$ as with the specific volume v. The lines corresponding to equal values of $v-v_{35^0/_{00},\,0^0C.}$ play, consequently, the same part in this investigation as the lines which correspond to equal values of v, namely, the isosteres. If, then, in Figure 4, the lines corresponding to equal values of $v-v_{35^0/_{00},\,0^0C.}$ be drawn for each unit of that value, and if the isobars in like manner be drawn for each metre of depth, then each one of the parallelogram-like figures enclosed by the intersection of these two systems of lines represents a solenoid. In Figure 4, however, the lines are not drawn so numerously as this, but only for every 10 units of $v-v_{35^0/_{00},\,0^0C.}$, and correspondingly for every 10 metres depth; accordingly, each one of the parallelogram-like figures represented by the intersections in the figure, comprehends 100 solenoids. If, then, one inscribe upon Figure 4, any closed curve whatsoever, we may very easily count the number of solenoids that this curve contains, by simply counting the parallelogram-like figures within the curve and multiplying by 100.

Professor Nansen has given a formula which enables us to reckon the number of solenoids within a curve composed of two isobars and two verticals, provided we know the density of the water in the two verticals.* Let us denote by G the acceleration of gravity, by h the length of the verticals and by q_1, q_2 the average density of the water in the two verticals ; let q_1 be less than q_2. Then

$$A=h\left(\frac{q_2}{q_1}-1\right)G.$$

This formula may be derived from formula (19) by employing the differential relation between pressure and depth, $dp=Gqdh$. In cases where the density is known, this formula is well adapted for the enumeration of the solenoids.

(2) On the Acceleration due to Terrestrial Rotation.

The acceleration of the circulation due to terrestrial rotation in a closed curve of water-particles, is, according to (15)

$$2\omega\frac{dS}{dt},$$

where ω is the angular velocity of the earth and $\frac{dS}{dt}$ the change in the area S of the projection of the closed curve on the equatorial plane. We must, therefore, first project the closed curve upon the equatorial plane, in order to calculate $\frac{dS}{dt}$

* Fridtjof Nansen : The Norwegian North Polar Expedition, 1893–1896. Oceanography of the North Polar Basin, p. 355.

from the observed velocities of the current. The normal components of the projected velocities must be integrated along the projected curve. It is evident that the change of area is only affected by the normal components and not at all by the tangential components. If we multiply by 2ω the value $\frac{dS}{dt}$ that we have thus obtained, we get the influence of terrestrial rotation upon the circulation of the curve.

In most cases, the vertical velocity of the water is so small in comparison with the horizontal that it may be neglected, and thereby the calculation is considerably facilitated. We first calculate the change $\frac{d\sigma}{dt}$ of the projection of the closed curve upon the sea-level, and multiply it by the sine of the latitude λ, and by 2ω, for we have :—

$$\frac{dS}{dt}=\frac{d\sigma}{dt}\sin\lambda,$$

and, therefore, the influence of the earth's rotation upon the circulation is given by

$$2\omega\frac{dS}{dt}=2\omega\frac{d\sigma}{dt}\sin\lambda. \qquad (29)$$

It is convenient to express the quantity $\frac{d\sigma}{dt}$ in square metres per sec., that is to say in $10^4\,\frac{cm^2}{sec}$. In order to have the whole expression in c.g.s. units, we must then express 2ω in units of $10^{-4}\,\frac{1}{sec}$.

$$\text{Now, } 2\omega=0{\cdot}0001458\,\frac{1}{sec}=1{\cdot}458\times10^{-4}\,\frac{1}{sec}.$$

Equation (29) stands accordingly

$$2\omega\frac{dS}{dt}=1{\cdot}458\frac{d\sigma}{dt}\sin\lambda. \qquad (30)$$

Table 12 gives 1·458 sin λ for each complete geographical degree of latitude.

TABLE 12.

1·458 sin λ.

λ	0	1	2	3	4	5	6	7	8	9
0	0	0·026	0·051	0·076	0·102	0·127	0·152	0·178	0·203	0·228
10	0·253	0·278	0·303	0·328	0·353	0·377	0·402	0·426	0·451	0·175
20	0·499	0·523	0·546	0·570	0·593	0·616	0·639	0·662	0·685	0·707
30	0·729	0·751	0·773	0·794	0·815	0·836	0·857	0·878	0·898	0·918
40	0·937	0·957	0·976	0·995	1·013	1·031	1·049	1·067	1·081	1·101
50	1·117	1·133	1·149	1·165	1·180	1·195	1·209	1·223	1·237	1·250
60	1·263	1·275	1·287	1·299	1·311	1·322	1·332	1·342	1·352	1·361
70	1·370	1·379	1·387	1·395	1·402	1·409	1·415	1·421	1·426	1·431
80	1·436	1·440	1·444	1·447	1·450	1·453	1·455	1·456	1·457	1·458

To illustrate the calculation of the value of $2\omega\frac{dS}{dt}$, let us suppose that the water between stations 10 and 11 of the "Heimdal's" cruise in May, 1901, was being carried northwards at the rate of $6{\cdot}4\,\frac{cm}{sec}$ at the surface, and at the rate of $5{\cdot}1\,\frac{cm}{sec}$ at 200 metres depth. The projection on the sea-level of the closed curve, which lies between the verticals at stations 10 and 11, and the horizontals at the surface and at 200 metres depth, has, at the outset, no area. After one second, the projection is a rectangle whose length is equal to the distance, 90,000 metres, between the two stations, and whose breadth is 6·4 — 5·1, *i.e.*, 1·3 cm.; the area of this rectangle is equivalent to 1,170 square metres. The projection of the closed curve upon the sea-level has therefore increased by 1,170 square metres in one second, that is to say, $\frac{d\sigma}{dt}$ amounts to 1,170 $\frac{m^2}{sec}$. This quantity must now, according to (30), be multiplied by 1·458 sin λ. In this case λ is 64° 35′, and Table 12 gives, correspondingly, 1·458 sin λ = 1·317. Thus, granting the assumptions we have made, the circulation-acceleration due to the earth's rotation amounts to $1{\cdot}317\times1{,}170=1{,}540\,\frac{cm^2}{sec^2}$

If we apply the rules concerning the significance of the acceleration due to terrestrial rotation that we have already given under the head of theoretical explanations, we find that the water is being driven downwards at station 10 and upwards at station 11.

The determinations of velocity which are to serve us as a basis for the determination of $2\omega\frac{dS}{dt}$ must be executed with exceptional care. For if we, for example, take the velocity at 200 metres depth to be $5{\cdot}0\frac{cm}{sec}$ instead of $5{\cdot}1\frac{cm}{sec}$, and in other respects perform the calculation as above, we get, in this case, $2\omega\frac{dS}{dt}=1659\frac{cm^2}{sec^2}$ instead of $1540\frac{cm^2}{sec^2}$ in the other case; that is to say, in this instance an error of $0{\cdot}1\frac{cm}{sec}$ in the velocity is tantamount to an error of $119\frac{cm^2}{sec^2}$ in the circulation-acceleration.

(3) On the Circulation-Acceleration R due to Friction.

In calculating the value of R, it is convenient to choose a "stationary" case; that is to say, a case in which the circulation round the closed curve is constant and consequently $\frac{dC_r}{dt}=0$. Equation (15) becomes, in this case, the following:—

$$0=A-2\omega\frac{dS}{dt}-R.$$

If A and $2\omega\frac{dS}{dt}$ be calculated in the manner already described, then this equation gives us the value of R. Let us consider once more for the purposes of illustration, the curve formed by our vertical lines at the "Heimdal" stations 10 and 11, May, 1901, and of the horizontals at 0 metres and 200 metres. For this curve, according to Figure 4, $A=1357\frac{cm^2}{sec^2}$, and if we assume for $2\omega\frac{dS}{dt}$ the value $1540\frac{cm^2}{sec^2}$, then

$$0=1357-1540-R;$$

and therefore $R=-183\frac{cm^2}{sec^2}$. The significance of the circulation-acceleration due to friction is always to be discovered by bearing in mind that the friction acts in a direction contrary to the larger of the two other elements. Hence in the example given above, friction acts in the opposite direction to the influence of the earth's rotation, while it supports or accentuates the action of the solenoids. This comes out clearly if we introduce $R=-183$ into the last equation. We then have $p=1357-1540+183$. Here, the values for the solenoids and for the influence of friction have the same sign, while the values for the influences of terrestrial rotation and of friction have opposite signs.

(4) Calculation of the Total Circulation-Acceleration $\frac{dC_r}{dt}$.

Let us assume that we have gained so much information regarding the value of R, by estimating it in numerous "stationary" cases, that we may ascribe a value to it for given closed curves. Suppose, for example, that the value $R=-183$ in the curve selected above be too small, and that the true value be somewhere about $R=-400$. In that case, the circulation of this curve is not constant, as it would be for $R=-183$, but it alters with the time. The amount by which the circulation changes is found by introducing into formula (15) $A=1357$, $2\omega\frac{dS}{dt}=1540$, and $R=-400$. From this we obtain

$$\frac{dC_r}{dt}=1357-1540+400=217.$$

Hence it results that the circulation of the curve is accelerated in the direction required by the solenoids. The movement of the water is therefore being accelerated, in the example chosen, towards Iceland at the surface and towards the Norwegian coast in the deep water. The mean acceleration of individual particles of the curve is found by dividing the circulation-acceleration 217 by the whole length of the curve, namely 18,040,000 cm. By so doing we obtain

$$\dot{u}_{tm}=0{\cdot}000012\frac{cm}{sec^2}.$$

This acceleration leads to an increase of velocity amounting to $1{\cdot}04\frac{cm}{sec}$ in 24 hours.

We have now demonstrated how the circulation-acceleration is calculated for a closed curve constituted by two vertical and two horizontal lines. To perform this calculation the density, velocity, and friction of the water must be known. Knowing these, we can calculate from them the circulation of the water for the next succeeding interval of time. Hence the circulation formula (15)

$$\frac{dC_r}{dt}=A-2\omega\frac{dS}{dt}-R,$$

is primarily of a prophetic or prognostic nature and will very probably find, in the future, its most important applications in the drawing of anticipatory inferences.

III.—ON THE QUALITATIVE DEDUCTION OF THE OCEAN CURRENTS OF THE NORTH ATLANTIC.

It is evident, from the calculation of friction under stationary conditions, as shown above, that formula (15) may also be used for other purposes. To close our discussion, we will give one other application of this law, illustrating it by deducing from the distribution of the specific volume in Figure 2, the great circulation north of the Equator in the Atlantic. Let us neglect, in the first instance, the earth's rotation and make use of the rule regarding the significance of the circulation due to the solenoids, in the case of the solenoids of Figure 2. Hence follows the circulation indicated by the arrows in Figure 5.

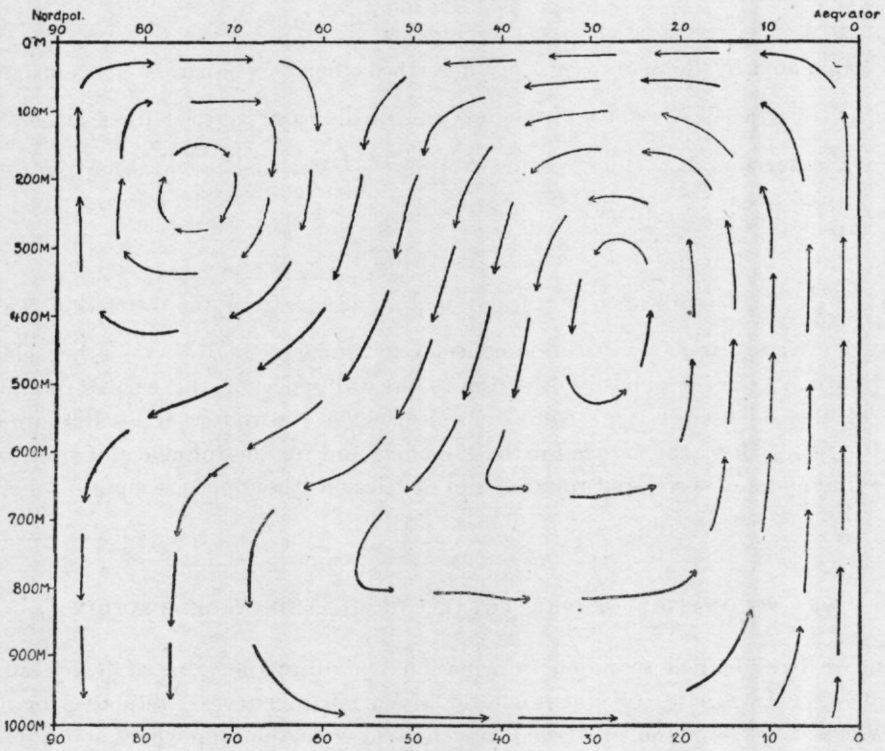

FIG. 5.

This circulation tends to increase, because the solenoids produce an effect of acceleration. But the more the circulation increases, the more also does the retarding influence of friction increase, and finally, solenoids and friction come into equilibrium one with another, so that the circulation remains constant. The ultimate condition of things is, accordingly, a stationary circulation such as is shown in Figure 5.

It is evident, from Figure 5, that the surface-water in the neighbourhood of the Equator is being carried northwards. If we now consider a closed curve on the surface of the ocean, constituted by horizontal lines along the African and American coasts, and horizontal lines running from continent to continent along the Equator and along the 30th parallel of North latitude, then the projection of this curve upon the equatorial plane expands as the horizontal boundary in 30° N. Lat. is carried northwards. The circulation of the curve is of course influenced by terrestrial rotation. If we apply in this case our rule for finding the direction of this influence, it will be found that the water is moving westwards at the Equator and eastwards at the latitude of 30°. On the American coast it is moving northwards, and on the African coast southwards. The direct observation of currents proves to be in harmony with this theoretical circulation.

According to Figure 5, the surface-water is driven northwards by the solenoids in lower latitudes, and southwards in higher latitudes. If we consider the change of the projection upon the equatorial plane of such closed curves as are constituted of two verticals and two horizontals in these currents, we discover that the northerly surface-stream is guided by terrestrial rotation towards the European coast, while the other moving southwards is deflected under the same influence towards the coasts of Greenland and America. This is the qualitative explanation of the Gulf Stream and of the cold East Greenlandic current. We perceive also, from Figure 2 and from Figure 5, that the larger number of solenoids is employed in impelling the movement of the Gulf Stream, a result in conformity with our knowledge of its strength and size. In deep water, as we see from Figure 5, the movement established by the solenoids is in a southward direction. This current is deflected by the earth's rotation towards the American coast.

These explanations illustrate how the ocean-currents may be deduced qualitatively, that is to say, according to their general direction, from the known distribution of density. In order to find their velocity, it is necessary to take into consideration the further influence of friction.