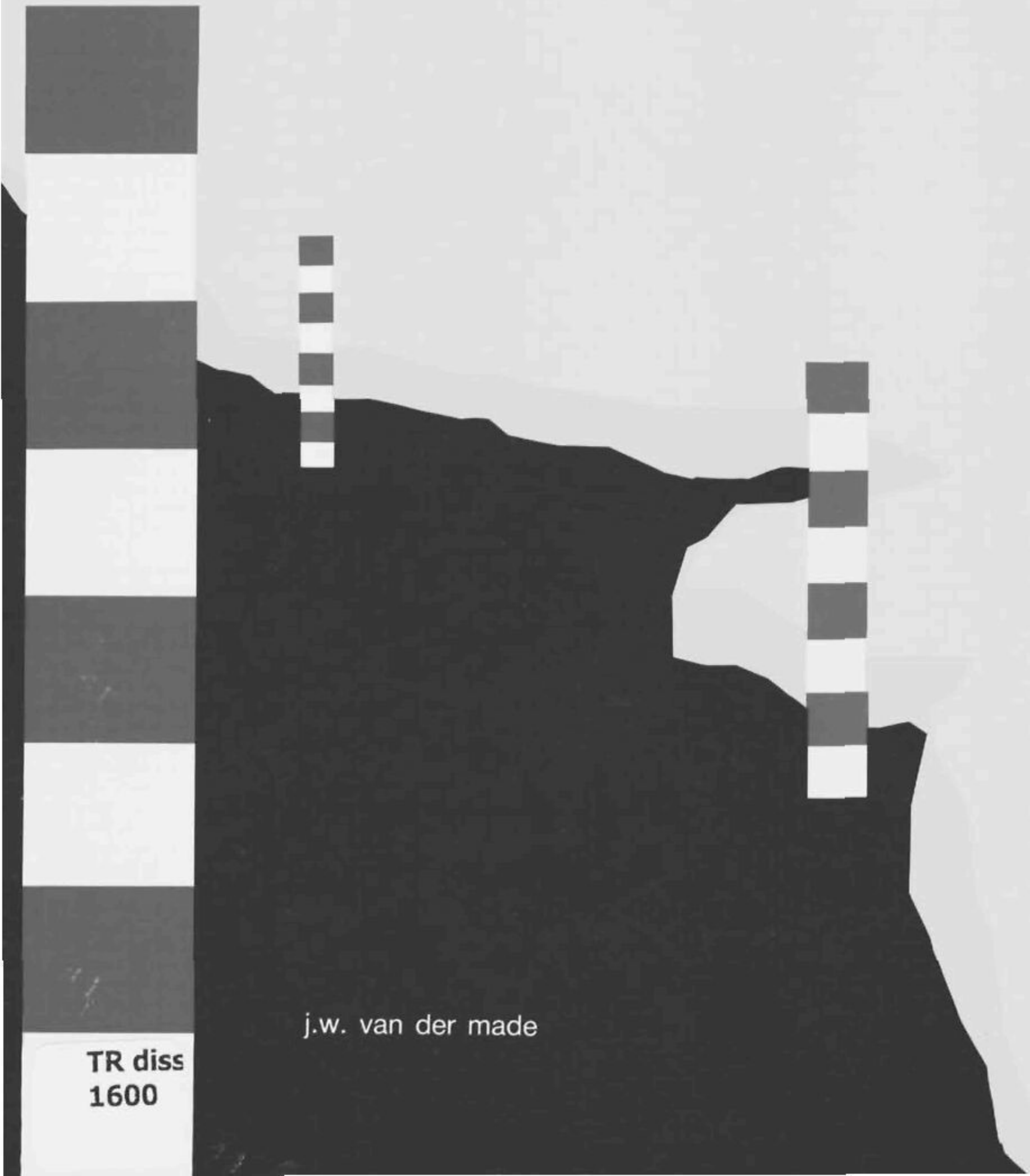
analysis of some criteria for design and operation of surface water gauging networks



j.w. van der made

Analysis of some criteria for design and operation of surface water gauging networks

PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR
AAN DE TECHNISCHE UNIVERSITEIT DELFT,
OP GEZAG VAN DE RECTOR MAGNIFICUS,
PROF.DR. J.M. DIRKEN,
IN HET OPENBAAR TE VERDEDIGEN TEN OVERSTAAN
VAN EEN COMMISSIE DOOR HET
COLLEGE VAN DEKANEN DAARTOE AANGEWEZEN
OP DINSDAG 5 JANUARI 1988
TE 16.00 UUR

DOOR

JOHANNES WILLEM VAN DER MADE

GEBOREN TE ROTTERDAM

CIVIEL INGENIEUR

1987
Van Gorcum, Assen

Dit proefschrift is goedgekeurd door de promotor
Prof.dr.ir. J.C. van Dam

STELLINGEN

bij

Analysis of some criteria for design and operation of surface water gauging networks

J.W. van der Made.

1

Bij interpolatie tussen waterstanden op rivieren of kustwateren geeft in veel gevallen de lineaire interpolatie de meest betrouwbare resultaten.

2

Het ontwerpen van een hydrologisch meetnet vormt een kip-ei probleem: het ontwerp wordt voornamelijk gebaseerd op gegevens van een reeds bestaand meetnet.

3

Omdat de optimale dichtheid van een meetnet van verschillende statistische, economische en andere factoren afhangt is het voorschrijven van algemeen geldende dichtheden niet juist. Word Meteorological Organization, 1982: **Guide to Hydrological Practices**, Geneva.

4

De 'ware waarde' van een waterstand heeft geen betekenis: het vermelden van een waterstandswaarde is slechts zinvol indien men ook vermeldt voor welk gebieds- en tijdsinterval deze representatief is.

5

Om het goed functioneren van een peilmeetnet te verzekeren is naast een primair net een aanvullend net gewenst.

6

De variantie van de meetfouten, als ontwerpcriterium voor een optimaal meetnet, is in feite alleen geschikt als de meetfouten normaal verdeeld zijn en geen autocorrelatie vertonen.

7

Een te bemeten grootheid, bijv. een waterstand, zal veranderen als gevolg van de meting. Hoeveel hangt af van de toegepaste meetmethode.

8

Omdat de meetvariantie ε^2 van een getoetst station meetelt in de variantie van het verschil tussen de gemeten en berekende waarde (Var Δy) bestaat de mogelijkheid dat een slecht functionerend station (met grote ε) ten onrechte wordt gehandhaafd of zelfs wordt vervangen door een goed station. Dit proefschrift, formule (2-17):

Var Δ y = ε^2

+ variantie voortgeplante meetfouten

+ variantie modelfouten.

9

De vraag van D.R. Dawdy: 'Has a stream flow gaging station or a precipitation gage ever been added to or dropped from a network based primarily on its worth to the network?', kan, als hij ook betrekking heeft op peilmeetstations, met 'ja' worden beantwoord. Dawdy, D.R., 1979: **The worth of Hydrologic Data**, Water Res. Res. Vol 15, No 8.

10

De benaming 'transformatie matrix' is, bij toepassing van het Kalmanfilter op niet-lineaire modellen onjuist. In feite moet men dan spreken van 'gevoeligheidsmatrix'.

11

Een gebleken geldigheid van een bepaalde frequentieverdeling voor een gemeten reeks waterstanden of afvoeren zegt niets over de geldigheid voor een zeer veel langere reeks van hetzelfde verschijnsel.

12

Een bepaalde drempelwaarde kan overschreden worden, zowel in positieve als in negatieve zin.
Het woord 'onderschrijding' wijst op een verkeerd taalgebruik.

13

Een verbeterde bescherming tegen natuurrampen, zoals overstromingen, veroorzaakt tevens een grotere kwetsbaarheid daarvoor.

14

Het verdient de voorkeur de meteorologische, hydrologische en maritieme voorspellings- en waarschuwingsdiensten in één landelijk instituut onder te brengen.

15

Een belangrijk energieprobleem wordt niet veroorzaakt door een tekort aan energie maar door een overschot, dat op ondoelmatige wijze wordt verbruikt en daardoor schade toebrengt aan de samenleving.

16

Bij ontwapeningsbesprekingen zou men niet het verschil tussen de aantallen wapens van beide partijen als uitgangspunt moeten kiezen maar de som ervan.

17

Files vormen een terugkoppeling in het autoverkeerssysteem.

18

Het plaatsen van verkeerslichten aan het benedeneinde van een helling bevordert het verschijnsel van het rijden van fietsers door rood licht.

19

Het digitaliseren van muziek is niet nieuw. Het wordt reeds lang toegepast in het draaiorgel.

20

Een grafische voorstelling is een vorm van surrealistische kunst.

21

Als er een permanente zons- of maansverduistering zou heersen zou de berekening van de getijtafels aanzienlijk eenvoudiger zijn dan nu het geval is.

22

Het feit dat Sint Nicolaas reeds eeuwenlang met een volbeladen schip jaarlijks, op nagenoeg dezelfde dag, in Nederland aankomt wijst erop, dat hij vanaf het begin van zijn activiteiten over goede waterstandsgegevens moet hebben beschikt.

Voorwoord

Toen ik in 1968 van de WMO, de Meteorologische Wereldorganisatie, het verzoek kreeg een bijdrage te leveren aan het, onder leiding van de legendarische Amerikaanse hydroloog W.B. Langbein, samen te stellen 'Casebook on Hydrological Network Design Practices' had ik nog weinig over ontwerpmethoden voor hydrologische meetnetten nagedacht. Al schrijvende echter werd mijn belangstelling gewekt en na het gereedkomen van de gevraagde hoofdstukken en de publicatie van het Casebook in 1972 heb ik mij verder in het onderwerp verdiept. In die tijd deden de behoeften van de waterhuishouding, de scheepvaarteisen en de opkomende automatisering de wenselijkheid ontstaan het peilmeetnet langs de grote rivieren en de getijwateren aan een grondige toetsing te onderwerpen. De daartoe gemaakte studies waren in 1978 voor ir. H.M. Oudshoorn, Hoofdingenieur-Directeur in de Directie Waterhuishouding en Waterbeweging van de Rijkswaterstaat, aanleiding aan mij voor te stellen over dit onderwerp een proefschrift te schrijven.

Ik vond toen prof.dr.ir. J.C. van Dam bereid hiervoor als promotor op te treden. In de jaren die volgden, en waarin de toetsing van het peilmeetnet plaats vond, heb ik veel steun ondervonden van de medewerkers van de Operationele Afdeling van genoemde directie, in het bijzonder van ir. J. van Malde, hoofd van deze afdeling. Bij de toetsing werden de in deze studie ontwikkelde methoden toegepast, waardoor een optimaal doelmatig meetnet werd verkregen. De methode heeft aldus zijn practische bruikbaarheid bewezen. Ik wil dank zeggen aan prof.dr.ir. J.P.Th. Kalkwijk voor de ondervonden hulp bij de behandeling van de hydrodynamisch-numerieke modellen. Daarnaast is de steun, verleend door dr.ir. A.W. Heemink en door dr. R. Helmers van grote waarde geweest. Verder dank ik de verschillende systeemanalysten, waaronder vooral de heer C. Heins, die behulpzaam waren bij het uitvoeren van de vele berekeningen, de medewerkers van de Afdeling Visuele Vormgeving voor het uitstekende tekenwerk, met name de heer W. Storm, de dames die het typewerk verzorgden, waaronder mevr. J. Mondé en de medewerksters van de Interdepartementale Typekamer te Winschoten. Verder dank ik de heer P. van Elk en mevr. K. Walzberg-van Kranenburg voor de typografische verzorging.

I also acknowledge mr. R.J. Moore of the Institute of Hydrology, Wallingford, UK, for his suggestions for revision of the English text.

Tenslotte wil ik allen danken, die in de loop der jaren van hun belangstelling voor de voortgang van dit proefschrift hebben blijk gegeven. In het bijzonder mijn vrouw Joop, die moest toezien hoeveel nachtelijke uren aan dit proefschrift werden besteed.

Des te meer betreur ik het, dat mijn ouders en mijn schoonouders deze bekroning niet meer konden beleven.

Aan Joop,
Jan en Ingrid.

Contents

1 Introduction

1.1 Motivation for measurement

Generally spoken natural science is based on measurements and observations. Any hypothesis or theory can be confirmed only if information, observed from occurring phenomena is available. Measurements can be carried out occasionally, e.g. for a specific study, or more regularly with fixed time intervals or even continuously.

The motivation to measure phenomena on a regular time base are twofold:
a. Interest in the phenomena itself;
b. Variability of the phenomena is so great that regular observations, or measurements, are desirable in order to gain adequate insight into the behaviour of those phenomena.

For water, both these two reasons are relevant. Water is closely related to life on earth, and is even a condition for the existence of life. Consequently man has an interest in water for his very survival.

Interest in variability concerns the overall behaviour of the phenomena but in particular the extreme conditions, which can conflict with man‘s interests. It is then especially important to be aware of what is going on. On the other hand, in situations where plenty of good quality water, is assured and the amounts are not excessive, then there will be less reason to monitor water. If this is not the case, then information on variability will be useful, desirable or necessary, depending on the gravity of the situation. Such information may be obtained through measurement of one or more characteristic features of water.

1.2 Aspects of measurement

Measurement can be considered from several points of view. It can concern:
a. The state in which the water is examined. In this connection one can distinguish:
   - water in the atmosphere: water vapour (content and transport);
   - water exchange between the atmosphere and the earth surface: precipitation, evaporation and evapotranspiration;
   - water on the earth surface: in rivers, lakes, seas and oceans, and also water in the form of snow and ice;

- water in the unsaturated zone: soil moisture (content and transport);
- water in the underground and water bearing rocks: groundwater (level, volume and transport);

b. Quantity or quality;

c. The measured units;
- volume $[l^3]$,
- level $[l]$,
- velocity $[lt^{-1}]$,
- discharge $[l^3\ t^{-1}]$,
- runoff (or specific discharge) $[l^3\ t^{-1}\ l^{-2}] = [lt^{-1}]$,
- density $[ml^{-3}]$,
- temperature [°C]

d. The dimension of the water body considered:
- river or canal $[l]$,
- land, lake or sea area $[l^2]$,
- soil, lake or sea volume, space of the asmosphere $[l^3]$.

All four of the above aspects should be considered when planning a measurement programme. In many cases the aspects are related. For instance precipitation, evaporation and runoff will be examined for a two dimensional plane and will be expressed in the dimension $[lt^{-1}]$ e.g. mm per year. Water levels of seas and lakes, as well as groundwater levels are also considered two dimensionally. But for rivers and canals a one dimensional approach can be adequate. In deep water bodies even a three dimensional examination may be needed, in particular where water quality problems are involved.

Surface water data can concern water levels, velocities, discharges and runoff, and each form may be of more relevance to particular fields of interest. Water levels are of importance for navigation, flood protection, reservoir capacity and level control in polders. Discharges are related to water management, water supply, water balances and water quality problems, whereas velocities concern safety aspects of structures and of ships and quality problems influenced by mixing.

For demonstrating the relative importance of water level, discharge and velocity data reference is made to Fig. 1-1.

Following a river downstream from the source one will see that in the upper reaches the importance of the discharges and the velocities is dominant. Water levels, if measured here are used as an indication of the streamflow, but they have little value in their own right. In these regions there is a strong relation between discharge and local precipitation.

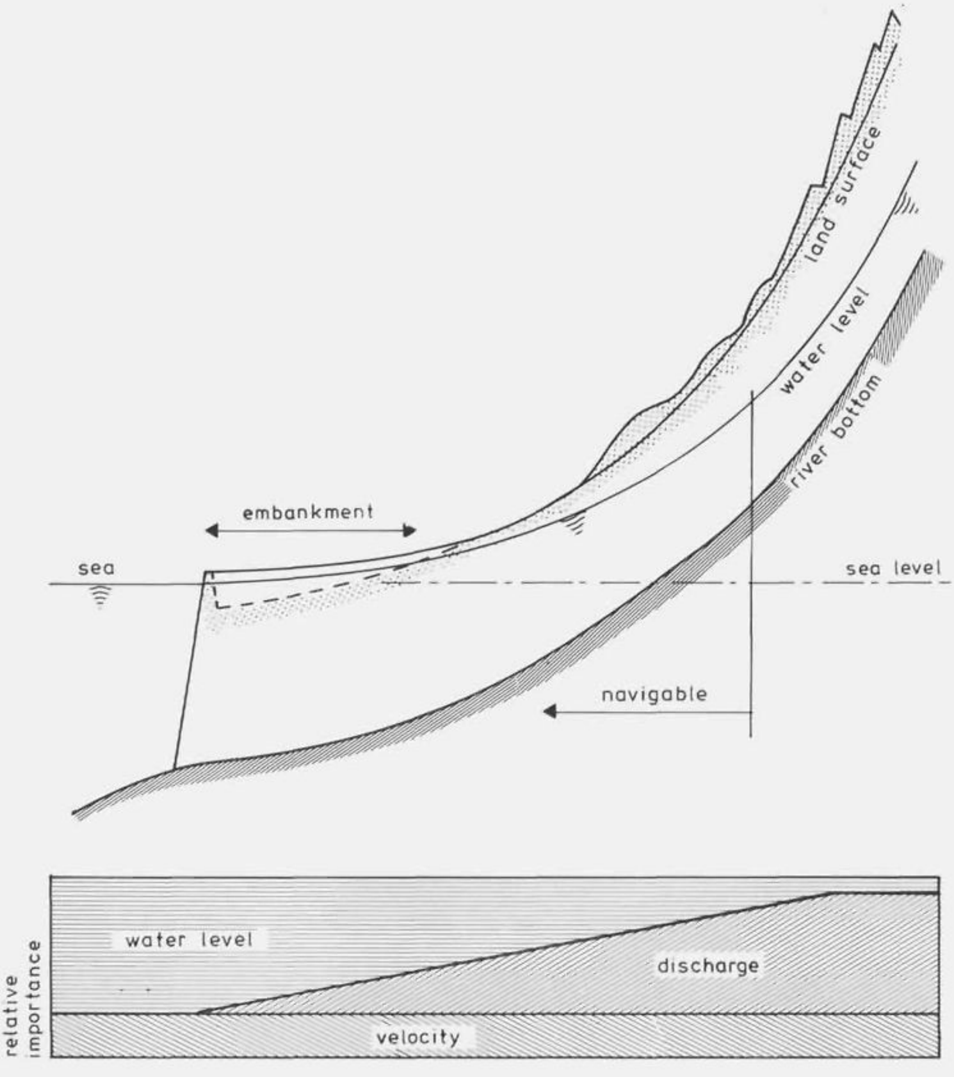


Fig. 1-1 Relative importance of water level, discharge and velocity data.

Going downstream, in particular from the point where the river becomes navigable, information on water levels as such becomes progressively more important. The adjacent land is lower with respect to the water levels than in the higher regions. Also navigation becomes progressively more important.

Water level data are very important in the tidal lowlands, in particular where the river is bordered by embankments and dykes. Finally in the sea only the water levels are of interest: the importance of discharge loses its meaning.

Velocity data have importance along the whole length of the river, including its seaward extension, since velocity is related to quality problems, which affect all places, to the safety of structures like dams and bridges and, finally, to navigation for both professional and recreational use.

The changing importance of water level, discharge and velocity data along a river course is shown schematically in Fig. 1-1.

Finally surface water data may be presented as runoff values, and in this form relate to the drainage basin as a whole or parts of it. Runoff expresses the depth of the water layer that is carried off in a certain interval of time. As such it is comparable with precipitation and evaporation; and its main use is in water balance studies.

The present study is focussed on the measurement of water level data, mainly along one dimensional water courses like rivers and tidal streams, but some attention is also paid to two dimensional water areas.

1.3 Definition of a measurement network

The historical development of water monitoring has led to a great number of level gauges, which, certainly in the beginning, were rarely adjusted to each other, but were mainly installed for local needs.

However, if a set of gauges is considered as a network, a certain coherence between the phenomena measured at these gauges is assumed to exist. Such a coherence inevitably developed as the number of gauges increased, and, at the same time, the mutual distances between the gauges decreased.

The coherence was used, for instance, to develop dependence relations between two gauging stations. With such relations the phenomena (e.q. levels) at one station can be estimated, within certain limits, when the data at the other station are known.

As the gauge numbers increased over time their spheres of influence more and more overlapped each other. Here the term 'sphere of influence' is used to mean the area within which the phenomena show some correlation or coherence with the phenomena at the gauging station considered.

Now, let such a sphere of influence be expressed by a circle; see Fig. 1-2. Case A shows a situation of two isolated gauging stations at which the phenomena behave fully independently. In case B there are some areas in which the phenomena are related to measured values at two stations. In this case the set of gauging stations may be regarded as a network. It is apparant that one may distinguish between:

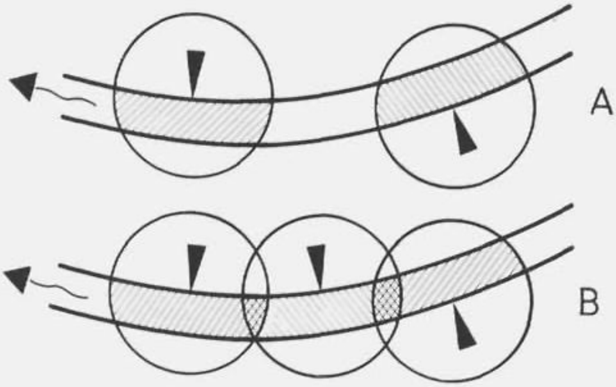


Fig. 1-2 Gauging stations and their spheres of influence

- a set of isolated gauging stations, which can provide local information around these stations only;
- a network of gauging stations which is able to provide information for every place within an area, consisting of several overlapping spheres of influences.

In the course of development a network may pass through the following phases:

a. Isolated stations phase. Single stations are established in order to meet local needs. The number of stations increases with the socio-economic development of the region;
b. Network phase 1. Station density becomes such that the measured values, although not intended, are beginning to show some coherence;
c. Network phase 2 or the phase of consolidation. The degree of coherence between measured values is strong. Many stations are maintained 'out of habit', although they produce superfluous information.
d. Network phase 3 or the phase of reduction. One becomes aware that much superfluous (i.e. expensive) information is being produced. Coherence relations are determined and a start is made to abandon certain stations.

Generally there is an initial period of growth followed by a period of consolidation and, finally, a reduction in the number of stations. An example of such a development is shown in Fig. 1-3 (Fontaine et al, 1983).

The foregoing discussion indicates that a measurement network constitutes a set of gauging stations, distributed over an area (or along a line) in such a way that anywhere within that area a value for the variable considered can be determined. In this context an isolated station is in fact a 'network' for its own sphere of influence.

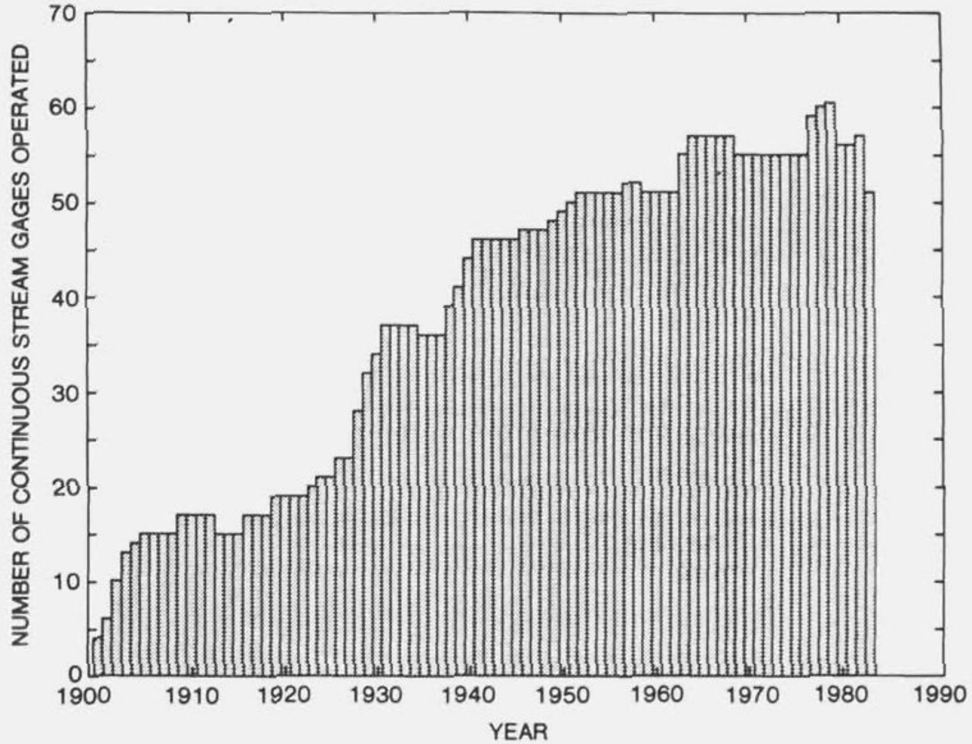


Fig. 1-3 History of continuous stream gauging in Maine.

But a set of stations, producing values which are independent from one station to the next do not constitute a network.

The effect of the network is, that the variability of the value of a phenomenon is reduced from the total or natural variability (assumed known a priori) to the variability of the estimate. If this concerns a measured value, then the remaining variability lies in the uncertainties of the measurements. If it concerns a value derived from measurements elsewhere (i.e. an interpolated value), then the remaining variability of the estimate is also related to the existing correlations and to the feasibility of the interpolation method. In both cases the network reduces the variability of the known values, schematically expressed in Fig. 1-4.

So the measurement network reduces the uncertainties and also improves the knowledge about the measured and interpolated values.

The existence of coherence between the values of a variable somewhere in the area considered, and that of the value of the corresponding variable at a gauging station,

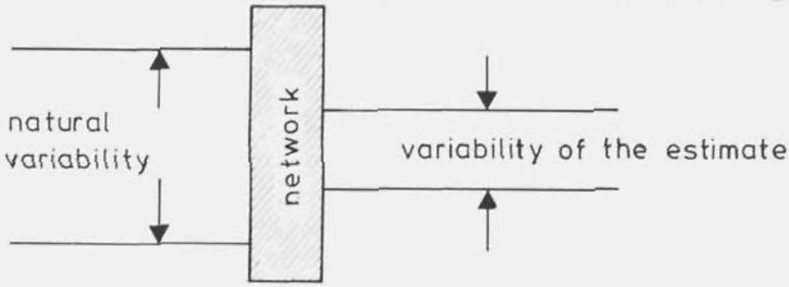


Fig. 1-4 Effect of a measurement network on the variability of the known values.

is essential for the network designer. Without such coherence, only values at the stations themselves could be determined. Coherence is described in the Explanatory Hydrological Glossary (TNO, 1986, in dutch), where it used to define a measurement network as follows: 'A system of coherent gauging stations, measurement or sampling posts'. In the definition of a network given in the International Glossary of Hydrology (Unesco - WMO, 1974) the need for coherence is somewhat obscured. A network is understood to mean an: 'Aggregate of hydrological stations and observing posts, situated within a given area in such a way as to provide the means of studying the hydrological regime'.

The coherence of the phenomena is used for the design of a measurement network and later on, after its completion for the operation of the network. Then coherence is used for checking the correct functioning of the gauges, for reconstructing missing data and for estimating non-measured values. In this respect the network forms the first link in an overall information system.

1.4 Requirements and criteria for network design

The effect the network may be expressed in the reduction of the variability of the value of concern (Fig. 1-4). The original variability may be expressed in terms of the standard deviation of the phenomena (water levels) concerned, the resulting variability in the standard error of the estimate or the standard error of measurement, depending on the value of concern is an interpolated (or estimated) value, or a measured value. The quality of the network could be described by the ratio of 'input' standard deviation to the 'output' standard error.

In general, a dense and carefully managed network can produce data with smaller standard error than one of low density with a poor level of organization. However the former as a rule, will be more expensive.

The question is, what standard error is most appropriate to the purpose for which the data are to be used. A hypothetical standard error of zero, which implies exact data, would give the most reliable result. But this can physically never be realized.

One could try to minimize the standard error as much as possible by perfecting the measurement equipment, increasing the density and improving the organization more and more. However, here the question is whether the additional cost of those actions will counterbalance the worth of the improvement of the data.

An attempt should be made to find a certain balance between the gains and the costs. In fact the question of what standard error, or, more directly, of what network density, is to be aimed at, is a cost-benefit problem. In this problem the following factors play a rôle:

- the socio-economic value of the data;
- the costs of installation and operation of the network as a part of the whole information system;
- the variability of the data;
- the coherence of the data in space and time.

A simplified, tentative approach will now be developed to establish how an impression about the desirable distance between stations can be obtained.

To gain an understanding of the concept 'socio-economic value' consider firstly the concept of information content. Without giving an exact definition, information content could be considered as a value, opposite to that of the standard error of estimate. Under ultimate favourable conditions this standard error will be a minimum, e.g. when there are no interpolation errors, as will be the case for directly measured values. Under other conditions this standard error is greater, which implies a loss of information.

A certain financial value might be assigned to a unit of the dimension in which the standard error of estimate is expressed. For water levels, for instance, a unit length in the standard error of estimate represents an economic value of A Dfl. This should also be considered to hold for a unit lenght of the river. In fact the value A is an indication of the economic importance of the variable of concern. The dimension of A is, in this case, [Dfl. l^{-2}].

Assume the standard error of estimate to increase with the square of the station distance:

$$SE = E + Dz^2 \qquad (I\text{-}1)$$

where:

SE = standard error of estimate,
E = minimum value of SE for $z = 0$,
z = station distance,
D = a coefficient

The coefficient D depends on the total variability of the water levels and on the measure of coherence between the levels along the river. A great variability implies a high D value, a strong coherence a low D value.

If, for instance, the natural standard deviation of the levels is SD and the distance over which no more coherence can be found is L, then the coefficient D may be determined as:

$$D = \frac{SD}{L^2} \tag{1-2}$$

The dimension of D is $[l^{-1}]$. The length L will be called the correlation scale.

The total value of the information loss along a river reach of length R amounts to

$$\begin{aligned} C_l &= A.R.\ (SE\text{-}E) \\ &= A.R.\ Dz^2 \\ &= \frac{A.R.\ SD}{L^2} z^2 \qquad (0 < z < L) \end{aligned} \tag{1-3}$$

Let the annual cost of installation and operation of one gauging station be S. Along the reach of length R there are R/z stations costing

$$C_s = SR/z \tag{1-4}$$

The total macro-economic cost of the information loss together with the gauging stations amounts to the sum of eqs. (1-3) and (1-4), namely

$$C = \frac{A.R.\ SD}{L^2} z^2 + S.R/z \tag{1-5}$$

The distance z_m for which the total cost is minimum is found by

$$\frac{dC}{dz} = \frac{2\ A.R.\ SD}{L^2} z - \frac{S.R.}{z^2} = 0 \text{ for } z = z_m \tag{1-6}$$

and

$$z_m^3 = \frac{S.L^2}{2A.SD} \tag{1-7}$$

Without giving too much weight to the exponents and coefficient in this equation, it is clear that the desirable station distance is made greater by high station costs (S) and by the presence of strong coherence, i.e. a large correlation scale L. On the other hand it is reduced by a high socio economic importance (A) and by a large variability (SD). These are the two factors introduced in Section 1.1 as providing the main motivation for measurement.

It should be noted that for values of z in the vicinity of z_m the total cost C of eq (1-5) may not vary a great deal. It is even imaginable that a more or less constant C value will hold for a rather broad range of z values. However, the two terms of which C is comprised according to eq (1-5) are not constant at all.

It is the society as a whole, or at least the users of the data, who are paying directly to the first term, the information loss, whereas the hydrological institute or the government (so finally also the society) is paying for the second term, the network cost. Since the cost, given by the second term are more transparant there may be a tendency to decrease this preferentially (e.g. the number of gauging stations) at the expense of the first. This can continue until the information loss, indicated by the first term, becomes so large, that pressure from interested users may lead to more information, i.e. to more stations.

Now a numerical example of the application of eq (1-7) will be given. It should be stressed that this is only meant as an example and that no general conclusion should be drawn from it.

Let S be assessed at Dfl 100 000,-/year. This figure is based on an installation cost of Dfl 500 000,- against 10% depreciation and interest, and an additional Dfl 50 000/year for maintenance, operation and data processing.

The correlation scale L is tentatively chosen as 200 km, a value which corresponds, to a certain extent, with values found in practice.

The value A is most difficult to quantify, and in many cases may even be impossible. In Section 2.2, for the example of river navigation, a value of the annual information loss for 1 cm standard error was estimated at Dfl 400 000,-. This concerns a river reach of 250 km. Thus it could be used here to give

$$A = \text{Dfl } 400.000,\text{-/cm. } 250 \text{ km}$$
$$= \text{Dfl } 160,\text{- } m^{-2}.$$

Finally let the water levels have a variation described by a standard deviation of 1 m which seems to be a realistic value:

$$SD = 1 \text{ m}$$

Now inserting the above values in eq (1-7) gives the result

$$z_m^3 = \frac{100\ 000.\ (200.10^3)^2}{2.160.1}$$

$$= 12{,}5.10^{12} \text{ m}^3,$$

or

$$z_m = 23208 \text{ m} = 23 \text{ km}.$$

A difficulty in the application of a cost-benefit approach is the practical impossibility of expressing the entirety of socio-economical worth in terms of money (A) or, more directly, into a network feature like density. Therefore it is often expedient to employ surrogate criteria such as a fixed maximum allowable standard error of estimate, for which, for instance, the standard error of measurement might be chosen. See Section 2.3.

In fact a prescribed value for station distance might also be considered as a surrogate criterion. However this is a very bad one, since it fails to take account of the quality of the data. Also in this category of criteria are the requirements, given for two dimensional variables, in the form of the number of stations per unit area, e.g. km^2. Tables, showing such requirements have been published in guidebooks (WMO, 1981). It seems desirable to replace such criteria by more feasible ones.

In this study the design criterion is a value which the standard error of estimate may not exceed. The standard error of estimate depends on the standard error of measurement and on the interpolation method or the model that is used to derive data for the non-gauged sites.

The standard error of measurement is an important information statistic in the design procedure. Although this statistic can differ from gauge to gauge, it is feasible to use it as a basis and to arrive at some general value, which can be used in the design and evaluation of networks.

1.5. Interpolation methods

The following methods can be distinguished:

a. Interpolation by means of a chosen mathematical relation, like:
   - linear interpolation,
   - interpolation by a higher power function (square, cubic, etc.),
   - interpolation by a transcendental function (harmonic, exponential, etc.),
   - interpolation by spline functions.

   For these functions it is possible to derive the influence of the measurement errors on the error of estimate at non-gauged sites. In order to identify the model errors the interpolation results should be compared with additional measurements 'in situ'.

   In all above methods the interpolation curves can either fit exactly the measured data ('enforced' interpolation) or approximate them according to a certain criterion, for instance the least sum of squares (smoothing).

b. Interpolation by means of statistical methods.

   In this case use is made of an assumed or measured correlation structure between the phenomena, either along the line or within the area considered. Besides the influence of measurement error on the error of estimate, one can also give a value to the error caused by the interpolation method (model error). Methods like Kriging (Delhomme, 1978) and Optimal Interpolation (Gandin, 1970) fall under this category. In the present study this kind of interpolation is developed as a special form of multiple linear regression. A distinction will be made between:
   - the sites where finally the network stations are established or maintained,
   - the sites where in the past, or in the design stage, measurements have been carried out for the assessment of the interrelations,
   - all other sites, where measurements have never been or never will be carried out. For these sites the correlation structure, mentioned in the beginning of this section, is to be used for the derivation of the required relations.

c. Interpolation by physically-based, mathematical modelling.

   Here, use is made of physical relations, expressed as mathematical formulae. For water level and streamflow problems these could be the laws of mass conservation (equation of continuity) and of energy conservation (equation of motion). Measured data could act as boundary conditions and for calibration of the model parameters.

As in the case for mathematical interpolation (case a), model errors can only be determined by comparison with actual measurements.

d. Combination of statistical methods and physically-based mathematical models.

In this case the behaviour of the phenomena (water levels, and if possible discharges and velocities) is simulated as accurately as possible by a physically-based mathematical model. Subsequently a statistical interpolation method is applied to the remaining deviations between model results and measured data, for instance multiple linear regression. This can be extended to non-measured data as well, using an assumed or measured correlation structure. A refined method, but still based on these same principles, is the Kalman filter technique.

e. Interpolation, using patterns obtained by alternative measurement techniques.

Under this category can be classified radar rainfall measurements, adjusted to surface measurements through raingauges. Also in this category are sea level measurements by satellite radar altimeter, which are being developed by ESA (ERS-1 satellite) and by NASA (TOPEX/POSEIDON satellite), and offer promising possibilities.

In this study attention is paid in particular to some methods of type a (linear, square, cubic and spline interpolation), type b (multiple linear regression), type c (a simple one-dimensional streamflow model) and type d (Kalman filter, combined with the model of type c).

1.6 The design process

All interpolation methods described in Section 1.5 refer to a certain physically based and statistically confirmed coherence between the values of the phenomena concerned in the area covered by the network. As was remarked in Section 1.4 the use of coherence is one of the basic principles of network design, since it is the coherence that determines the internal relations. Relations and coherence can only be determined or tested on the basis of measurements. As a consequence the design of a network should be based on earlier measurements.

In the very first stage of design, and assuming that there are no measurements available at all, then an attempt should be made to plan a provisional network with common sense, locating stations at places which are considered of major importance. This can be done with the aid of maps, surface elevation profiles or field surveys. In addition it should be borne in mind that certain factors can lead to gauging stations being located at specified places. Those factors may be:

a. Hydraulic factors:
   - near inflows of major tributaries into the main river or near other important inflows,
   - at sites where rivers bifurcate or confluence, as is the case in deltaic areas,
   - at the mouth of a river into a sea, lake or reservoir,
   - at upstream and downstream sides of weirs, dams or sluices. It should be remarked that during flood conditions, when the weirs are opened it is often sufficient to use only one of these gauges,
   - where water projects are planned, long before these become operationally, in order to examine their influence on the river regime;

b. Political factors:
   - at river crossings of borders between countries or administrative units,
   - at contractually determined places, e.q. for water distribution purposes;

c. Socio-economic and phychological factors (local needs, established customs, status):
   - at important cities and harbours.

On the basis of a provisional network the variability and the coherence can be determined. This information can be used to examine whether more stations are required in order to meet the socio-economic needs or the design criteria of accuracy and reliability derived from these needs, e.g. a given maximum standard error of estimate. The network costs should also be accounted for. Techniques outlined in section 1.5 and further described in the following chapters may be used to obtain a theoretical optimum network. Such a theoretical network must be adjusted to satisfy practical conditions and possibilities for gauge locations. This can lead to some modifications, before a final plan is adopted.

In the course of time, coherence and variability, the socio-economic needs and local conditions, as well as the costs of networks maintenance and operation may change. These changes can necessitate a revision of the network, so that the whole process starts again. Network design is thus a continuous process. In Fig. 1-5 this fact is expressed in a never ending flow diagram.

In the detection of changes of coherence, important information can be acquired from measurements at sites not belonging to the main network. These could form an additional set of stations, the requirements of which might be somewhat lower than those of the network stations.

Besides the purpose of following the development of the relations, there are two other reasons for maintaining additional stations. These are:

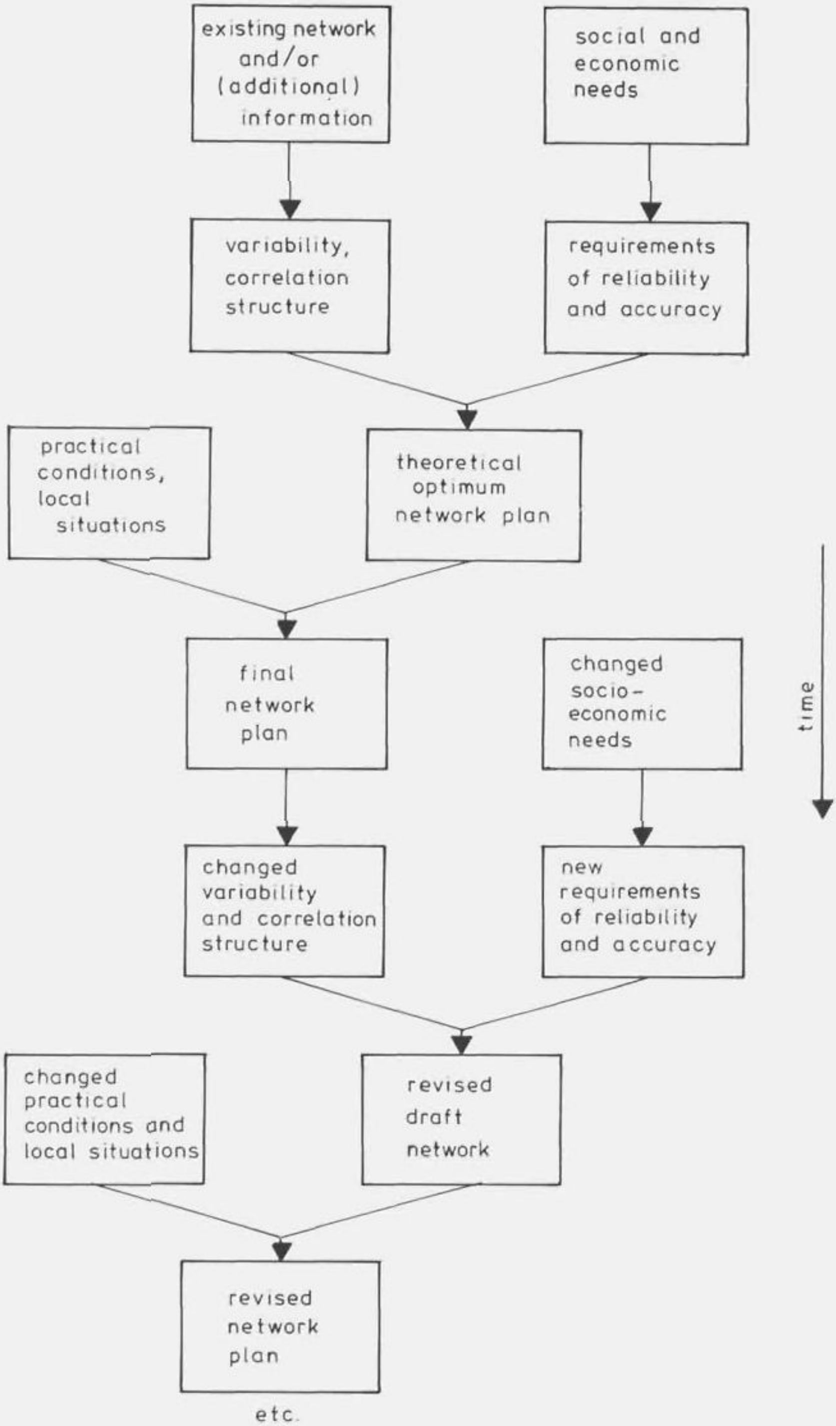


Fig. 1-5 Flow diagram of permanent network design.

- If the design is based on a criterion which can be expressed in terms of the standard error of the deviations between calculated values and measured values, it is assumed implicitly that the frequency distribution of these deviations is normal or practically normal. In many cases this is a reasonable assumption. However, under extreme conditions very great deviations can be found sometimes, leading to unusually long tails of the frequency distribution. In order to better describe cases yielding errors greater than twice the standard error, additional data from intermediate stations is desirable.
- In order obtain a continuous data set without missing values – even if gauges of the main network are temporarily disturbed or interrupted – a certain redundancy of information is required, in order to obtain data to replace missing values. These could be provided by a set of additional stations.

For these reasons an additional set of gauging stations besides the main network, but more simply constructed and equipped, is recommended. As a rule one additional station might be located between each pair of main network stations. In that case each network station is covered by two spare gauges.

1.7 A glance into history

Probably the oldest water level measurements will have been carried out in ancient civilizations which were strongly oriented to water. In the cases of the Egyptians and the Sumerians for instance, irrigation systems formed the backbone of the public organization. For a good functioning of these systems, information, produced by measurements must have been indispensable. It is known that water level measurements formed the basis for tax levies (Caesperlein, 1974).

A well known series of water level measurements is the one at Roda Gauge, starting in the year 620 and still continuing to this day (Hurst, 1965).

In Europe the earliest water level measurements known occurred in the 16^{th} century, and these were restricted to flood levels only. Special marking stones of historical flood levels can be found in many places, including the Netherlands as well as other countries.

Water level measurements on a routine bassis have been carried out in Amsterdam from 1 January 1700 up to 1932, the year in which the local waters were no longer subject to tidal influences. Other early tidal water observations were made elsewhere in the Netherlands (Brielle, Katwijk), as well as in France (Cartwright, 1972) and England (Glen, 1983), but these series concerned only short periods.

Measurements of water level along non tidal rivers commenced gradually, and in many cases measurements were later abandoned. Around 1770 daily water level measurements of the river Rhine in the border region between Germany and the Netherlands were initiated and published by the 'Societeit van Nijmegen'. These measurements are continued to this day (Fig. 1-6).

In 1818 it was ordered by a Royal Decree that the water level gauges were to be related to the same reference plane, i.e. the 'Amsterdam Peil'. This indicates the official starting point of the water level gauging network in the Netherlands.

Meanwhile in the tidal zone gauges were installed in Brielle (1814), Den Helder (1832) and Vlissingen (1833). From 1850 recording gauges began to appear to complement or replace the old staff ganges, beginning with Den Helder (1850), and soon followed by other stations, primarily in the tidal zone. Here, because of the great water level variability the need for continuous recording was evident. In the river zone, where the water is less variable, staff gauges were maintained in the main network until recently.

In 1854 the first 'Yearbook of Water levels' was published, giving water levels for all main rivers and tidal waters at 42 gauging stations. In the 1860's and 1870's the number of gauging stations was extended considerably, in particular to support important hydraulic engineering projects. The 1900-Yearbook includes the water level data of 97 staff gauges and 58 recording gauges. A water level measurement network could be said to exist in the Netherlands from that time.

This network was not the result of special planning or design, but it had developed historically. The gauging stations themselves exhibited a multiplicity of instruments and equipment. In order to rationalize this situation, in 1921 the 'Peilschaalcommissie' (Water Level Gauge Commission) was installed. One of the questions this Commission was requested to answer was:

- Is it necessary to maintain all existing observation points, and if not, what locations have to be continued, either with recording gauges or with staff gauges?

The final report of this Commission, published in 1926, gives for the first time a design for a network, covering all main waters in the country. This plan is still the base for the present network, although in the course oftime a lot of changes have been carried out in connection with the construction of major water projects like the Delta Works.

The present needs of water management, together with new developments in the field of instrumentation, data transmission and processing, provide the main reasons for the current review and reconstruction of the network in the Netherlands. It is this

AANTEEKENINGEN

GEHOUDEN

DOOR DE

SOCIETEIT TE NYMEGEN,

VAN DE

PYLS-HOOGTENS

EN

MERKWAARDIGSTE GEBEURTENISSEN,

OP DE

RIVIEREN DEN RHYN, MAAS, WAAL, NEDER-RHYN EN YSSEL;

Beginnende met primo January

1770.

Te NYMEGEN,

Gedrukt by *ISAAC VAN CAMPEN*, Societeits-Drukker.

Fig. 1-6a Front page of the first publication of water level measurements made for rivers in the Netherlands (1770).

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AANTEEKENINGEN

Van de hoogte der Wateren, volgens de Pyl-paalen in de Rivieren den RHYN te *Keulen*, de MAAS te *Grave*, de WAAL te *Nymegen*, den NEDER-RHYN te *Arnhem*, en den YSSEL te *Doesborgh*, welke Pyls-hoogtens des Morgens ten 8. uuren genomen zyn.

January 1770.

1771	Rhyn		Maas		Waal		Neder Rhyn		Yssel		Winden te Nymegen		Lugts-Gesteldheit te Nymegen	Verschil van hoogte tusschen		Hoogste Water in de					Laagste Water in de				
	Voet	Duim	Voet	Duim	Voet	Duim	Voet	Duim	Voet	Duim	Voor-mid.	Na-mid.		Waal duime	neder Rhyn duime	Rhyn	Maas	Waal	neder Rhyn	Yssel	Rhyn	Maas	Waal	neder Rhyn	Yssel
1			10	5	10	.5	17	3	14	5	[illegible]	[illegible]	[illegible]	14	"		10.5	10.5	17.3	14.3					
2			17	9	17	10½	17	·	14	"	[illegible]	[illegible]	[illegible]	10½	"										
3			16	11	17	2½	16	10	13	9	[illegible]	[illegible]	[illegible]	4½	"										
4			16	6	16	9½	16	8	13	6	[illegible]	[illegible]	[illegible]	-	½										
5			16	6	16	-	16	2	13	3	[illegible]	[illegible]	[illegible]	"	2										
6			16	6	15	5½	16	·	13	"	[illegible]	[illegible]	[illegible]	·	6½										
7			16	7	14	9½	15	6	12	9	[illegible]	[illegible]	[illegible]	·	8½										
8			16	11	14	3	14	11	12	6	[illegible]	[illegible]	[illegible]	·	8										
9			16	9	13	6	14	2	12	·	[illegible]	[illegible]	[illegible]	·	8										
10			15	9	12	8	13	3	11	3	[illegible]	[illegible]	—	·	9										
11			14	6	11	9½	12	1	10	3	[illegible]	[illegible]	—	·	3½										
12			14	·	11	1½	11	2	9	3	[illegible]	[illegible]	[illegible]	·	½										
13			12	6	10	9	10	9	8	3	[illegible]	[illegible]	—	·	"										
14			11	6	10	3½	10	4	7	5	[illegible]	[illegible]	[illegible]	·	½										
15	11	-	11	-	10	1½	10	3	10	"	[illegible]	[illegible]	[illegible]	-	1½						11			10.3	
16	11	2	10	6	10	4½	11	3	8	6	[illegible]	[illegible]	[illegible]	-	10½										
17	11	3	10	2	10	4½	11	2	8	4	[illegible]	[illegible]	[illegible]	·	9½							10.2			
18	11	8	10	6	10	·	11	·	8	·	[illegible]	[illegible]	[illegible]		12										
19	11	11	11	9	9	11½	10	11	7	11	[illegible]	[illegible]	[illegible]	·	11½								9.11½		
20	12	2	12	2	10	"	10	10	8	3	[illegible]	[illegible]	[illegible]	·	10										
21	12	5	12	3	10	2	10	11	8	10	[illegible]	[illegible]	[illegible]	-	9										
22	12	9	13	4	10	4½	10	11	7	4	[illegible]	[illegible]	[illegible]	-	6½										7.4
23	15	2	14	8	10	11½	11	3	8	4	[illegible]	[illegible]	[illegible]	·	3½										
24	17	1	15	7	12	3	12	6	9	3	[illegible]	[illegible]	[illegible]	·	3										
25	20	9	16	2	14	"	14	11½	10	4	[illegible]	[illegible]	[illegible]	-	½										
26	21	9	16	8	15	9	16	3	11	10	[illegible]	[illegible]	[illegible]	-	6	21.9									
27	21	6	16	11	16	6	16	7	12	7	[illegible]	[illegible]	[illegible]	-	1										
28	19	8	17	2	17	1	16	9	13	3	[illegible]	[illegible]	[illegible]	4	"										
29	18	6	17	3	17	3	16	9	13	5	[illegible]	[illegible]	—	4	-										
30	17	9	16	7	16	7	16	8	13	5	[illegible]	[illegible]	[illegible]	-	3										
31	16	-	15	11	16	·	16	5	13	3	[illegible]	[illegible]	Regen	·	5										

Den 27 [illegible] 131 1½ 131 9½

De Merkwaardigste Gebeurtenissen deeser Maand zyn [illegible]

Fig. 1-6b Earliest recorded water level measurements for rivers in the Netherlands (January 1770).

review and reconstruction to which this study is particular devoted. The spatial distribution of the gauging stations was examined and improved along the lines described in the following chapters.

1.8 The rôle of the World Meteorological Organization in hydrological network design

The World Meteorological Organization (WMO), one of the specialized agencies of the United Nations, is, besides its activities in the field of meteorology, also involved in hydrology, in particular in the operational aspects. One of the main items of operational hydrology is the subject of hydrological network design.

The hydrological questions which are of importance to WMO are dealt with by a special Commission, the Commission on Hydrology (CHy), which was established in 1960.

Even at the first session of CHy in 1960 the problem of network design was raised. It established a working group, which produced a section on network design for the Guide to Hydrological Practices, which is one of the main hydrological publications of WMO. A general review of the subject was given by Rodda (1969).

In the 4thedition of this Guide (1981) the requirements for a network are described as follows:
'The aim of a network is to provide a density and distribution of stations in a region such that, by interpolation between data sets at different stations, it will be possible to determine with sufficient accuracy for practical purposes, the characteristics of the basic hydrological and meteorological elements anywhere in the region'.

In Chapter 2 this point will be further discussed.

In 1968 the CHy assigned W.B. Langbein, a pioneer in the field of stream gauging networks (Langbein, 1954), as a rapporteur with the task to compile a Casebook on Hydrological Design Practices. This Casebook was published in 1972 and was updated several times afterwards.

A later publication was the report 'Concepts and Techniques in Hydrological Network Design' (Moss, 1982) and most recently 'Design Aspects of Hydrological Networks' (Van der Made, et al, 1986). The latter includes consideration of the design procedure in general, the variability of the phenomena considered, statistical and socio-economic aspects, discussions of networks for measuring precipitation/ evaporation, surface water and ground water, both with respect to data on water quantity and water quality and, finally, on conjunctive network design.

Recently WMO has initiated an intercomparison project of hydrological network design techniques, with a view to develop general guidelines for network design.

A number of symposia and seminars on hydrological network design were held in course of the years. Those of Quebec (IAHS, 1965), Newcastle-upon-Tyne (WMO, 1976) and Tucson, Ariz. (AGU, 1979) be entitled to careful consideration.

1.9 Plan of this study

After the present introductory Chapter, the general philosophy behind the assessment of network density, in terms of the station distance, is discussed in Chapter 2. In this chapter it becomes clear that important information for network design lies in the accuracy of the measurements, which can even serve as the basic criterion for the design procedure. The examination of the accuracy of the measurements is the subject of Chapter 3.

Subsequently, interpolation methods are discussed. The pure mathematical methods are dealt with in Chapter 4, e.g. polynomials and spline functions. Chapter 5 deals with optimum interpolation, based on multiple linear regression. This concerns those sites where measurements are available. It is this chapter which discusses the most frequently used method for examining the Dutch water level network.

In Chapter 6 the method of Chapter 5 is extended to deal with non-measured river reaches and water areas, and illustrated using a number of examples.

Chapter 7 deals with a physically-based, mathematical model to derive water levels along river reaches between gauging stations. Finally in Chapter 8, it is discussed how such a mathematical model can be improved by making adjustments using additional measurements. This is done by means of a Kalman filter technique.

The purpose of this study is not to recommend a special method. Rather, a number of methods are examined, and the advantages and disadvantages of each of them are considered. Which method is to be used in a practical case can be judged in the light of the available information.

2 Background for the determination of station distance

2.1 The need for information

In Chapter 1 the following statement of the aim of a network was cited (WMO, 1981):
'The aim of a network is to provide a density and distribution of stations in a region such that, by interpolation between data sets at different stations, it will be possible to determine with sufficient accuracy for practical purposes, the characteristics of the basic hydrological and meteorological elements anywhere in the region'. Here two questions arise:
1. how can required information be derived from the data acquired?
2. what is sufficient accuracy?

These questions are also relevant to the design of a water level network - the type of network with which this thesis mainly deals. Question 1 will be considered first in this context.

It is obvious that some kind of interpolation procedure will be required to derive values for ungauged sites*). This is true if data are really required for every site in the region considered. However, if only local information is required - e.g. the water level at a lock or at a harbour entrance - then local observations at one station may suffice. In this case what is realized is a separate observation station, rather than a network.

If information is required for more sites, then the question is raised as to whether it is necessary to make observations at all sites or whether it is possible to derive information from observations elsewhere. If the latter is the case then data from the stations are transferred to other sites, as a rule after some transformation. If, for an ungauged site, data from more than one station are used, these stations need to be considered in a certain coherent way; now the concept of a 'network' is established. The derivation of data for ungauged sites will, by definition, be called 'interpolation'. However other methods, which do not come under what is normally understood by interpolation, can also be applied, such as hydraulic computation. In some cases extrapolation may be necessary, but generally this is not recommended because of the uncertainties that are introduced in doing so. Various methods of

*) If interpolation in time is applied this concerns the intervals between the instants of measurement.

interpolation are given in Chapters 4 to 7. In this study these methods will in general be applied for spatial interpolation.

The second question, 'what is sufficient accuracy?', should be considered in connection with the use of the data. Users of water level data are, for instance, those responsible for water management (e.g. to determine the river discharge, or to judge the intake or discharge requirements for the operation of sluices and pumping stations), for safeguarding against flooding or for flood risk evaluation, for navigation (navigable depth and clearance) for energy production (hydropower or cooling water supply for thermal power) and others. The question is, what are the requirements of these users in terms of the quality of the water level data, obtained from the network. In this context quality might be expressed in terms of the 'root mean square error' of the given value with respect to the 'true' value, and called the standard error of estimate. Although the 'true' value is not clearly defined - it is even questionable whether a true value really exists - the concept of a value expressing the most probable state of an element, and representative of a certain site, will be used here. This matter will be dealt with further in Chapter 3. The standard error of estimate will give the best representation of the quality of the data if the deviations between calculated and 'true' values are independent and normally distributed. This is not always the case. However, the standard error of estimate will be used as a 'calculation quantity'; if the above conditions are not fulfilled, corrections might be applied, as described in Section 2.4.

Returning to the question of users requirements, generally speaking it can be said that the smaller the standard error of estimate, the better the quality of the data and the better the fulfilment of these requirements. Thus the network designer should aim at a reduction in the standard error of estimate.

The next point is how great a reduction is desirable. This problem is connected with the costs of the technical measures that have to be taken in order to arrive at a certain reduction in the standard error of estimate, e.g. by establishing more gauging stations. It is also important to know what benefits are associated with such a reduction. These questions could be solved by a cost-benefit analysis, given that adequate information on the economic value of the benefits is available.

It should be noted that a reduction in the standard error of estimate is possible only down to a certain minimum level, because the influence of measurement errors will always remain. The difference between the standard error of estimate and that minimum can be considered as a measure of information loss. If relevant economic data are available this difference (in cm) could be converted into an equivalent amount of money. This amount should be added to the costs of construction and operation of the network. The economically optimum network is attained if the sum of the monetary equivalent of information loss and network costs is minimized.

If it is not possible to carry out a cost-benefit analysis one has to make use of other criteria, commonly called surrogate criteria (Dawdy, 1979; Moss, 1982). This might be a fixed design value, which the standard error of estimate may not exceed anywhere. Such a design value might be chosen in the order of magnitude of the standard error of measurement. In Section 2.3 this matter will be discussed further.

The standard error of estimate for an arbitrary site within the area considered, is built up of several components. First consider the gauged sites. Here the standard error of estimate of the data is due mainly to the measurements and only to a minor extent to the method of interpolation (which for the gauged sites might preferably be called smoothing). The determination of the standard error of measurement is the main subject of Chapter 3. This standard error relates not only to the quality of the instrument itself, but to the whole complex of local conditions at and around the gauging station. Depending on the range (time and area) over which the data are considered to be representative, the standard error of measurement of the water level can amount to several cm's.

The derived data for non-gauged sites are subject to the propagated measurement errors from the gauges used for interpolation and to the deviations due to the *correlation structure of the water levels along the river reach*. *As a rule the effect of* these errors will increase with station distance.*) However, when using two or more stations for the interpolation of intermediate data, the propagated standard errors of measurement can lead to a standard error of estimate, smaller than the standard error of measurement itself, because interpolation in fact corresponds with a kind of averaging, thus reducing the original standard error. This fact produces some scope for allowing other error sources, so that, when choosing a certain station distance the resulting standard error of estimate might be equal to, or even smaller than the standard error of measurement. This phenomenon will be used as an important tool in the examination and assessment of the network locations.

2.2 Economical implications of a reduction in the standard error of estimate

Assume that the standard error of a value for a certain area can be reduced by carrying out measurements at more sites than previously, i.e. by extension of the existing network. Such an action will be profitable if the profits counterbalance the expenses incurred in implementing and operating this extension. These expenses include construction of stations, installation, operation and maintenance of equipment and instruments, and transmission, processing and storage of data. If the possible profits remain too small, it is then better not to extend the network. It might

*) Tidal phenomena can locally cause reductions of these deviations (see Section 6.4.5.).

even be possible to reduce the number of stations if the existing network results in standard errors smaller than those really required.

An impression of the economic implications of a reduction of the standard error of estimate is given in the following example, which focusses attention on the importance of the actual water levels for inland navigation. Of course the actual water levels do not constitute the only component of importance: waves, vertical movements of ships and the water level drawdown due to return flow are also important. The bottom configuration is also crucial as it effects the ship's clearance. Finally, when planning future transport, then there is a need to estimate water levels some time in advance. This means that the quality of the forecast water levels also needs to be considered.

However, for reasons of simplicity all these effects will be set aside in the following discussion, where only the actual water levels are taken into account.

Suppose the water level y at a certain site has been determined by interpolation from measurements at one or more stations elsewhere, and has a standard error of σ_1 cm. If the deviations are normally distributed this means that there is a probability of about 2,5% that the 'true' water level is higher than $y+2\sigma_1$. There is also a probability of 2,5% that it is lower then $y-2\sigma_1$. If one does not want to accept a risk of more than 2,5% that a ship will touch the bottom because of heavy loading one should take into account for the acceptable draught a water level of $y-2\sigma_1$. If, by increasing the network density such, that the standard error is reduced to a value of σ_2 (smaller than σ_1), then the critical level can be assessed at $y-2\sigma_2$. This means an increase in draught of:

$$D = (y-2\sigma_2) - (y-2\sigma_1) = 2(\sigma_1-\sigma_2) \qquad (2\text{-}1)$$

which, in this case, corresponds to a certain quantity of freight, which can be loaded additionally.

For one case the following tentative calculation was made. Information received from an inland navigation company showed that a change in depth of 10 cm corresponds to a cargo quantity of 320 t in a pushing tug combination. At a (1982) rate of Dfl. 5,-/t for the transport from Rotterdam to the Ruhr area this is equal to a value of Dfl. 1600,-. Considering that annually on an average 170 of such journeys are carried out by about 40 combinations, and also considering that in general in roughly 20% of time, draught restrictions are necessary, one arrives at an annual amount per cm change of depth, of

$$0{,}2 \times 40 \times 170 \times \text{Dfl. } 1600{,}\text{-} \times 0{,}1 = \text{Dfl. } 200\,000{,}\text{-}.$$

In accordance with eq (2-1) a reduction in the standard error of 1 cm implies an increase of draught of 2 cm, which roughly corresponds to Dfl. 400 000,-/year. This amount was used in the example of Section 1.4.

If adequate information is available, similar considerations can be applied to other uses of the water level data and the results can be totalized. In this way, an impression of the economic value of an improvement of the quality of the data can be obtained. Rigorous results from such integrated studies are not known, although studies in the field of the worth of hydrologic data have been carried out for water resources planning (Haimes et al, 1979; Davis et al, 1979).

2.3 The standard error as a design criterion

Because of the many practical difficulties attached to a network design procedure based on cost-benefit analysis, other approaches have been applied. These consist mainly of the assessment of a criterion, not immediately connected with direct economical gains. These are called surrogate criteria (Dawdy, 1979; Moss, 1982). An example of a surrogate criterion is the value, considered as acceptable for standard error of estimate, now being used as a design value. The problem can thus be formulated as follows: to design a network, such that the standard error of estimate nowhere exceeds a given value.

Here the question is: what is an acceptable limit value for the standard error of water level data? Again the different uses of the data have to be considered. For navigation purposes, for instance, one may question whether it is realistic to express the water level in cm's, taking into account the ships vertical movements due to pitching, waves, level drawdown, etc. In flood risk studies the liveliness of the water surface should be accounted for when assessing heights of dikes and quays. For discharge computation, using rating curves, the inaccuracy of the water level data should be compared with that of the discharge measurements on which these curves are based: the latter are, as a rule, much less precise than the water levels (Herschy, 1978).

Here also it is clear that it is difficult, if not impossible, to arrive at a definite statement about the design value. However, taking into account the above considerations, a design value of two or more cm's does not seem unrealistic.

An approach to arrive at a satisfactory solution is the following. If data were required at some site, then measurements at that site must be accepted, which are subject to the inherent inaccuracy of the measurement procedure. If that inaccuracy is adopted now as a base for network design, computed data would be acceptable only if the standard error of estimate would not exceed the standard error of measurement. As was stated earlier, one of the features of interpolation is that such a result is attainable.

The standard error of measurement will be adopted for use as a network design criterion in the following chapters. How this standard error might be determined will be discussed in Chapter 3.

One point should be stressed here. When examining a number of gauging stations and their standard errors of measurement it is obvious that in general they will have different values. However, for practical reasons it is desirable to use only one value as a design criterion. Therefore a value should be chosen which is of the order of magnitude of the standard errors of measurement, and preferably a simple value, expressed either in whole or in half cm's. This value will be indicated in the following by E, whereas the individual standard errors of measurement are given by ε for the station under examination and by ε_i for the stations used in the computation. In the following the latter set of stations will be called the 'network stations'.

It should be noted that this procedure is based on the assumption that the measurement errors and the errors of estimate are both normally distributed. If however the errors of estimate are not normally distributed, but the measurement errors are, then the standard errors of their distributions do not represent comparable values. If this is the case it might be desirable to apply some correction to the design value. This will be done using a correction coefficient. (See also Section 2.5).

The network design requirement will, when expressed as a formula, read as follows:

$$\varphi\sigma \leqslant E \tag{2-2}$$

where:

- σ = standard error of estimate.
- E = design criterion for the standard error of estimate anywhere within the network area.
- φ = correction coefficient.

Now consider the following level data values at the station under examination:
- the 'true' level y_t
- the measured level $\underline{y}$
- the computed level $\hat{y}$, obtained using the relation:

$$\hat{y} = f(\underline{x}_1 \ldots \underline{x}_m) \tag{2-3}$$

where:

$\underline{x}_1 \ldots \underline{x}_m$ = measured water level data at the network stations.

Consider also the differences:

$$\Delta y = \underline{y}\text{-}y_t \text{ (error of measurement)} \tag{2-4}$$

$$\Delta \hat{y} = \hat{y}\text{-}y_t \text{ (error of estimate)} \tag{2-5}$$

$$\Delta y = \underline{y}\text{-}\hat{y} \text{ (measurement-estimate)} \tag{2-6}$$

These differences will be considered as realizations of a stationary process with fixed mean (expected) values and variances.

A network can be evaluated by comparison of the calculated values $\hat{y}$ at a certain site with the values $\underline{y}$, measured at that site by a gauge, and not belonging to the network stations. The differences Δy between the measured and the calculated values are subject to:

- measurement errors at the site considered;
- propagated measurement errors of the network stations;
- errors due to deviations between the results of the relation given by eq (2-3) and the 'true' values; these will be referred to as 'model errors'.

All these components are included in the variance of Δy. This can be explained as follows.
Let:

$$\varepsilon^2 = \text{Var}\Delta \underline{y} \tag{2-7}$$

where:

ε = standard error of measurement at the site and:

$$\sigma^2 = \text{Var}\Delta y \tag{2-8}$$

where:

σ = standard error of estimate

From eqs (2-4), (2-5) and (2-6) it follows:

$$\begin{aligned} \Delta y &= \underline{y}\text{-}\hat{y} \\ &= y_t + \Delta \underline{y} - (y_t + \Delta \hat{y}) \\ &= \Delta \underline{y} - \Delta \hat{y} \end{aligned} \tag{2-9}$$

Assuming independence between the two right hand terms (i.e. the measurements at the site are not influenced by the calculation results), then the variance of Δy can be expressed by:

$$\mathrm{Var}\Delta y = \mathrm{Var}\Delta \underline{y} + \mathrm{Var}\Delta \hat{y} \tag{2-10}$$

or, with regard to eqs (2-7) and (2-8):

$$\mathrm{Var}\Delta y = \varepsilon^2 + \sigma^2 \tag{2-11}$$

Further, as was stated by eq (2-3), the computed value $\hat{y}$ is based on measured values $\underline{x}_1 \ldots \underline{x}_m$. If true values $x_{1t} \ldots x_{mt}$ could be used, then the calculated value of y would still differ from its true value, but this time only on account of model error. Measurement error would play no rôle. Now let the model error r be:

$$r = f(x_{1t} \ldots x_{mt}) - y_t \tag{2-12}$$

The influence of the measurement errors of $\underline{x}_1 \ldots \underline{x}_m$ on the result of eq (2-3), i.e. $\underline{m}$ can be found by considering the propagated measurement errors:

$$\underline{m} = f(\underline{x}_1 \ldots \underline{x}_m) - f(x_{1t} \ldots x_{mt}) \tag{2-13}$$

The expression of eqs (2-3), (2-12) and (2-13) can be combined to give:

$$\hat{y} = y_t + \underline{m} + r \tag{2-14}$$

and, with regard to eq (2-6):

$$\Delta y = \underline{y} - y_t - \underline{m} - r$$

$$= \Delta \underline{y} - \underline{m} - r \tag{2-15}$$

Assuming again independence of the measurement errors, both mutually and with respect to the model errors then the variance of Δy can be expressed by:

$$\mathrm{Var}\Delta y = \mathrm{Var}\Delta \underline{y} + \mathrm{Var}\ \underline{m} + \mathrm{Var}\ r \tag{2-16}$$

or, because of eq (2-7):

$$\mathrm{Var}\Delta y = \varepsilon^2 + \mathrm{Var}\ \underline{m} + \mathrm{Var}\ r \tag{2-17}$$

where:

Var $\underline{m}$ = variance of the propagated measurement errors $\underline{m}$

Var r = variance of the model errors r (model noise).

This demonstrates the influence of the three components mentioned before, on VarΔy.

The variance of estimate follows from a comparison between eq (2-11) and eq (2-17):

$$\sigma^2 = \text{Var } \underline{m} + \text{Var } r \tag{2-18}$$

If the relation of eq (2-3) is linear:

$$y = A_o + \sum_{i=1}^{m} A_i \underline{x}_i \tag{2-19}$$

The variance of the propagated measurement errors reads:

$$\text{Var } \underline{m} = \sum_{i=1}^{m} A_i^2 \varepsilon_i^2 \tag{2-20}$$

where:

ε_i = standard error of measurement of the network stations concerned.

Therefore, according to eq (2-17)

$$\text{Var}\Delta y = \varepsilon^2 + \sum_{i=1}^{m} A_i^2 \varepsilon_i^2 + \text{Var } r \tag{2-21}$$

Now, looking back at the network design criterion of eq (2-2), and, considering that in practice the standard error of estimate cannot directly be determined, but that the variance of Δy can be found by comparing calculated and measured values, (see eq (2-6)), the following expression for the design criterion is to be preferred:

$$\text{Var}\Delta y = \varepsilon^2 + \sigma^2 \leqslant \varepsilon^2 + E^2/\varphi^2 \tag{2-22}$$

If $\varphi = 1$ this becomes:

$$\mathrm{Var}\Delta y \leqslant \varepsilon^2 + E^2 \qquad (2\text{-}23)$$

If the design value E is put equal to the standard error of measurement ε then the design criterion reads simply:

$$\mathrm{Var}\Delta y \leqslant 2\varepsilon^2 \qquad (2\text{-}24)$$

2.4. Determination of the desirable station distance in a special case

As a rule the variance VarΔ y will increase with the distances between each of the network stations and the site under examination and also with the distance between the network stations. This is due to the fact that generally the longer the reach considered the weaker the correlation.

Now consider the case where the water level at a site is derived from two network stations, located at equal distances $^1/_2$ d from the site. Assume at the same time that a station exists at the site to be examined, which will not be used in the calculations, but which will serve as the basis for comparison. Now Var Δy can be calculated for a great number of such pairs of network stations at different distances d between these network stations. The results may be plotted to produce a graph of the kind shown in Fig. 2-1.

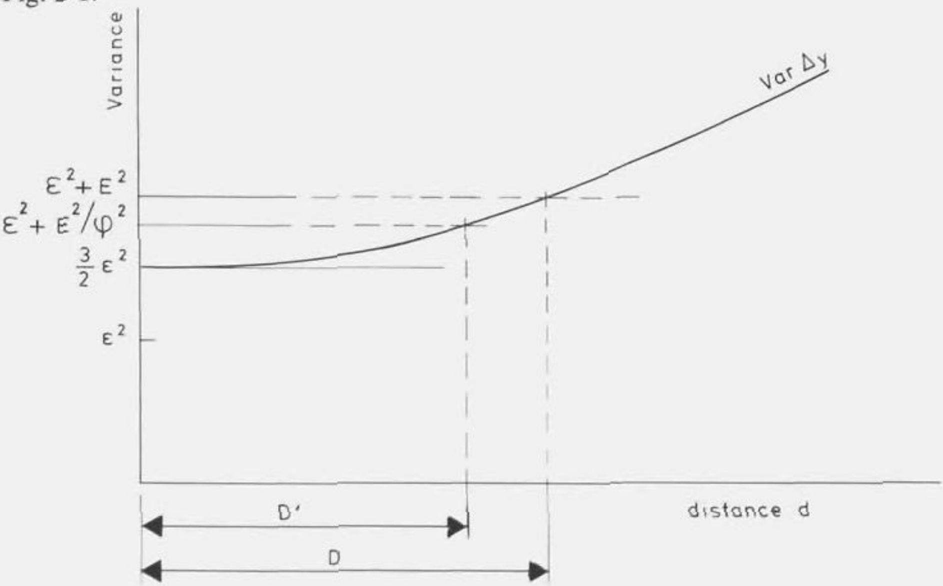


Fig. 2-1. VarΔy vs. station distance d. This graph shows at what distance, D or D', the required variance is found.

At a distance d = 0 the term Var r in eq (2-17) does not exist, and therefore VarΔ y will only be influenced by the errors of measurement, both at the network stations and at the station under examination.

For d = 0 the deviation Δy can be considered as the difference between the average level of the two network stations and that of the station under examination. From eq (2-6), after substitution of that average, it follows that:

$$\Delta y = \underline{y} - \tfrac{1}{2}(\underline{x}_1 + \underline{x}_2), \qquad (2\text{-}25)$$

whereas, always for d = 0:

$$y_t - \tfrac{1}{2}(x_{1t} + x_{2t}) = 0 \qquad (2\text{-}26)$$

Subtraction of eq (2-26) from eq (2-25) yields (see eq (2-4))

$$\Delta y = \Delta\underline{y} - \tfrac{1}{2}(\Delta\underline{x}_1 + \Delta\underline{x}_2). \qquad (2\text{-}27)$$

If the measurement errors are independent, then

$$\underset{d=0}{\mathrm{Var}\Delta y} = \varepsilon^2 + \tfrac{1}{4}\varepsilon_1^2 + \tfrac{1}{4}\varepsilon_2^2. \qquad (2\text{-}28)$$

Assuming

$$\varepsilon = \varepsilon_1 = \varepsilon_2, \qquad (2\text{-}29)$$

then

$$\underset{d=0}{\mathrm{Var}\Delta y} = \tfrac{3}{2}\varepsilon^2. \qquad (2\text{-}30)$$

If no correction is necessary ($\varphi = 1$) then the design requirement, according to eq (2-23), is

$$\mathrm{Var}\Delta y \leqslant \varepsilon^2 + E^2.$$

Now the design distance D can easily be taken from the graph of Fig. 2-1. If a correction (in general $\varphi > 1$) is necessary, then the design distance will be reduced from D to D'.

The case described above is a hypothetical one. In practice it will be unlikely for the site under examination to be exactly halfway between the two network stations. Also, in many cases, the tests will be based on more than two network stations. In addition, the influence of the measurements errors to VarΔy will be different for each case considered, since the standard error of measurement may vary from station to station. Also the model errors will be different for each case considered.

For these reasons a correction coefficient φ, valid for an entire network, does not seem very feasible for design purposes. In the next section a more structured approach will be described.

Often what is required in practice is not the optimum station distance but rather some test of the existing network, such as whether some stations may be discontinued, or whether some new stations are required. These will be located at those places between existing stations which fulfil certain practical conditions, like accessibility, technical facilities and locally favourable conditions for construction and maintenance.

2.5 Application of corrections

As was stated above, the value Var Δy can be used as the key value for network design procedures. It should not exceed a certain limit value, which may be based on the variance of measurement. This requirement is most appropriate if the frequency distributions of both the errors of measurement (at the network stations as well as at the stations under examination) and the model errors are normal, and the deviations Δy are mutually independent. In this case only, the frequency distributions of the errors of measurement Δy and those of the errors of estimate $\Delta\hat{y}$ are comparable, the latter including, besides model errors, also (propagated) errors of measurement.

This means that if these two constraints are not fulfilled, VarΔy cannot be used reliably as a key value. The approach, described in the following, might be applied in such circumstances.

Suppose a series of water level data is to be analysed, which is long enough to include alle possible conditions of the river regime. If VarΔ y is used as a key value it is of importance to know how associated sample variances might behave.

If the differences Δy are normally distributed, then the sums of squares of groups of such differences are proportional to a χ^2-distribution (Yevjevich, 1972; Papoulis, 1965). If not, these sums of squares will be of another distribution type. To the author's knowledge distributions of sums of squares of groups of elements which are not distributed normally, have not yet been investigated. However, for an actual series of Δy-values, a distribution for the sums of squares can be derived by dividing the series into groups of k consecutive elements Δy, and then computing their sums of squares. Plotting the resulting sums of squares values as a frequency histogram will indicate the shape of the distribution which turns out to be an alternative χ^2-distribution. This alternative χ^2-distribution should be compared with the pure χ^2-distribution of k-1 degrees of freedom, which corresponds to the variance of the Δy- series as a whole.

It should be remembered that the χ^2-value of a group of k elements is defined as

$$\chi^2 = \frac{\sum_{i=1}^{k} (\Delta y_i - \bar{\Delta} \bar{y})^2}{\mathrm{Var}\Delta y} = \frac{\sum_{i=1}^{k} (\Delta y_i)^2}{\mathrm{Var}\Delta y} \qquad (2\text{-}31)$$

since the populations means $\bar{\Delta}\bar{y} = 0$. Further

$\sum_{i=1}^{k} (\Delta y_i)^2$ = the sum of the squared Δy-values of the group

$\mathrm{Var}\Delta y$ = the variance of the Δy-series as a whole.

Thus the χ^2-values are the contributions of the groups to the variance Var Δy. Such a partial variance is

$$\mathrm{Var}\Delta y_k = \frac{1}{k} \sum_{i=1}^{k} (\Delta y_i)^2 \qquad (2\text{-}32)$$

so that

$$\chi^2 = k \frac{\mathrm{Var}\Delta y_k}{\mathrm{Var}\Delta y.} \qquad (2\text{-}33)$$

For this reason the χ^2-values can be used to examine the behaviour of separate samples.

It should be noted that the probability density of the χ^2-distribution can be expressed as

$$f(\chi^2) = \frac{1}{2^{k/2}\Gamma(k/2)} e^{-\chi^2/2} (\chi^2)^{k/2-1} \qquad (2\text{-}34)$$

The corresponding distribution function is

$$F(\chi^2) = \int_0^{\chi^2} f(\chi^2)d\chi^2. \quad (2\text{-}35)$$

If serial correlation exists between the Δy values, then the χ^2-distribution remains valid as long as the Δy-values are normally distributed, but the χ^2-values will be different from those in a population without serial correlation. In case of serial correlation the χ^2-values follow from the formula (Box and Jenkins, 1976):

$$\chi^2 = \frac{(\Delta Y)R^{-1}(\Delta Y)^T}{\mathrm{Var}\Delta y} \quad (2\text{-}36)$$

where:

ΔY = the $1 \times k$ matrix of Δy-values in the sample
= $[\Delta y_1, \Delta y_2 \ldots \Delta y_k]$
R = the $k \times k$ serial correlation matrix between the Δy-values

If there is no serial correlation, then the matrix R is equal to the unit matrix denoted by I: this leads to eq (2-31).

When examining an actual case, a graph should be made of the distribution of sums of squares of the groups, as was described earlier, and this compared with the theoretical χ^2-distribution given by eq (2-35). If the actual distribution does not coincide reasonably with the theoretical χ^2-curve, then the series of Δy-values is either

- not normally distributed, or
- serial correlated, or
- both.

Subsequently the modified χ^2-values should be calculated, according to eq (2-36) and these plotted again, together with the curve of the theoretical χ^2-distribution. If the modified χ^2-values and the theoretical curve coincide then the given series of Δy-values might be considered to be acceptable for the network configuration which has produced this series. In that case the serial correlation could be taken into account in the calculation of the y-values.

But, if neither the original, nor the modified χ^2-values correspond with the theoretical χ^2-distribution, the reason must be that the series of Δy-values is not normally distributed. This is in particular of importance if the extreme values of Δy occur more frequently than according to the normal distribution, i.e. if the distribution has long tails. This will lead to differences between the distribution of the calculated χ^2-values and the theoretical χ^2-distribution, especially in the area of the highest values, i.e. those values, which are most crucial for the feasibility of the network.

In this case another χ^2-distribution is proposed, this time based on a corrected variance, to be called $\mathrm{Var}\Delta y^*$. This alternative χ^2-distribution should be assessed such that it gives, at a certain high frequency level (e.g. at $F(\chi^2) = 0.95$), the same χ^2-values as those based on the χ^2-values of the groups. This procedure will be explained later on, with the help of an example (see Section 2.6).

The variance $\mathrm{Var}\Delta y^*$ should be chosen now as the new key value for the network design procedure, for practical use leading to the correction coefficient φ of eq (2-2). The requirement for the acceptance of the network is

$$\mathrm{Var}\Delta y^* \leqslant \varepsilon^2 + E^2 \tag{2-37}$$

or, if $E = \varepsilon$

$$\mathrm{Var}\Delta y^* \leqslant 2\varepsilon^2 \tag{2-38}$$

A coefficient, γ^2, is now introduced, such that when it is multiplied by the original variance $\mathrm{Var}\Delta y$, then the corrected variance is obtained; that is

$$\mathrm{Var}\Delta y^* = \gamma^2.\mathrm{Var}\Delta y. \tag{2-39}$$

It may be assumed that the possible non-normality of the distribution of the series Δy derives from the model errors, rather than from the measurement errors. Thus the correction will only concern the term Var r in eq (2-17). Let this term be corrected by a coefficient λ^2; then

$$\mathrm{Var}\Delta y^* = \varepsilon^2 + \mathrm{Var}\ \underline{m} + \lambda^2\ \mathrm{Var}\ r. \tag{2-40}$$

Finally, there is the correction coefficient φ of eq (2-2), with which the standard error $\sigma\Delta y$ should be multiplied. With regard to eq (2-11) this means

$$\mathrm{Var}\Delta y^* = \varepsilon^2 + \varphi^2\sigma^2 \tag{2-41}$$

The relation between the coefficients γ^2, λ^2 and φ^2 follows from a comparison between eqs (2-39), (2-40) and (2-41), taking into account eq (2-17):

$$\left.\begin{array}{l}\gamma^2 \text{ Var } \Delta y = \varepsilon^2 + \varphi^2 (\text{Var } \underline{m} + \text{Var } r) \\ \text{and} \\ \gamma^2 \text{ Var } \Delta y = \varepsilon^2 + \text{Var } \underline{m} + \lambda^2 \text{ Var } r\end{array}\right\} \qquad (2\text{-}42)$$

If γ^2 is known (e.g. graphically derived), then the other coefficients follow from

$$\varphi^2 = \frac{\gamma^2 \text{ Var}\Delta y - \varepsilon^2}{\text{Var}\Delta y - \varepsilon^2} \qquad (2\text{-}43)$$

and

$$\lambda^2 = \frac{\gamma^2 \text{ Var}\Delta y - \varepsilon^2 - \text{Var } \underline{m}}{\text{Var}\Delta y - \varepsilon^2 - \text{Var } \underline{m}} \qquad (2\text{-}44)$$

2.6. Example

This example concerns the River IJssel, one of the branches of the River Rhine in the Netherlands. A situation map is shown in Fig. 2-2, which includes the water level gauging stations located along this river. In this example the water levels of the station Doesburg were calculated, using the measured levels of a network, consisting of the stations:

1. Kampen.
2. Katerveer.
3. Olst.
4. Zutphen.
5. IJsselkop.

For the calculation a linear relation was used, according to eq (2-19):

$$\hat{y} = A_0 + \sum_{i=1}^{5} A_i \underline{x}_i \qquad (2\text{-}45)$$

The coefficients A_0 and A_i (i=1...5) were calculated by multiple linear regression analysis (see Chapter 5). The water level data were taken at 3-day intervals during the years 1977-1978, yielding a total of 240 data items, starting at 3 January 1977. The variance of the differences Δy, between the measured and calculated values, was found to be

$$\text{Var } \Delta y = 10{,}82 \text{ cm}^2$$

Fig. 2-2 Water level gauging stations along the River IJssel.

If the design criterion E is chosen as E=2,5 cm (which corresponds to E=ε; see chapter 3), the value used for the water level network in the Netherlands, then this result seems acceptable when testing according to eq (2-24).Specifically

$$\text{Var } \Delta y \leq 2\varepsilon^2$$

in this case becomes

$$10{,}82 \text{ cm}^2 \leq 12{,}5 \text{ cm}^2.$$

Thus, as far as station Doesburg is concerned, the gauging station network, not including station Doesburg, seems to be acceptable.

Subsequently the distributions of the χ^2-values of groups were considered in the manner described in Section 2.5. The 240 Δy-values were divided into 48 groups of 5 elements each, and their χ^2-values were calculated according to eq (2-31) with k=5. These χ^2-values were plotted against their cumulative frequency, as shown in Fig. 2-3. The curve 'a' in this figure represents a χ^2-distribution for 5-1=4 degrees of freedom, and it is based on the variance of the Δy series as a whole, i.e.

$$\text{Var } \Delta = 10{,}82 \text{ cm}^2$$

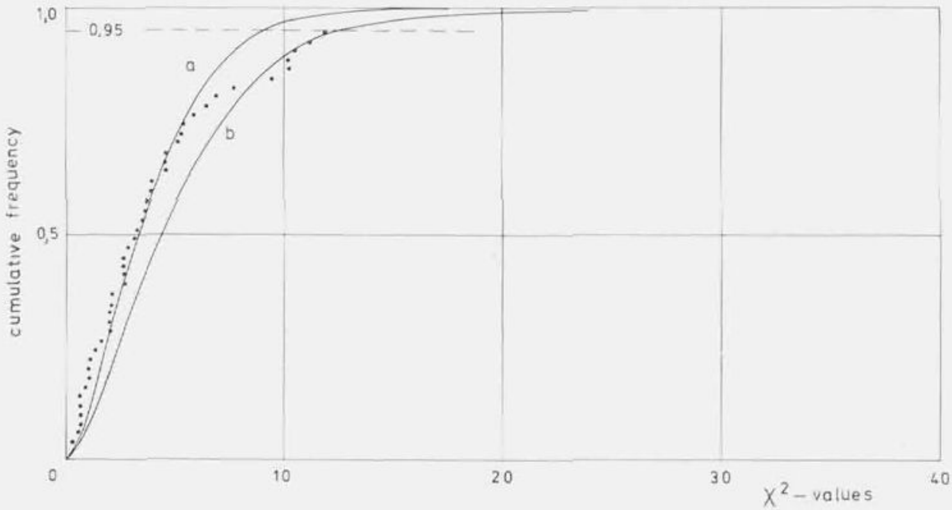


Fig. 2-3 χ^2 values of 5-element Δy groups compared with two χ^2-distributions.

The fit looks reasonable for the lower and central part of curve 'a', but it is poorer for the upper part, which is also the most critical part, since it concerns the large deviations of the calculated values from the measured ones. In order to meet this objection a solution was found in selecting another χ^2-distribution, which is expressed by the curve 'b', also drawn in Fig. 2-3. This curve represents a χ^2-distribution, based on a hypothetical series having a variance of 1,33 × the calculated series variance VarΔ y = 10,82 cm. The curve 'b' fits the plots around the level $F(\chi^2)=0{,}95$. The alternative variance will be assigned as the corrected variance Var Δy*, given by

$$\text{Var } \Delta y^* = 1{,}33 \times 10{,}82 \text{ cm}^2 = 14{,}39 \text{ cm}^2$$

Now regarding the requirement of eq (2-36):

$$\text{Var } \Delta y^* \leqslant 2\varepsilon^2$$

the network considered would not be acceptable since

$$14{,}39\ cm^2 > 12{,}50\ cm^2.$$

The procedure described above might be carried out for other stations, not belonging to the network. Another, more approximate, approach might be based on the assumption that the model errors for all other intermediate stations behave in the same way as for the station examined in the case described (i.e. Doesburg), and that only these errors, cause a deviation from the theoretical χ^2-distribution. But the measurement errors, both direct and propagated, are assumed to be strictly normally distributed, and therefore do not contribute to the deviations from the theoretical χ^2-distribution when considering groups of Δy-values. Now for one station a correction coefficient λ^2 is calculated acccording to eq (2-44) and, subsequently, this coefficient is applied to other stations for the calculation of Var Δy^*. This is done according to the formula

$$\mathrm{Var}\Delta y^* = (1-\lambda^2)(\varepsilon^2 + \mathrm{Var}\ \underline{m}) + \lambda^2 \mathrm{Var}\Delta y \qquad (2\text{-}46)$$

which follows from eqs (2-40) and (2-17).
Since a linear relation is used the propagated error variance follows from eq (2-20)

$$\mathrm{Var}\ \underline{m} = \sum_{i=1}^{m} A_i^2 \varepsilon_i^2$$

For the network stations a general standard error of measurement, ε_i = 1,5 cm (i = 1...5) was applied. The following regression coefficients A_i (i= 1...5) were obtained for the period considered

1.	Kampen	:	A_1 =	0,0359
2.	Katerveer	:	A_2 =	- 0,2194
3.	Olst	:	A_3 =	0,1163
4.	Zutphen	:	A_4 =	0,6683
5.	IJsselkop	:	A_5 =	0,3665

Thus

$$\mathrm{Var}\ \underline{m} = \sum_{i=1}^{5} A_i^2 \varepsilon_i^2 = 1{,}45\ cm^2.$$

Consequently eq (2-44) gives

$$\lambda^2 = \frac{\gamma^2 \text{ Var } \Delta y - \varepsilon^2 - \text{Var } \underline{m}}{\text{Var } \Delta y - \varepsilon^2 - \text{Var } \underline{m}}$$

$$= \frac{\text{Var } \Delta y^* - \varepsilon^2 - \text{Var } \underline{m}}{\text{Var } \Delta y - \varepsilon^2 - \text{Var } \underline{m}} \qquad \text{(see eq (2-39))}$$

$$= \frac{14{,}39 - 2{,}25 - 1{,}45}{10{,}82 - 2{,}25 - 1{,}45} = 1{,}50$$

and, according to eq (2-43):

$$\varphi^2 = \frac{14{,}29 - 2{,}25}{10{,}82 - 2{,}25} = 1{,}42.$$

The values of $\sigma\Delta y^*$ were then calculated for the other sites examined, i.e. the stations Wijhe, Deventer, Dieren and De Steeg. Two alternative networks were investigated in addition. In Table 2-1 the data required, ($\sigma\Delta y$ and the regression coefficients $A_1 \ldots A_n$) are given for the original network (network 1), as well as for the two alternatives (networks 2 and 3). For reasons of uniformity, now $n=6$ for all three alternatives. Values for the constant value A_0 are also given.

In network 2 the station Doesburg is included, but the station Zutphen is omitted. In network 3 both the stations Doesburg and Zutphen are included.

The standard errors of measurement for the sites to be examined were assessed as follows:

Wijhe	:	= 1 cm
Deventer	:	= 2 cm
Zutphen	:	= 1,5 cm
Dieren	:	= 1,5 cm
Doesburg	:	= 1,5 cm
De Steeg	:	= 1,5 cm

Reference should be made to Section 5.6.2 (example) with regard especially to the deviating values for Wijhe and Deventer.

Finally the values of Var Δy^* were transformed into standard errors $\sigma\Delta y^*$. The results are also given in Table 2-1. These results have to be checked against the design criterion. This criterion according to eq (2-35) is

$$\mathrm{Var}\Delta y^* \leqslant \varepsilon^2 + E^2$$

or:

$$\sigma\Delta y^* \leqslant \sqrt{\varepsilon^2 + E^2} \tag{2-47}$$

which, if E = 2,5 cm, gives

$$\sigma\Delta y^* \leqslant \sqrt{\varepsilon^2 + (2{,}5)^2} \tag{2-48}$$

The values of $\sqrt{\varepsilon^2+E^2}$ are also given in Table 2-1. The table shows that only for network 3 is the requirement of eq (2-48) fulfilled for all sites considered.

It is network 3 that was selected as the main water level gauging network along the river IJssel.

Sites to be examined by network 1	$\sigma\Delta y$	A_0	Kampen A_1	Katerv. A_2	Olst A_3	Zutphen A_4	Doesburg A_5	IJsselk. A_6	$\sigma\Delta y^*$	$\sqrt{\varepsilon^2+E^2}$
	cm	cm							cm	cm
Wijhe	2,13	16,0517	0,0025	0,0779	0,9794	- 0,0764	----	- 0,0423	2,28	2,69
Deventer	2,71	- 3,9823	0,0777	- 0,2754	0,9054	0,3268	----	- 0,0215	2,81	3,20
Dieren	3,31	33,0221	0,0534	- 0,2492	0,1557	0,7519	----	0,2559	3,81	2,91
Doesburg	3,29	18,2205	0,0359	- 0,2194	0,1163	0,6683	----	0,3665	3,79	2,91
De Steeg	2,96	- 5,2637	0,0346	- 0,1321	0,0188	0,4375	----	0,6246	3,37	2,91
by network 2										
Wijhe	2,14	9,7264	- 0,0333	0,1832	0,8152	----	0,0582	- 0,0838	2,36	2,69
Deventer	2,82	- 1,5576	0,1140	- 0,3439	1,0878	----	0,2564	- 0,0883	2,89	3,20
Zutphen	3,16	15,6555	0,1502	- 0,3374	0,7319	----	0,6155	- 0,1228	3,56	2,91
Dieren	2,25	22,7056	0,0614	- 0,1603	0,2398	----	0,9161	- 0,0555	2,33	2,91
De Steeg	2,42	- 13,4453	0,0289	- 0,0465	0,0218	----	0,5778	0,4218	2,66	2,91
by network 3										
Wijhe	2,04	12,7939	- 0,0039	0,1171	0,9586	- 0,1959	0,1799	- 0,1078	2,14	2,69
Deventer	2,69	- 5,6922	0,0743	- 0,2548	0,8945	0,2641	0,0938	- 0,0559	2,79	3,20
Dieren	2,13	18,9908	0,0258	- 0,0802	0,0661	0,2373	0,7701	- 0,0264	2,22	2,91
De Steeg	2,41	- 14,8124	0,0158	- 0,0171	- 0,0422	0,0873	0,5241	0,4326	2,66	2,91

Table 2-1: Standard errors of Δy and regression coefficients for three network alternatives along the river IJssel.

3 Accuracy of measurements

3.1 General

In Chapter 2 the standard error of measurement appeared to be of central importance to the network design procedure. In the present chapter the accuracy of the measurements will be discussed, and in particular how this may be quantified. Accuracy is related specifically to errors of measurement, and not to mistakes in measurement. The greater these errors, the lower the accuracy. Therefore the use of the concept 'inaccuracy' or 'uncertainty' seems more relevant than'accuracy', since this corresponds better with the magnitude of the errors.

The error of measurement $\Delta \underline{y}$ can be defined as the difference between the measured value $\underline{y}$ and the so called 'true' value y_t, (see eq (2-4)):

$$\Delta \underline{y} = \underline{y} - y_t$$

However, what is normally understood by the true level, i.e. the water level as it really occurs at the site of the gauging station, or, more precisely, at the inlet of the tube to the stilling well, is not the water level of prime interest to hydrologist. The actual water level is subject to a number of wave actions, due to wind, navigation and other causes which are changing continuously. As a rule these movements are of no importance to the purpose for which the water level is measured, unless specific wave studies are of concern. Besides, the actual water level relates to the stations site only and not to other sites, even though these may be only a few metres away.*) Generally of most interest is a water level value, which is representative of a certain area around the station, and as a rule which extends over the whole cross section of the river. For all these reasons one may wish to eliminate those phenomena that are of no interest for the purpose concerned, namely to assess the water level, which is representative of a certain area and a certain time span.

*) It is impossible to consider a true water level without first defining at what scale the measurements are to be carried out. If high frequency waves are suppressed then a smoothed value of the water level is obtained. Indeed, the frequency of measurement itself already imposes a certain limit, above which no waves are found. By increasing the frequency of measurement then more movements will be found. At the micro scale even molecular movements will be recognized. However, this is not of direct concern here. Of most interest are motions which are occuring over minutes, rather than seconds. The problem touched on here is related to the fractal problem, in which context the length of a coastline (Mandelbrot, 1982) is a more familiar example.

Due to the characteristics of the station's location, its construction and its dimension, a certain smoothing and transformation of the water level movement will already take place, even before the level is registered on a chart or a computer compatible medium. On the one hand this is a benefit, since a number of disturbing phenomena have been eliminated, but on the other hand other disturbances may be introduced, such as time lags, level set up by velocity reduction against the station, density differences between the open water level and the level in the stilling well, gauge datum levelling errors, instrument errors etc. On account of these disturbances the registered water level will differ from the actual water level.

Although the magnitude of the newly introduced errors will not be known exactly, some attempts may be made to correct the registered level in accordance with one's insight and knowledge. Partly this can be done by application of a filtering procedure, partly by other corrections. These can be carried out either manually or by computer. The hydrograph, resulting from these actions will be regarded as the definitive 'measured water levels', to be indicated by $\underline{y}$.

Even now, some uncertainty will remain as to whether these derived levels are really the water levels that are required, since undesirable disturbances, either real or introduced by measurement and further processing, may remain. Probably it will be impossible to determine the representative water level, to be denoted by y_t; a certain difference $\Delta\underline{y}$ between the measured and the representative (or 'true') water level will always remain.

3.2 Order of magnitude of the errors of measurement

Errors of measurement of water levels can arise from several sources. (See Section 3.1). These include:

1. errors due to the location of the gauging station at the river and to the hydraulic conditions in the vicinity;
2. errors due to the construction of the station house, the stilling well and the inlet tube;
3. errors due to the differences in density between the open water and the water in the stilling well;
4. instrument errors;
5. levelling errors in the gauge reference datum and the zero of the gauge;
6. observation and processing errors.

These errors will differ from case to case and it will be difficult, if not impossible, to quantify the partial errors of the different sources, and in turn the total errors, in terms of the standard error of measurement and the mean error of measurement. In the following an attempt will be made to consider these components separately in order to gain an idea of the order of magnitude.

3.2.1 Errors due to the location of the gauging station at the river and the hydraulic conditions of the adjacent area

Differences in water levels around a station will be caused by differences in velocity head. If, for instance, the velocity in the middle of the river is 1 m/s and close to the bank it is 0,5 m/s the respective velocity heads are 5 cm and 1 cm using the relation:

$$\Delta h = \frac{v^2}{2g} \tag{3-1}$$

where:
v = velocity
g = acceleration due to gravity

This implies a difference in water stage of 4 cm. Thus locally the water levels can differ by up to about 2 cm from the mean level over the cross section of the river.

If the station house impedes the flow, then over a short distance the velocities can vary from 0 to say 1 m/s, thus causing differences in velocity head, and at the same time differences in water level of about 5 cm.

Curved river stretches cause a transverse slope in water level to develop, which will depend on the velocity and the radius according to:

$$I_t = \frac{v^2}{gr} \tag{3-2}$$

where:
I_t = the transverse slope of the water surface
r = radius of the stream line

If $v = 1$ m/s and $r = 500$ m this will cause a slope of:

$$I_t = \frac{1}{5000}$$

A width of 100 m is already associated with about a 2 cm water level difference between the two banks. Thus a deviation of about 1 cm from the mean level can easily occur. In wider rivers or at higher velocities this deviation will be bigger.

Without giving a specific value it can be stated that the contribution to the total error of measurement, for reasons connected with the location of the gauging station, can amount to several cm's. A part of the measurement errors will vary with the flow velocity. Since these will vary around a value, proportional to the mean value of v^2 - and not with regard to zero - these departures too will have a mean value, that is different from zero. This leads to a systematic error in the measured water levels, which, if important, can be eliminated or reduced after a careful examination of the local situation.

3.2.2 Errors due to the construction of the station house, the stilling well and the inlet tube

In a float driven gauge, the measured levels are usually the levels in the stilling well and not the water levels outside. However, differences can occur between the two levels, depending on the dimensions of the inlet tube and the stilling well. Often these dimensions are chosen such that certain differences are caused intentionally in order to dampen out wave movements for which there is no interest. However this can cause a time lag in recording the water level. If the tube is sufficient wide this lag can be neglected, but if sediment is deposited within the tube, then the narrowing may increase the flow resistance and hamper the flow in and out. This will cause a water level variation in the well, which is completely different to that of the open water. It is therefore strongly recommended that the well and the tube are cleaned adequately and frequently if the sediment content of the water is high. If adequately maintained the effects can be neglected; if not, then differences of several cm's can occur. This is particularly important in areas with strong water level variations such as in the tidal zone.

3.2.3 Errors due to differences in density between the open water and the water in the stilling well

This kind of error will occur in salt and brackish water zones such as coastal areas. If the salinity is changing due to outflow of rivers, sluices or pumping stations, then the density of the open water can vary markedly whereas that in the stilling well retains more or less an average value. If, for instance, the density outside varies from 1000 kg/m^3 to 1020 kg/m^3 and the inside density is 1010 kg/m^3, then differences between the inner and outer water levels of 1% of the pressure head above the tube inlet can occur. If the tube inlet is at 4 m below the water level, this corresponds to a 4 cm difference in water level inside and outside the stilling well.

3.2.4 Instrument errors

Because the measuring instrument forms a most essential link in the data acquisition chain much care is usually taken in its development and construction. Therefore its contribution to the error will be relatively small, being of the order of some mm's. Greater inaccuracies however can occur in the time scale if the rotation of the drum is incorrect. This is of most importance if the vertical movement is strong. Therefore much care should be paid to this aspect of the recording. Temperature variations can cause variations in the length of the suspension wire of the float, which has an effect on the levels recorded.

3.2.5 Levelling errors in the gauge reference datum and the zero of the gauge

These errors can be caused by the installation of the reference datum mark and by the adjustment of the gauge to that level. The soil conditions are important in this respect because of possible bottom subsidence. As a rule, the levels of the reference datum and of the gauge should be checked frequently and at least once annually. Errors of some cm's may derive from this cause.

3.2.6 Observation and processing errors

In this phase of data acquisition many kinds of errors may occur, although these can be reduced by careful handling of the information. However, unforeseen errors can be introduced by reading, digitizing, and erroneous 'improvements' when checking. Finally, if the data are given in whole cm's, then rounding standard errors of 0,28 cm can be included.*)

3.3 Random errors and systematic errors

Some kinds of errors associated with a series of measurements can have a constant value. This can be the case for levelling errors and some instrument errors. Other errors may vary around a mean value, unequal to zero, such as errors due to density

*) The standard error of estimate, due only to the use of gauge units of 1 cm amounts to:

$$\sqrt{\frac{\int_{-0,5\text{ cm}}^{0,5\text{ cm}} x^2 dx}{1\text{ cm}}} = \sqrt{\frac{1/3\, x^3 \Big|_{-0.5\text{ cm}}^{0.5\text{ cm}}}{1\text{ cm}}} = \sqrt{1/12}\text{ cm} = 0,28\text{ cm}$$

differences and errors due to velocity reduction. In fact such a mean value is also a constant component of the error.

In summary the errors are made of constant components and variable components, as a rule of stochastic nature. The constant components are called 'systematic errors', and the variable components will be referred to as 'random errors'.

The random errors can be handled more easily than the systematic errors. Consider for instance a series of n measurements y_i (i = 1...n) of a phenomenon having a constant value y_0. An impression of the random errors can be obtained by calculating the mean value of the measurements

$$E(y_0) = \bar{y} = \frac{\sum_{i=1}^{n} y_i}{n} \tag{3-3}$$

and their variance

$$\text{Var } y = \frac{\sum_{i=1}^{n} (y_i - \bar{y})^2}{n-1} \tag{3-4}$$

The standard error of measurement is given by the square root of the variance:

$$\sigma y = \sqrt{\text{Var } y} \tag{3-5}$$

The investigation can be extended still further by examining the distribution function of the random errors.

However, for the systematic errors such an examination is impossible. Averaging gives no answer, since the reference value ('zero value') remains unknown. The only way to reduce this kind of error is through a good site selection procedure which considers the local conditions, and through a careful maintenance of the station house and the instrument, frequent levellings and a careful execution of the data processing procedures. Possibly comparison with similar cases will give some idea about the order of magnitude of systematic errors.

A favourable feature of the systematic errors may be that they partly compensate for each other by some having negative and others positive values. Random errors, in contrast, always have a cumulative effect on the total error of measurement. If, for instance, there are three systematic errors of 1 cm each, two positives and one negative, the result is a systematic error of 1 cm. If there are also three random errors, expressed in standard errors of 1 cm each, their contribution amounts to $\sqrt{(1^2+1^2+1^2)}$ = 1,7 cm. The range, within which 95% of the errors occur, if a normal distribution is assumed, is from 1 - 2×1,7 = - 2,4 cm up to 1 + 2×1,7 = 4,4 cm, thus the band width, which is due to the random errors, namely 6,8 cm, largely dominates the magnitude of 1 cm, due to the systematic errors.

If the systematic errors were not to compensate each other, but were all of the same sign, say positive, then the result would still show a greater influence due to the random errors: the mean error now amounts to 3 cm, but the 95%-band of the random error would still be 6,8 cm, that is more than twice the mean error.

3.4 Determination of the error of measurement

As was discussed in Section 3.1. the concept of the true water level is not clearly defined. It is even questionable if such a level can be considered to be realistic.

As a consequence it is impossible, in particular for individual water level measurements, to derive an error of measurement which is the difference between the measured and the 'true' water level. But an attempt could be made to estimate characteristic values of the inaccuracy, such as the mean difference between measured and what were called representative levels, together with the standard error of these differences.

The methods for estimation concern comparisons of measurements at the gauge considered with other measurements for which the conditions are somewhat different from those which are under examination. The different conditions can be found in distance and time. The relation between changing conditions and the differences of the two measurements can give an indication of possible measurement errors, but no exact solution. For this reason it is recommended that various approaches are applied and their results compared. From those, a final conclusion might be drawn.

The following approaches will be considered:
1) comparison of measurements at the same station at different times;
2) comparison of measurements at the station under examination with measurements at a nearby station;
3) comparison of measurements at the station under examination with measurements at other stations at different distances from the station under examination.

3.4.1 Comparison of measurements at the same station at different times

Consider a measurement $\underline{y}(t)$ and a measurement $\underline{y}(t-\tau)$, made at the same station, but at a time interval τ apart. Now consider a linear regression model

$$y(t) = A_0 + Ay(t-\tau) \tag{3-6}$$

Let $\hat{y}(t)$, denote an estimate of y(t), calculated using the measured value $\underline{y}(t-\tau)$, according to the relation

$$\hat{y}(t) = A_0 + A\underline{y}(t-\tau) \tag{3-7}$$

The difference between the measured value $\underline{y}(t)$ and this calculated value is

$$\begin{aligned}\Delta y(t,\tau) &= \underline{y}(t) - \hat{y}(t)\\ &= \underline{y}(t) - A_0 - A\underline{y}(t-\tau)\end{aligned} \tag{3-8}$$

If the model is fitted according to the minimum sum of squared errors principle, the variance of Δ y(t,τ) amounts to:

$$\text{Var } \Delta y(t,\tau) = \{1-\varrho^2(\tau)\} \text{ Var } \underline{y}(t) \tag{3-9}$$

where:

$\varrho(\tau)$ = the correlation coefficient between $\underline{y}(t)$ and $\underline{y}(t-\tau)$

When Var $\underline{y}(t)$ is assumed to be equal to Var $\underline{y}(t-\tau)$ for all τ, then the overall variance of the measured values for the full series of measurements is a single value, not influenced by t. Let it be denoted by Var $\underline{y}$. From eq (3-9) it follows that the variance of the differences Δy is a function of τ only. Eq (3-9) transforms into:

$$\text{Var } \Delta y(\tau) = \{1-\varrho^2(\tau\} \text{ Var } \underline{y} \tag{3-10}$$

The correlation coefficient $\varrho(\tau)$ is defined as

$$\varrho(\tau) = \frac{\text{Cov } \underline{y}(t)\, \underline{y}(t-\tau)}{\text{Var } \underline{y}} \tag{3-11}$$

Consider the numerator of this expression. By analogy with eq (2-4) one may write

$$\Delta \underline{y}(t) = \underline{y}(t) - y_t(t) \tag{3-12}$$

Replacing the y-values by y_t-values in eq (3-11) and using eq (3-12), then the numerator may be written

$$\text{Cov } y(t)\, y(t-\tau) =$$

$$= \text{Cov } \{y_t(t) + \Delta y(t)\}\ \{y_t(t-\tau) + \Delta y(t-\tau)\} \qquad \text{3-13)}$$

This may be further expressed as follows:

$$\text{Cov } \{y_t(t) + \Delta y(t)\}\ \{y_t(t-\tau) + \Delta y(t-\tau)\} =$$

$$\text{Cov } y_t(t)\, y_t(t-\tau) + \text{Cov } y_t(t).\ \Delta y(t-\tau) +$$

$$+ \text{Cov } y_t(t-\tau)\, \Delta y(t) + \text{Cov } \Delta y(t).\ \Delta y(t-\tau) \qquad (3\text{-}14)$$

Assuming independence between $y_t(t)$ and $\Delta y(t-\tau)$ and also between $y_t(t-\tau)$ and $\Delta y(t)$, then the second and third term of eq (3-13) become 'zero', and:

$$\text{Cov } y(t).y(t-\tau) = \text{Cov } y_t(t) y_t(t-\tau) + \text{Cov } \Delta y(t)\, \Delta y(t-\tau) \qquad (3\text{-}15)$$

Assuming also independence between the level y and the error of measurement Δy then:

$$\text{Var } y = \text{Var } y_t + \text{Var } \Delta y \qquad (3\text{-}16)$$

Since $\Delta y(t)$ and $\Delta y(t-\tau)$ can be considered to have the same variance, Var Δy, since practically it concerns the same series, it can be stated that

$$\text{Cov}\Delta y(t).\ \Delta y(t-\tau) = \varrho_\Delta(\tau).\ \text{Var } \Delta y \qquad (3\text{-}17)$$

where:

$\varrho_\Delta(\tau)$ = the correlation coefficient between the subsequent errors of measurement.

For the same reason as holds for eq (3-17) a fixed value of the variance, Var y_t, may be assumed for the true values. Now it follows that

$$\text{Cov } y_t(t)\ .y_t(t-\tau) = \varrho_t(\tau)\text{Var } y_t \qquad (3\text{-}18)$$

where:

$\varrho_t(\tau)$ = the correlation coefficient between the 'true' values $y_t(t)$ and $y_t(t-\tau)$

Substitution of eqs (3-18) and (3-17) into eq (3-15) and subsequent substitution of eqs (3-15) and (3-16) into eq (3-11) yields:

$$\varrho(\tau) = \frac{\varrho_t(\tau)\ \mathrm{Var}\ y_t + \varrho_\Delta(\tau)\mathrm{Var}\Delta \underline{y}}{\mathrm{Var}\ y_t + \mathrm{Var}\Delta \underline{y}}$$

$$= \varrho_t(\tau) - \{\varrho_t(\tau) - \varrho_\Delta(\tau)\} \frac{\mathrm{Var}\Delta \underline{y}}{\mathrm{Var}\ y_t + \mathrm{Var}\Delta \underline{y}} \tag{3-19}$$

With reference to eq (3-16) this is equal to:

$$\varrho(\tau) = \varrho_t(\tau) - \{\varrho_t(\tau) - \varrho_\Delta(\tau)\} \frac{\mathrm{Var}\ \Delta \underline{y}}{\mathrm{Var}\ \underline{y}}$$

$$= \varrho_t(\tau) \left\{ 1 - \frac{\mathrm{Var}\Delta \underline{y}}{\mathrm{Var}\ \underline{y}} \left(1 - \frac{\varrho_\Delta(\tau)}{\varrho_t(\tau)} \right) \right\} \tag{3-20}$$

For practical applications the following inequality generally holds:

$\mathrm{Var}\Delta \underline{y} \ll \mathrm{Var}\ \underline{y}$,

and one can state approximately that

$$\varrho^2(\tau) \approx \varrho_t^2(\tau) \left\{ 1 - 2 \frac{\mathrm{Var}\Delta \underline{y}}{\mathrm{Var}\ \underline{y}} \left(1 - \frac{\varrho_\Delta(\tau)}{\varrho_t(\tau)} \right) \right\} \tag{3-21}$$

Substitution into eq (3-10) yields:

$$\mathrm{Var}\Delta\, y(t) \approx \left[1 - \varrho_t^2(\tau) \left\{ 1 - 2 \frac{\mathrm{Var}\Delta \underline{y}}{\mathrm{Var}\ \underline{y}} \left(1 - \frac{\varrho_\Delta(\tau)}{\varrho_t(\tau)} \right) \right\} \right] \mathrm{Var}\ \underline{y}$$

$$= \{1 - \varrho_t^2(\tau)\}\ \mathrm{Var}\ \underline{y} + 2\, \varrho_t^2(\tau)\ \mathrm{Var}\Delta \underline{y} \left(1 - \frac{\varrho_\Delta(\tau)}{\varrho_t(\tau)} \right) \tag{3-22}$$

For the purpose of the present discussion, i.e. to find a method to determine the variance of measurement $Var\Delta y$ or it square root, the standard error of measurement ε (eq (2-7)), the eqs (3-21) or (3-22) can be an important tool. Suppose that values of $Var\Delta y(\tau)$ have been calculated for number of intervals $\tau = \Delta t$, $2\,\Delta t$ etc. Now plot the values of $Var\,\Delta y\,(\tau)$, found from measurements y and calculations $\hat{y}$ (see eq (3-8)), or its square root $\sigma\Delta y(\tau)$ against the intervals (Fig. 3-1).

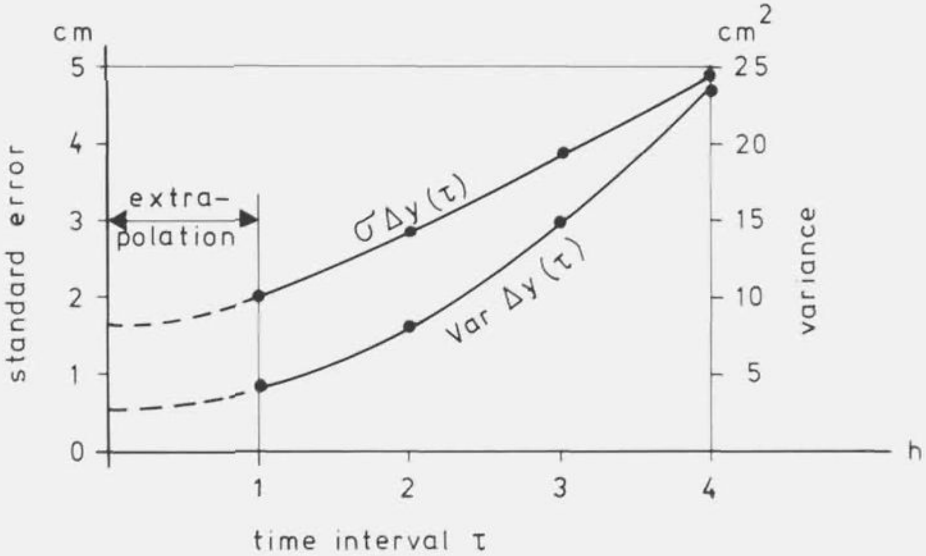


Fig. 3-1 Variance $Var\Delta\,y(\tau)$ and standard error $\sigma\Delta y(\tau)$ vs. time interval τ.

If there is no correlation between the subsequent measurement errors, i.e.

$$\varrho_\Delta(\tau) = 0 \tag{3-23}$$

then eq (3-21) transforms into:

$$\varrho^2(\tau) = \varrho_t^2(\tau) \left\{1\text{-}2 \frac{Var\Delta y}{Var\,y}\right\} \tag{3-24}$$

and eq (3-22) into:

$$Var\Delta y(\tau) = \{1\text{-}\varrho_t^2(\tau)\}\,Var\,y + 2\,\varrho_t^2(\tau)\,Var\Delta y \tag{3-25}$$

For $\tau = 0$ there are seen to be two simultaneous measurements from the same instrument: in other words,the same values are being compared. This implies that for the true values

$$\varrho_t(0) = 1 \tag{3-26}$$

Therefore, because of eq (3-24):

$$\tilde{\varrho}^2(0) = 1 - 2 \frac{\mathrm{Var}\Delta \underline{y}}{\mathrm{Var}\ \underline{y}} \quad {}^{*)} \tag{3-27}$$

and because of eq (3-25):

$$\left.\begin{array}{l} \tilde{\mathrm{Var}}\Delta y(0) = 2\ \mathrm{Var}\Delta \underline{y} \quad {}^{*)} \\ \text{and:} \\ \tilde{\sigma}\Delta y(0) = \varepsilon\sqrt{2} \quad {}^{*)} \end{array}\right\} \tag{3-28}$$

This value can be found by extension of the relation between $\mathrm{Var}\Delta y(\tau)$ and τ (see Fig. 3-1) as far as $\tau = 0$.

Thus an extrapolation should be carried out bearing in mind the likely behaviour of $\varrho_t(t)$ in the vicinity of $\tau=0$, although in general no formula for the relation between $\varrho_t(\tau)$ and τ can be given. A continuous curve is most likely, starting from $\varrho_t(0) = 1$ (eq (3-26)).

A second assumption might be that the direction of the curve for $\tau=0$ is horizontal, so that

$$\frac{d\ \varrho_t(\tau)}{d\tau} = 0 \text{ for } \tau = 0. \tag{3-29}$$

This assumption can be justified as follows. In Section 3.1. the transformation of the occurring water level movement into a data series $\underline{y}$ was discussed. Here, the aim was to derive a series of representative water level data, y_t, by filtering out waves up to a certain wave period. The unavoidable introduction of errors was the reason that $\underline{y}$ is obtained rather than y_t. Also short waves were not allowed in the representative data, y_t. Very short waves clearly will lead to a rapid decrease in the coherence between water levels occurring at the beginning and at the end of a certain time interval. Long waves, on the contrary, tend to maintain coherence, whereas in the case of 'waves' of infinite length (i.e. a constant water level) coherence will persist

*) The ~-sign indicates an extrapolated value.

indefinitely. Since short waves are suppressed, as being of no importance, reduction in coherence with increasing time interval, will be very slow. For this reason, a zero slope at the origin of the correlation curve (correlogram) was considered to be likely.

Figures 3-2 and 3-3 show correlograms of water levels at a river station (Lobith, hourly values, 1979) and at a tidal station (Huibertgat, 5 min. values, 6-27 October 1979). Despite the different characters of these curves, both show horizontal tangents for $\tau=0$.*)

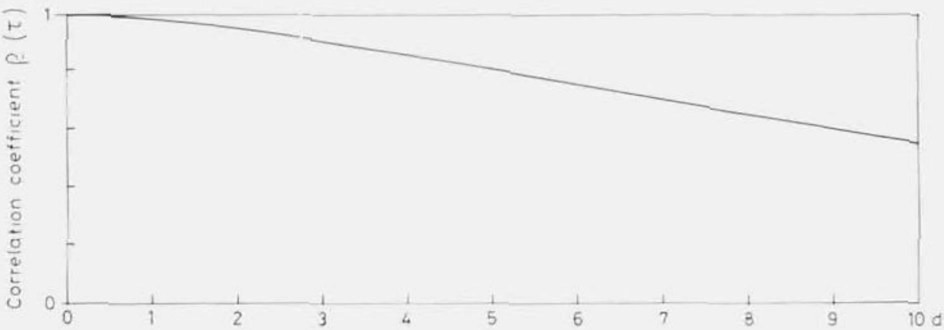


Fig. 3-2 Correlogram water levels of station Lobith (Rhine), based on 1-hourly measurements during the year 1979

Since Var $\Delta \underline{y}(\tau)$ is proportional to $1-\varrho^2$ according to eq (3-10) the variogram, as shown in Fig. 3-1, will also show a horizontal tangent for $\tau=0$. This can be explained by:

$$\frac{d\ Var\Delta \underline{y}(\tau)}{d\tau} = 2\varrho(\tau)\ \text{Var}\ \underline{y}\ \frac{d\varrho(\tau)}{d\tau} \tag{3-30}$$

Because the right hand differential quotient is 0, this is also the case for the left hand one.

If $\varrho(0) < 1$, then:

$$\frac{d\sigma\Delta \underline{y}(\tau)}{d\tau} = \frac{d\sqrt{\text{Var}\Delta \underline{y}(\tau)}}{d\tau} = \frac{\varrho(\tau)}{\sqrt{1-\varrho^2(\tau)}}\ \sigma \underline{y}\ \frac{d\varrho(\tau)}{d\tau} = 0 \text{ for } \tau = 0$$

*) Mathematically this correlogram can be calculated by a Fourier transform of the power spectrum. If short waves (high frequences) do not occur or are surpressed by filtering, the top of the correlogram at $\tau=0$ has a horizontal tangent. In Annex I this is explained in more detail.

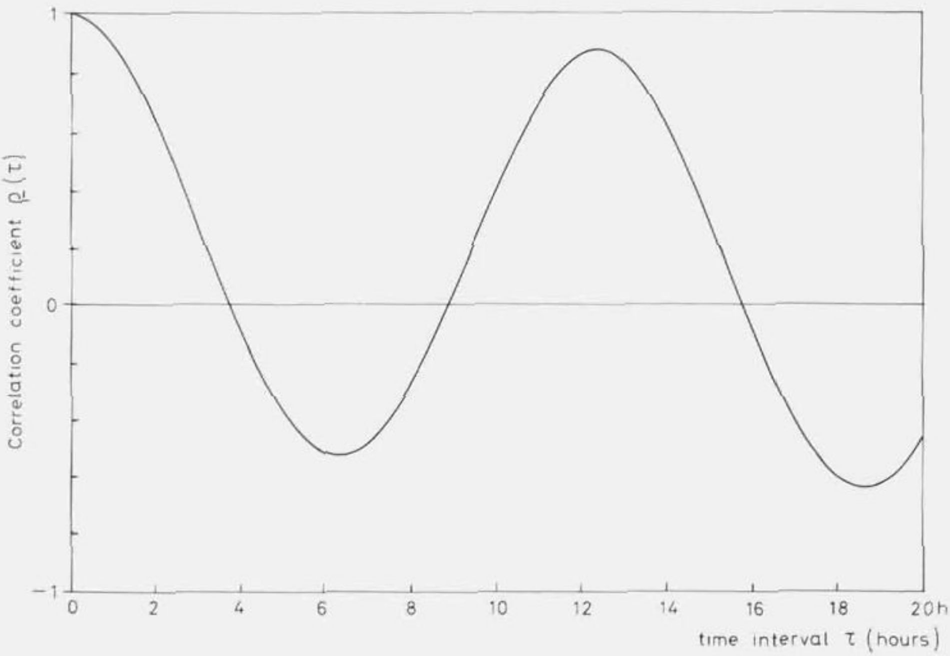


Fig. 3-3 Correlogram water levels of station Huibertgat (North Sea), based on 5-min measurements during the period 6-27 October 1979

Thus the graph of the standard error $\sigma\Delta \underline{y}(\tau)$ against τ also has a horizontal tangent for $\tau=0$.

A detail of the Lobith correlogram is given in Fig. 3-4. The values of $\underline{\varrho}$ (τ) for τ = 1...4 h are plotted. Their positions make a horizontal direction at τ= 0 very likely. The extrapolation up to that point, using spline functions, with the constraint of a horizontal tangent at the origin (see Chapter 4, Subsection 4.4.3.3.) produced a correlation coefficient for τ= 0 of

$$\tilde{\underline{\varrho}}(0) = 0,999946447$$

For the period considered (the year 1979) the following standard deviation holds:

$\sigma \underline{y} \quad = 155$ cm
or $\mathrm{Var}\ \underline{y} = (155)^2$ cm^2

Then eq (3-24) yields:

$$(0,999946)^2 = 1. \left\{ 1 - 2\ \frac{\mathrm{Var}\Delta \underline{y}}{(155)^2} \right\}$$

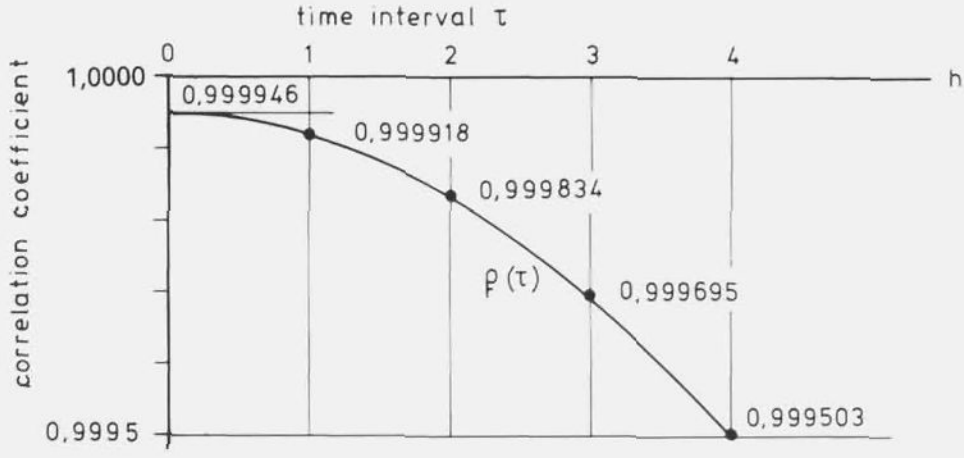


Fig. 3-4 Correlation coefficient $\varrho(\tau)$ of measured water levels vs time interval τ (Lobith station).

from which it follows that

$$Var\Delta \underset{\sim}{y} = 1{,}286\ cm$$

or:

$$\varepsilon = \sqrt{1{,}286}\ \text{cm} = 1{,}134\ \text{cm}.$$

This standard error of measurement is affected by disturbances in time, but not by disturbances associated with the location. Thus the ε-value, might include some more disturbances than were assumed in the above calculation.

Local disturbances may be considered to affect all measurements errors to about the same extent, causing some permanent correlation among them; so, in that case, the assumption of eq (3-23)

$$\varrho_\Delta(\tau) = 0$$

no longer holds. Let this value be R_Δ for small values of τ, but $\tau \neq 0$.

Now return to eqs (3-20) and (3-21). The quotient, that appears in these equations

$$\frac{\varrho_\Delta(\tau)}{\varrho_t(\tau)}$$

can without objection be replaced now by a constant term R_Δ, since in the vicinity of $\tau=0$ the denominator is so close to one that no deviations of importance are introduced. This yields:

$$\underset{\sim}{\varrho}^2(\tau) = \varrho_t^2(\tau) \left\{ 1 - 2 \frac{\text{Var}\,\Delta\underset{\sim}{y}}{\text{Var}\,\underset{\sim}{y}} (1 - R_\Delta) \right\} \tag{3-31}$$

and:

$$\text{Var}\Delta y(\tau) = \{1 - \varrho_t^2(\tau)\}\ \text{Var}\,\underset{\sim}{y} + 2\ \text{Var}\Delta\underset{\sim}{y}\,(1-R_\Delta) \tag{3-32}$$

For $\tau = 0$ these values become

$$\tilde{\underset{\sim}{\varrho}}^2(0) = 1 - 2 \quad \frac{\text{Var}\Delta\underset{\sim}{y}}{\text{Var}\,\underset{\sim}{y}} (1 - R_\Delta) \tag{3-33}$$

and

$$\bar{\text{Var}}\Delta y(0) = 2\ \text{Var}\Delta\underset{\sim}{y}\,(1-R_\Delta) \tag{3-34}$$

Thus for a given $\bar{\text{Var}}\Delta y(0)$, for instance resulting from an extrapolation, the value of the variance of measurement $\text{Var}\Delta\underset{\sim}{y}$ is inversely proportional to $1-R_\Delta$. If, for instance, there is correlation between the measurement errors, expressed as an autocorrelation coefficient $R_\Delta = 0{,}6$, then for the variance of measurement for the Lobith example one has:

$$\begin{aligned} \bar{\text{Var}}\Delta y(0) &= \{1-\tilde{\underset{\sim}{\varrho}}(0)^2\}\ \text{Var}\,\underset{\sim}{y} \\ &= (1-0{,}999946^2).(155)^2\ \text{cm}^2 \\ &= 2{,}573\ \text{cm}^2 \end{aligned}$$

From eq (3-34) it follows:

$$\text{Var}\Delta y = \frac{\bar{\text{Var}}\Delta y(0)}{2(1-R_\Delta)}$$

$$= \frac{2{,}573}{2(1-0{,}6)} = 3.216\ \text{cm}^2,$$

and therefore

$\varepsilon = \sqrt{\text{Var}\Delta y} = 1{,}793$ cm.

This value of the standard error is higher than the value calculated earlier for the correlation free case. The standard error of measurement ε increases with the correlation coefficient R_Δ. For the case of the station Lobith the relation between ε and R_Δ is depicted in Fig. 3-5.

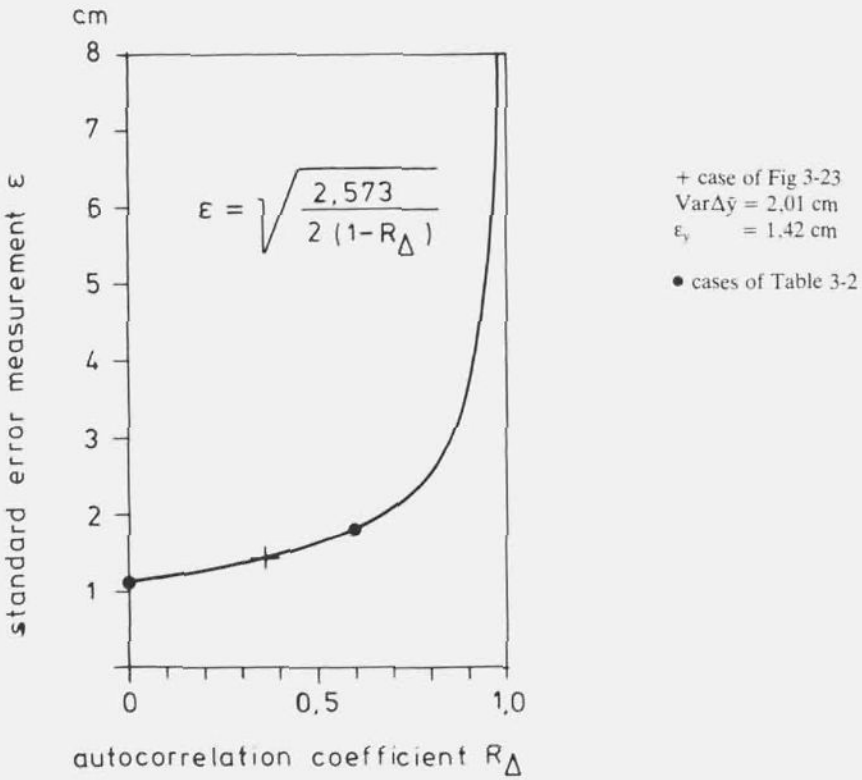


Fig. 3-5 Standard error of measurement ε vs. autocorrelation coefficient R_Δ. (Lobith example).

The correlation, expressed by R_Δ, will be caused by locally bounded disturbances, affecting consecutive measurements in a similar way. This can also be regarded as a systematic error, to be termed ε_s. Its square value can be derived by subtracting the variance Var Δy for the uncorrelated case, 1,286 cm², from that of the correlated case, 3,216 cm², giving:

$\varepsilon_s^2 = 3{,}216 \text{ cm}^2 - 1{,}286 \text{ cm}^2 = 1{,}930$ cm or:

$\varepsilon_s = 1{,}389$ cm

This value corresponds to a correlation coefficient of $R_\Delta = 0{,}6$ in the example considered. A general relation between the squared systematic error ε_s^2, and the correlation coefficient follows from the subtraction of the variance $\mathrm{Var}\Delta\, \underline{y}$ for the uncorrelated case from that of the correlated case. Using eqs (3-32) and (3-27) it follows that:

$$\varepsilon_s^2 = \mathrm{Var}\ \Delta\ \underline{y}\ (\mathrm{corr}) - \mathrm{Var}\ \Delta\ \underline{y}\ (\mathrm{uncorr})$$

$$= \frac{\bar{\mathrm{Var}}\ \Delta y(0)}{2(1-R_\Delta)} - \frac{\bar{\mathrm{Var}}\ \Delta y(0)}{2}$$

$$= \frac{\bar{\mathrm{Var}}\ \Delta y(0)}{2}\ \frac{R_\Delta}{1-R_\Delta} \qquad (3\text{-}35)$$

This may seem a useful formula. However, it cannot be applied, since neither the correlation coefficient R_Δ nor the systematic error ε_s can be derived from the measurements. In order to learn more about local influences, attention should not be restricted to one station. Comparisons with observations in the vicinity of the station should prove useful.

Behaviour of the correlation coefficient $\varrho\ (\tau)$ and the variance of measurement $Var\Delta\underline{y}(\tau)$ close to $\tau = 0$.

When τ decreases to values close to 0, temporal disturbances of short duration will affect the two measurements being compared more and more in the same way. This increases the serial correlation coefficient $\varrho_\Delta\ (\tau)$. Therefore in general

$$\lim_{\tau \to 0} \varrho_\Delta(\tau) = 1 \qquad (3\text{-}36)$$

For $\tau = 0$, ϱ_Δ as well as ϱ_t become 1, and as a consequence of eq (3-20), also $\varrho(0) = 1$*). This is a limit case since two simultaneous measurements with the same instrument cannot be carried out in practice, but are in fact replaced by one single measurement.

*) This value $\varrho(0)$ is to be distinguished from the extrapolated value $\tilde{\varrho}(0)$.

It will be tried now to give an impression of the behaviour of $\varrho_\Delta(\tau)$. If, for instance the measurement errors would be equally partitioned over all frequencies in the water level variations (i.e. the case which corresponds to Fig. I-9E of annex I) then the following function of ϱ_Δ can be adopted:

$$\varrho_\Delta(\tau) = \frac{\sin \dfrac{2\pi\tau}{\Delta T}}{\dfrac{2\pi\tau}{\Delta T}} = \frac{\text{dif } 2\pi\tau}{\Delta T} \qquad (3\text{-}37)$$

Here T denotes the period corresponding to the frequency at which the measurements are carried out. In the present example an arbitrary value of $\Delta T = 10$ min was choosen. Now substitute eq (3-37) into eq (3-20) and into eq (3-22) in order to find the curves of $\varrho(\tau)$ and Var $\Delta y(\tau)$ or its square root $\sigma\Delta y(\tau)$. The expression for $\varrho_t(\tau)$, needed for this calculation, can be derived from eq (3-24):

$$\varrho_t(\tau) = \frac{\tilde{\varrho}(\tau)}{1 - \dfrac{\text{Var}\Delta y}{\text{Var } y}} \qquad (3\text{-}38)$$

Since it concerns the extrapolated part of the $\varrho(\tau)$-curve of Fig. 3-4, in which the *influence of $\varrho_\Delta(\tau)$ has been eliminated, the notation $\tilde{\varrho}(\tau)$ is used here.* The formula to be applied is derived from the interpolation method used, as described in the explanation of Fig. 3-4. This formule reads

$$\tilde{\varrho}(\tau) = 0{,}999946 - 0{,}000028\,\tau^2 \; (0<\tau<1) \qquad (3\text{-}39)$$

According to eq (3-29) the denominator of eq (3-38) is the extrapolated correlation coefficient $\tilde{\varrho}(\tau)$ for $\tau = 0$. From eq (3-39) it follows that

$$\tilde{\varrho}\,(0) = 0{,}999946$$

Further elaborating of eq (3-38) yields

$$\varrho_t(\tau) = \frac{0{,}999946 - 0{,}000028\tau^2}{0{,}999946}$$

or

$$\varrho_t(\tau) = 1 - 0{,}000028\tau^2 \ (0<\tau<1) \tag{3-40}$$

For all elements, occurring in eq (3-21), expressions or values are now known. The curve, corresponding to eq (3-21) for different values of τ is shown in Fig. 3-6.

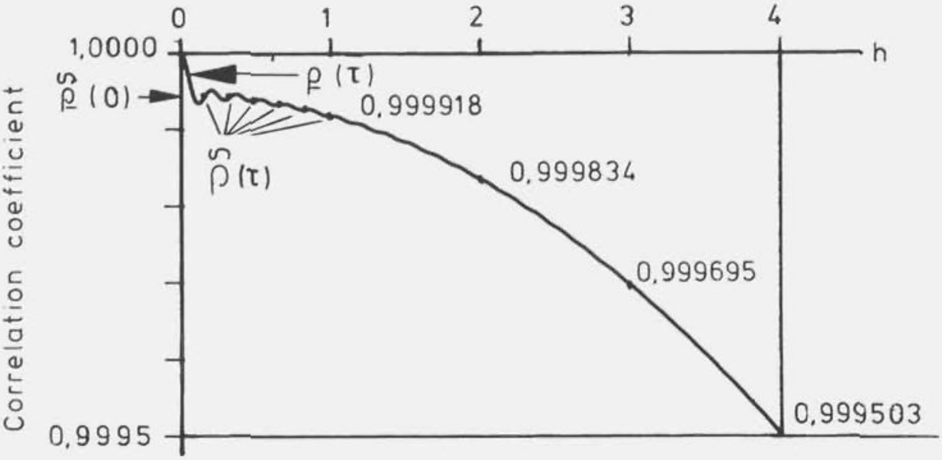


Fig. 3-6 Correlation coefficients $\varrho(\tau)$ and $\tilde{\varrho}(\tau)$ vs. τ.

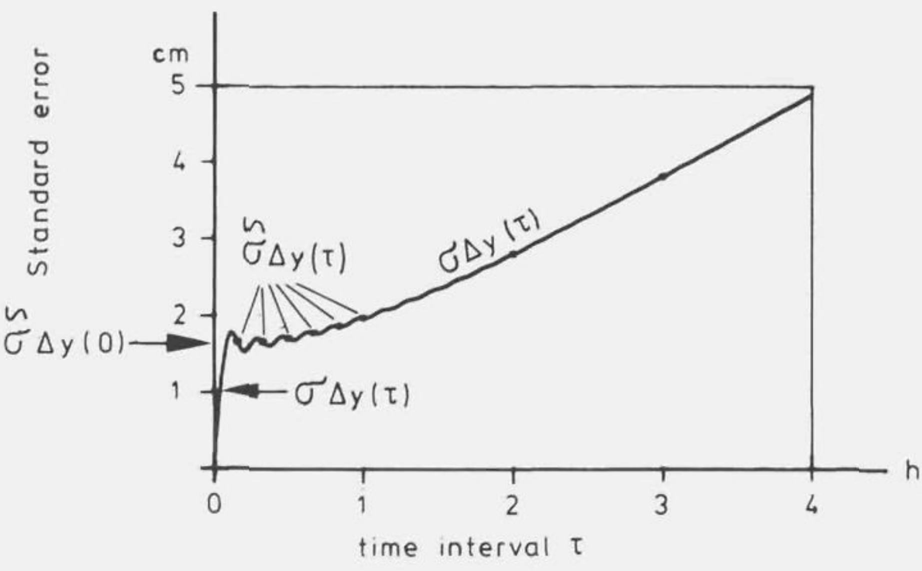


Fig. 3-7 Standard errors $\sigma\Delta y\ (\tau)$ and $\tilde{\sigma}\Delta y(\tau)$ vs. τ.

The curve of the standard error $\sigma\Delta y(\tau) = \sqrt{\text{Var}\Delta y(\tau)}$ can be calculated by means of eq (3-22), using the same elements as for eq (3-21). This curve is shown in Fig. 3-6.

Figures 3-6 and 3-7 clearly show the behaviour of ϱ (τ) and $\sigma\Delta y(\tau)$, and of $\tilde{\varrho}$ (τ) and $\tilde{\sigma}\Delta y(\tau)$ which are of direct interest since these determine the standard error of measurrement. The vicinity of $\tau = 0$ is strongly influenced by temporal disturbances. These disturbances are considered to form part of the errors of measurement.

It is worth remarking that the curves, shown in Figures 3-6 and 3-7 will never be able to be constructed using measurements. The positions of the values following from the measurement are such, that a curve drawn through these positions will be directed to the values of $\tilde{\varrho}$ (0) andy $\tilde{\sigma}\Delta y(0)$ respectively, as is shown on the graphs in Figures 3-6 and 3-7. Intensifying the measurements, e.g. by application of a measurement interval of $\frac{1}{2}$T instead of T, will lead to curves of $\varrho(\tau)$ and $\sigma\Delta y(\tau)$ with oscillations having half the length of those in the curves shown. But again the positions of the measurement plots will be on curves leading to $\tilde{\varrho}(0)$ and $\tilde{\sigma}\Delta y(0)$.

In the hypothetical case of (unfiltered) measurements with a measurement interval $\tau = 0$, the curves of $\varrho(\tau)$ and $\sigma\Delta y(\tau)$ really would lead to $\tilde{\varrho}$ (0) and $\tilde{\sigma}\Delta y(0)$, with a sudden jump to one, respectively zero for $\tau = 0$.

3.4.2 Comparison of measurements at the station under examination with measurements at a nearby station

In order to detect the influence of local disturbances on the measurement errors, measurements from the station under examination might be compared with measurements from another station located nearby. The distance between the two stations should be long enough for the local influences at one station to be completely different to those at the other station, but, on the other hand, short enough to exclude model errors. Although situations like this will be rarely found, sometimes good situations will arise, for instance when an existing station is replaced by a new one, and a certain period of overlap is maintained.

Now let y denote the measurements at the new station and x those at the old station. The mathematical model is now simply

$$y = x \tag{3-41}$$

The following notation will be used:

x_t, y_t = the representative ('true') values.
$\underset{\sim}{x}$, $\underset{\sim}{y}$ = the measured values.
$\hat{y}$ = the calculated values.

By definition, no model errors are considered; all errors that occur will be considered to be measurement errors. Thus

$$y_t = x_t, \tag{3-42}$$

$$\hat{y} = \underline{x}, \tag{3-43}$$

and

$$\Delta y = \underline{y} - \hat{y}, \tag{3-44}$$

which, because of eq (3-43) can be expressed as

$$\Delta y = \underline{y} - \underline{x}. \tag{3-45}$$

From eq (2-4) (also cited in Section 3.1) it follows that:

$$\underline{y} = y_t + \Delta \underline{y};\ \underline{x} = x_t + \Delta \underline{x} \tag{3-46}$$

Substituting these into eq (3-45) and regarding eq (3-42) it follows that:

$$\Delta y = \Delta \underline{y} - \Delta \underline{x} \tag{3-47}$$

Computing the squared sum of Δy for a series of values gives, using

$$(\Delta y_i)^2 = (\Delta \underline{y}_i)^2 + (\Delta \underline{x}_i)^2 - 2\Delta \underline{y}_i \Delta \underline{x}_i \tag{3-48}$$

$$\frac{\Sigma(\Delta y_i)^2}{n} = \frac{\Sigma(\Delta \underline{y}_i)^2}{n} + \frac{\Sigma(\Delta \underline{x}_i)^2}{n} - \frac{2\Sigma \Delta \underline{y}_i \Delta \underline{x}_i}{n} \quad {}^{*)} \tag{3-49}$$

Considering the different right hand terms then

$$\left.\begin{aligned} \frac{\Sigma(\Delta \underline{y}_i)^2}{n} &= (\bar{\Delta}\bar{\underline{y}})^2 + \mathrm{Var}\Delta \underline{y}, \\ \frac{\Sigma(\Delta \underline{x}_i)^2}{n} &= (\bar{\Delta}\bar{\underline{x}})^2 + \mathrm{Var}\Delta \underline{x} \end{aligned}\right\} \tag{3-50}$$

*) The summation $\sum_{i=1}^{n}$ will be written Σ for the sake of brevity

and:

$$\frac{\Sigma \Delta y_i \Delta x_i}{n} = \overline{\Delta y}\,\overline{\Delta x} + \mathrm{Cov}\Delta y \Delta x, \qquad (3\text{-}51)$$

where $\overline{\Delta x}$, $\overline{\Delta y}$ = the mean values of Δx_i, Δy_i

Substitution of eqs (3-50), and (3-51) into eq (3-49) yields:

$$\frac{\Sigma (\Delta y_i)^2}{n} = (\overline{\Delta y})^2 - 2\overline{\Delta y}\,\overline{\Delta x} + (\overline{\Delta x})^2 + \mathrm{Var}\Delta y + \mathrm{Var}\Delta x - 2\,\mathrm{Cov}\Delta y \Delta x \qquad (3\text{-}52)$$

Since

$$\mathrm{Var}\Delta x = \varepsilon_x^2$$

and

$$\mathrm{Var}\Delta y = \varepsilon_y^2$$

and further introducing the correlation coefficient ϱ_Δ between Δy and Δx, it can be stated that

$$\mathrm{Cov}\,\Delta y \Delta x = \varrho_\Delta \sqrt{\mathrm{Var}\Delta y\,\mathrm{Var}\Delta x} = \varrho_\Delta \varepsilon_y . \varepsilon_x \qquad (3\text{-}53)$$

Thus (3-52) transforms into

$$\frac{\Sigma (\Delta y_i)^2}{n} = (\overline{\Delta y} - \overline{\Delta x})^2 + \varepsilon_y^2 + \varepsilon_x^2 \,.\, 2\varrho_\Delta \varepsilon_y \varepsilon_x$$

or, considering eq (3-47) also to hold for the mean values, then

$$\frac{\Sigma (\Delta y_i)^2}{n} - (\bar{\Delta}\bar{y})^2 = \varepsilon_y^2 + \varepsilon_x^2 - 2\varrho_\Delta \varepsilon_y \varepsilon_x$$

or

$$\mathrm{Var}\Delta y = \varepsilon_y^2 + \varepsilon_x^2 - 2\varrho_\Delta \varepsilon_y \varepsilon_x \qquad (3\text{-}54)$$

This equation contains three unknows, i.e. ε_y, ε_x and ϱ_Δ. Additional information is therefore required to solve this equation, or assumptions have to be made. If, for instance,

$$\varepsilon_y = \varepsilon_x = \varepsilon \tag{3-55}$$

then

$$\mathrm{Var}\Delta y = 2\varepsilon^2 (1-\varrho_\Delta) \tag{3-56}$$

or, explicitely in ε

$$\varepsilon^2 = \frac{\mathrm{Var}\Delta y}{2\,(1-\varrho_\Delta)} \tag{3-57}$$

Here two unknowns still remain, but nevertheless it may be a useful formula, e.g. for the determination of ϱ_Δ if ε is derived by some other means. This knowledge of ϱ_Δ can later be used for similar cases.

Example

In 1983 the gauging station at Lobith at the River Rhine was replaced by a new one, because the old station no longer met certain technical requirements. In order to make the data series of both stations comparable, an overlap period of one year was maintained from 15 August 1982 to 14 August 1983 (Fig. 3-8).

Fig. 3-8 New and old gauging station Lobith at the River Rhine

For the six 2-months periods the statistics shown in Table 3-1 were calculated.

period during 1982-1983			$\bar{\Delta y}$ (cm)	$\sigma\Delta y$ (cm)
15 August	-	14 October	-0,72131	0,83927
15 October	-	14 December	-1,01634	0,99149
15 December	-	14 February	-0,78333	0,82527
15 February	-	14 April	-1,22034	0,87233
15 April	-	14 June	-0,81967	1,25841
15 June	-	14 August	-1,31148	0,95814
15 August	-	14 August	-0,97796	0,98864

Table 3-1. Statistics of the data series from the gauging stations Lobith-old and Lobith-new.

Now consider the standard error value for the whole year, as given in Table 3-1:

$$\sigma\Delta y = 0{,}98864 \text{ cm}$$

First consider the case that no spatial correlation exists ($\varrho_\Delta = 0$) between both series of measurement errors, Δy_i and Δx_i, whereas the standard errors of measurement are equal:

$$\varepsilon_y = \varepsilon_x = \varepsilon.$$

Then, on the basis of eq (3-57), and taking the square root, it follows that

$$\varepsilon = \frac{\sigma\Delta y}{\sqrt{2}} = \frac{0{,}98864}{\sqrt{2}} = 0{,}699 \text{ cm}$$

This is a rather low value, especially when compared with the result of Subsection 3.4.1, where the data of one station were compared with time shifted data from the same station. This led to a standard error of $\varepsilon = 1{,}134$ cm if there were no autocorrelation and to $\varepsilon = 1{,}793$ cm if some autocorrelation existed, and corresponding to a correlation coefficient of $R_\Delta = 0{,}6$.

The low value of ε, obtained assuming no spatial correlation in the present case suggests that correlation between the two stations might be expected. If correlation does exist, the given value of $\sigma\Delta y$ would lead to higher values of ε than 0,699 cm. as follows from eq (3-57), or its square root form

$$\varepsilon = \frac{\sigma\Delta y}{\sqrt{2(1-\varrho_\Delta)}} \tag{3-58}$$

Figure 3-9 depicts the relation between ε and ϱ_Δ when based on the value $\sigma\Delta y$ = 0,98864. The values, found in Subsection 3.4.1 are summarized in Table 3-2.

autocorrelation coefficent R_Δ	standard error of measurement ε	spatial correlation coefficient ϱ_Δ
0	1,134 cm	0,621
0,6	1,793 cm	0,848

Table 3.2 Some related values of R_Δ, ε and ϱ_Δ.

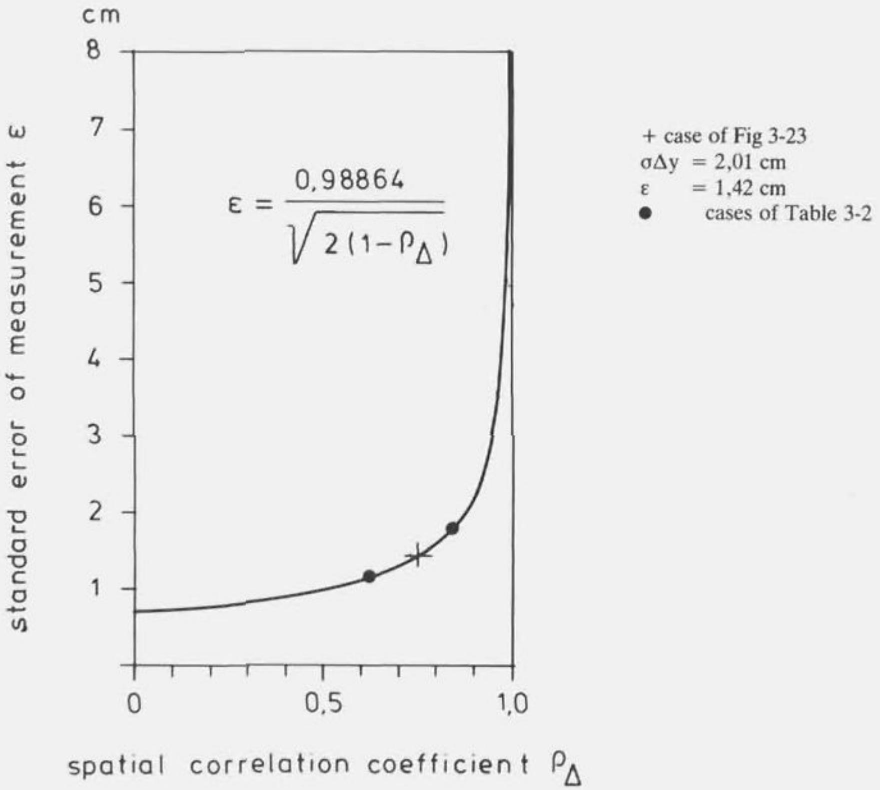


Fig. 3-9 Standard error of measurement ε vs. spatial correlation coefficient ϱ_Δ (Lobith example).

A direct relation between R_Δ and ϱ_Δ follows when combining eq (3-34) with eq (3-57). Eq (3-34) says that

$$\begin{aligned} \tilde{\text{Var}}\Delta y(0) &= 2\,\text{Var}\Delta y\,(1-R_\Delta) \\ &= 2\varepsilon^2\,(1-R_\Delta) \end{aligned}$$

Substituting of eq (3-57) gives

$$\bar{Var}\Delta y(0) = \frac{Var\Delta y}{1-\varrho_\Delta}(1-R_\Delta)$$

Since it was found that

$$\bar{Var}\Delta y(0) = 2{,}573 \text{ cm}^2$$

and

$$Var\Delta y = (0{,}98864 \text{ cm})^2 = 0{,}977 \text{ cm}^2$$

then it follows that

$$2{,}573 = \frac{0{,}977}{1-\varrho_\Delta}(1-R_\Delta)$$

or

$$\varrho_\Delta = 0{,}379\, R_\Delta + 0{,}621.$$

This relation is shown in Fig. 2-10.

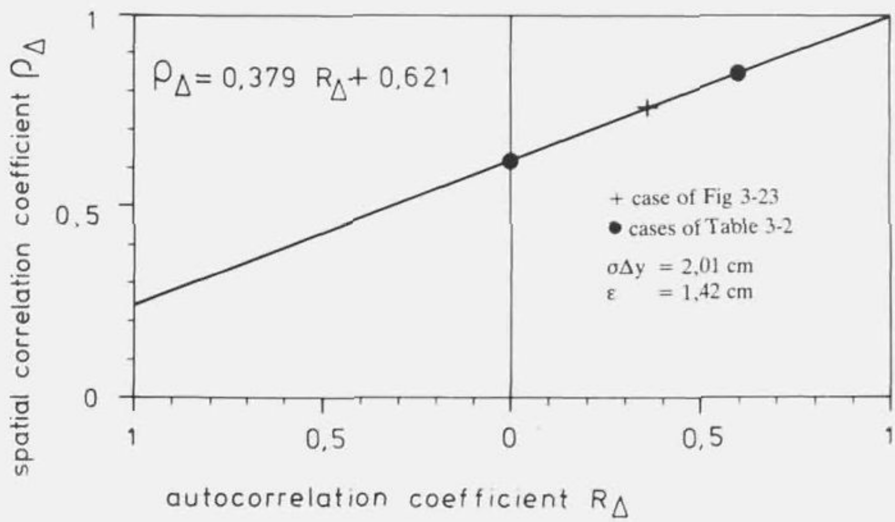


Fig 3-10 Spatial correlation coefficient ϱ_Δ vs. autocorrelation coefficient R_Δ (Lobith example).

Since the following inequality holds

$$-1 \leq R_\Delta \leq 1$$

it follows that

$$0{,}242 \leq \varrho_\Delta \leq 1$$

This means that in any case there exists some spatial correlation. Moreover, since it is very unlikely that there is a negative value of R_Δ, it may be assumed that

$$0 \leq R_\Delta \leq 1$$

and

$$0{,}621 \leq \varrho_\Delta \leq 1$$

This strengthens the conclusion that there is spatial correlation between the two neighbouring stations. Further,the presence of positive autocorrelation in the time series will lead to higher spatial correlation than in the autocorrelation free case.

In summary one may note that the comparison of the old and the new gauging station Lobith did not lead to an estimation of the standard error of measurement, but it showed that the measurement errors of the two stations were correlated. Therefore, if two stations are selected for comparison in order to determine the standard error of measurement, one should either take this phenomenon into account or one should select the two stations at such a distance that correlation between their water levels does not occur, or at least may be neglected.

3.4.3 Comparison of measurements at the station under examination with measurements at other stations at different distances from the station under examination

Here the measurements at the station under examination are to be compared with measurements at a series of other stations along the same river. Each time one of the other stations is considered, an interrelation will be determined between that station and the station under examination. This relation may be linear in the most simple case, but also square or higher power relations may be used. This may concern simultaneous measurements, but measurements shifted in time may be taken into account as well.

There are no principal differences between the present approach and the method of comparison using measurements at one single station, as described in Subsection 3.4.1. The main distinction is that the time interval is replaced now by a spatial, or distance, interval z. However, a complicating factor is that, whereas in time series the intervals can be chosen freely and preferable in such a way that all intervals are *exact multiples of the smallest interval that is used (in general the measurement or* the sampling interval), for distance series one is restricted to the station locations. As a rule the distances between the stations will differ, sometimes considerably.

The advantage of this distance approach is that local disturbances can be eliminated to a certain extent, because the conditions around the various stations, as a rule, will not be the same.

When the station under examination and the station with which it is compared*) are close together, some correlation between the measurement errors may be expected, as was shown in Subsection 3.4.2. At longer distances this correlation will be reduced and will finally disappear, although a clear limit to this correlation's influence is difficult to recognize. Gradually also model errors are being introduced. These are errors due to the differences between the established relation and reality. These *errors are induced by factors acting along the river reach between the two stations, a* fact that is related to correlation between the water levels themselves, but not to correlation between their measurement errors.

In Section 2.3 it has been shown that the variance of the differences between the measured data at the station under examination and the calculated data for that station, includes:
- model errors;
- propagated measurements errors of the network station;
- measurement errors of the station under examination.

This fact has been expressed by eqs (2-17) ad (2-21). For a linear regression model:

$$y = A_0 + A_1 x \qquad (3\text{-}59)$$

eq (2-21) becomes:

$$\mathrm{Var}\Delta y = \varepsilon_y^2 + A_1^2 \varepsilon_x^2 + \mathrm{Var}\ r \qquad (3\text{-}60)$$

when Var r = variance of the model errors.

As a rule the model errors will increase with increasing distance between the two

*) Further to be called the network station

stations. But at short distances the measurement errors will play a dominant role. At very short distances the measurement errors are hidden from view by the intercorrelation between these errors at both stations, although they remain in the measured data. The latter was discussed widely in Subsection 3.4.2.

Plotting the standard errors $\sigma\Delta y$ against the stations distances for a number of network stations at various distances from the station under examination, a curve as shown in Fig. 3-11 might be obtained. For the reasons explained in Subsection 3.4.1, but now applied to distance instead of time, the curve is to be continued horizontally until $z=0$. At this point $\sigma\Delta y$ must take the value $\varepsilon\sqrt{2}$.

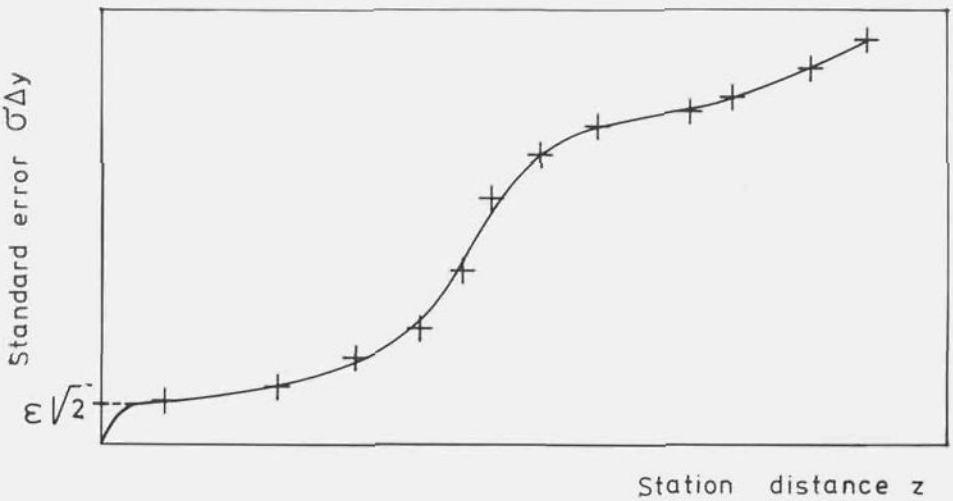


Fig. 3-11 Standard error $\sigma\Delta y$ vs. station distance z.

This diagram is comparable with Fig. 3-1, showing the relation between $\sigma\Delta y$ and the time interval τ for measurements made at one single station.

For the distance approach similar formulae can be derived to those given for the time approach in eqs (3-21) and (3-22). The corresponding equations are

$$\varrho^2(z) \approx \varrho_t^2(z) \left\{ 1\text{-}2 \frac{\text{Var}\Delta \underline{y}}{\text{Var}\, \underline{y}} \left(1\text{-} \frac{\varrho_\Delta(z)}{\varrho_t(z)} \right) \right\} \qquad (3\text{-}61)$$

and

$$\text{Var}\Delta y(z) \approx \{1\text{-}\varrho_t^2(z)\}\, \text{Var}\, \underline{y} + 2\varrho_t^2(z)\, \text{Var}\Delta \underline{y} \left(1\text{-} \frac{\varrho_\Delta(z)}{\varrho_t(z)} \right) \qquad (3\text{-}62)$$

The correlation coefficient $\varrho_\Delta(z)$ indicates the correlation between measurement errors at the network station at a distance z and those at the stations under examination. If z = 0, then $\varrho_\Delta(z)$ becomes the constant coefficient ϱ_Δ of Subsection 3.4.2.

The distance approach differs with respect to the time approach in that with the time approach the values of the abcissa are at equal distances, and a curve drawn through the values has a regular shape. In the distance approach the abcissa values are usually unequal since they depend on the actual locations of the gauging stations. A curve drawn through the data values will generally have an irregular shape, depending on all kinds of disturbances and discontinuities ocurring along the river. Example of these are inflows of tributaries, diversions, narrows, back water effects, changes in slope of the riverbed, etc. In upland rivers these irregularities generally will be stronger than in tidal rivers, where the correlation is influenced by the harmonic character of the tidal movement. It is not feasible in upland rivers to attempt to describe the shape of a curve of the type shown in Fig. 3-11 by a mathematical function, but in tidal rivers this might be succesful.

If gauging stations exist at both sides of the stations to be examined then a two-sided graph can be drawn, thus making use of more information. Of course, such two-sided graphs might also be set up for the time approach but they will be of no value, since the correlation between $\underline{y}(t-\tau)$ an $\underline{y}(t)$ on the one hand is equal to the correlation between $\underline{y}(t)$ and $\underline{y}(t+\tau)$ on the other hand. The only result would be a duplication in mirror image of the graph already known. This would not lead to more information.

There are cases in the distance approach, in where it is not feasible to make use of a two sided graph, namely when the conditions are affected by heavy disturbances at one side e.g. the confluence with an important tributary or a water fall, but not at the other side. In such cases, it is preferable to consider the side having the lowest standard errors only, in order to avoid disturbed results at distance 'zero'. Such cases will be shown in the following examples.

By extrapolating the curve of the standard errors, $\sigma\Delta y$, up to distance 'zero' an estimate for the standard error of measurement can be found. In the one sided graph this will be an extrapolation, in the two sided graph an interpolation. A condition is that the curve shows a minimum at distance 'zero', since in that case model errors can be disregarded (Var r = 0), whereas the standard errors of measurement will be equal. This is because these errors concern the same station. The coefficient A_1 will be 1 for the same reason. Now eq (3-60) becomes*)

*) At distance 'zero' alle types of relations, including higher power functions and transcendental functions, will be transformed to the linear relation y = x.

$$\hat{Var}\ \Delta y = 2\varepsilon^2 \qquad (3\text{-}63)$$

and

$$\tilde{\sigma}\Delta y = \varepsilon\sqrt{2} \qquad (3\text{-}64)$$

A condition for this procedure is to apply that the nearest network stations are beyond the range of the spatial correlation between the measurement errors. If not, eq (3-63) becomes

$$Var\Delta y = 2\varepsilon^2(1-\varrho_\Delta) \qquad (3\text{-}65)$$

which follows from eq (3-34). In this case Var Δ y will be reduced and will not yield the correct values of ε.

Spline functions might be applied in order to construct the interpolation curve. Such details are discussed in Section 4.4. The spline functions used here are separate 2nd-degree parabolas between the data plots. If desired, the parabolas can also be changed at sites where no measurements are carried out (dummy points; see Subsection 4.4.3.1).

Examples

1) RIVER IJSSEL STATIONS

Figs 3-12 A, B and C show two sided graphs of $\sigma\Delta y$ vs. the location of network stations, for the station under examination: in this case Zutphen on the IJssel river in the Netherlands. The plotted $\sigma\Delta y$-values are derived from 3rd-degree relations between the Zutphen station and each of the network stations. A situation map is given in Fig. 2-1. The results are based on water levels measured in the years 1977 and 1978, water levels being measured every 3rd day, starting from 3 January, 1977.

In Fig. 3-12A only the station locations are used as transition points of the partial parabolas. The resulting interpolation curve has a shape that seems rather obtuse at the site of the station under examination, which leads to too high a $\sigma\Delta y$-value for this site. Probably the distance to the nearest network station was too large for an acceptable reproduction of the $\sigma\Delta y$-curve. This was improved by the introduction of dummy points at both sides of the Zutphen station under examination. Fig. 3-12B shows the case of two dummy points 5 km away from the station under examination, in Fig. 3-12C this distance is only 100 m. In Fig. 3-12B the curve gradually goes to the site of the station under examination coming to a horizontal level at that site. In Fig. 3-12C the curve goes more or less straight on to that station, which seems to be in contradiction with the smooth character of the representative values y_t. Therefore the curve of Fig. 3-12B was selected for further consideration.*)

The interpolated value at the site of the station under examination ($\sigma\Delta y = 3{,}64$ cm) can be considered now to be equal to $\varepsilon\sqrt{2}$ (eq (3-63)). The corresponding ε-value was calculated as:

$$\varepsilon = 3{,}64 \text{ cm}/\sqrt{2} = 2{,}57 \text{ cm}$$

ε-values for some other stations along the IJssel river were found in a similar way as given in Table 3-3:

station	standard error if measurement ε (cm)	
	bases on linear relations	based on 3rd-power relations
Olst	1,57	1,37
Deventer	2,04	1,64
Zutphen	2,41	2,57
Dieren	1,79	1,75

Table 3-3 Estimated standard errors of measurement for various IJssel river stations.

There are no big differences between the results obtained from linear and from 3rd-degree relations. It is unlikely that other types of relations will change the picture in a major way. 2nd-degree relations, for instance, might be expected to yield results which will be somewhere in between.

The standard errors of measurement thus found are all of the order of 1,5 cm to 2,5 cm. This corresponds reasonably well with the values derived earlier.

*) Assume that by the measurement system and the following processing procedure oscillations up to a period of T = 10 min are surpressed. This corresponds with a wave length of
$L = cT = T\sqrt{gh}$
where:
c = celerity
g = acceleration due to gravity (9,8 m/s²)
h = mean water depth.
For h = 4 m holds:
$L = 10 \times 60\sqrt{9{,}8 \times 4} = 3700$ m
This length should be coverd at least by the partial curve around z=0. An extension of 5 km, as applied in Fig. 3-12B fulfils this requirement

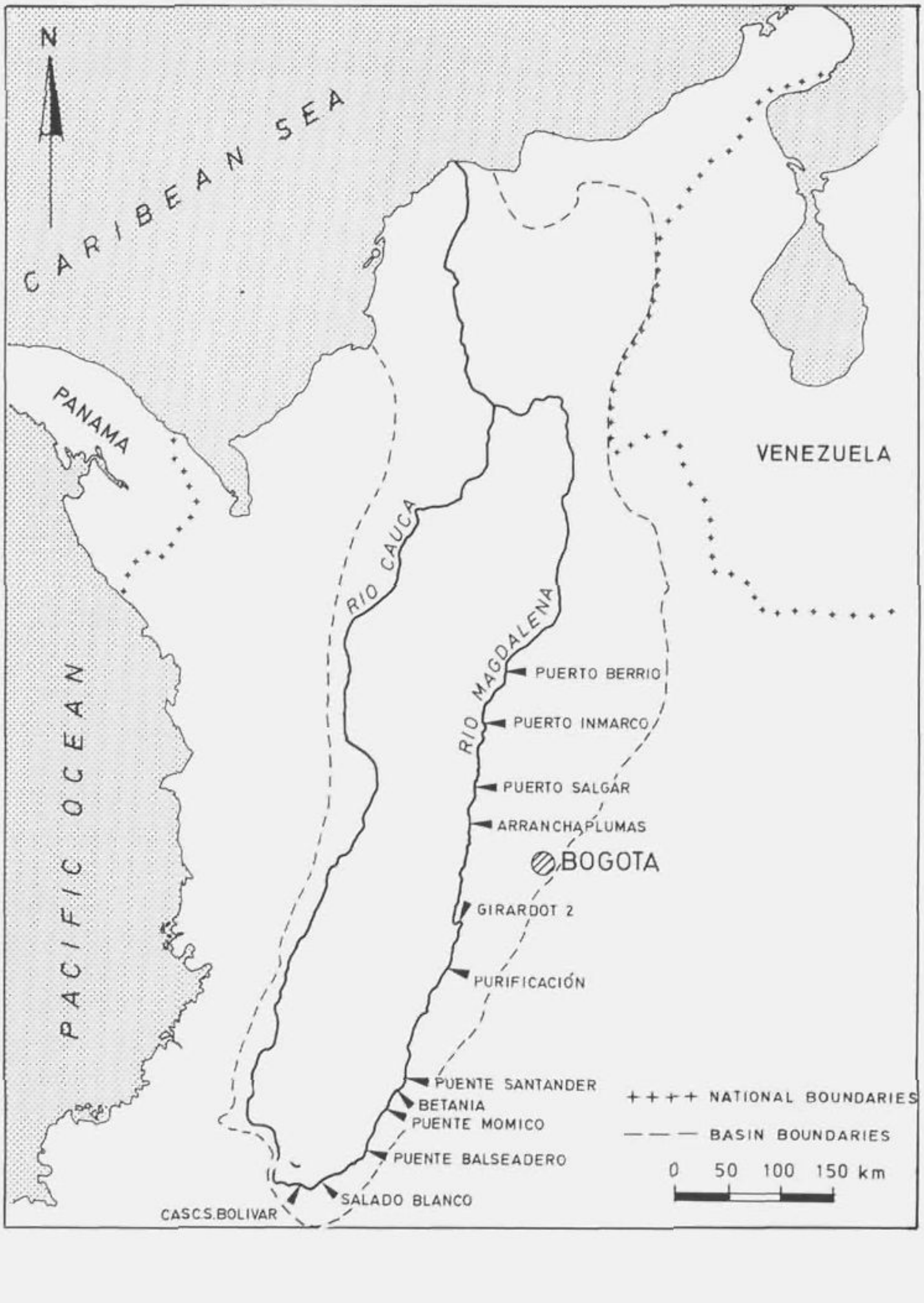


Fig. 3-13 Water level gauging stations along the upper Rio Magdalena (Republic of Colombia).

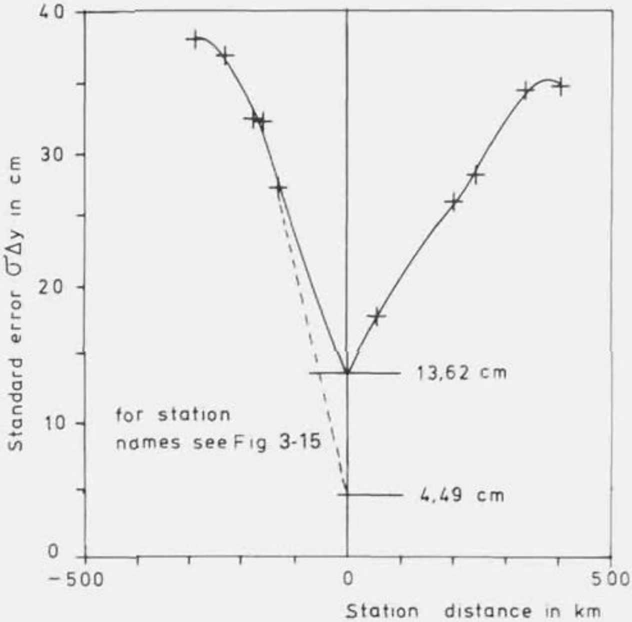


Fig. 3-14 Standard error σΔy of station Purificación vs. station distance.
(simultaneous measurements)
--------- one sided interpolation
——— two sided interpolation

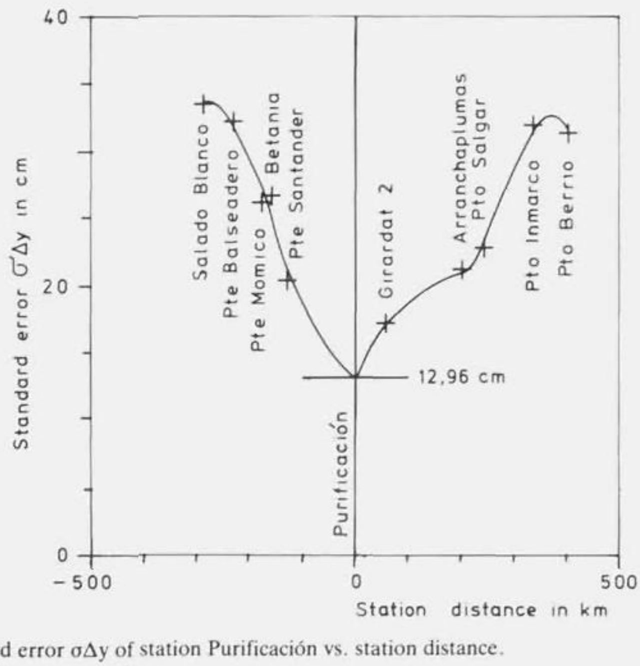


Fig. 3-15 Standard error σΔy of station Purificación vs. station distance.
(5 day measurements)

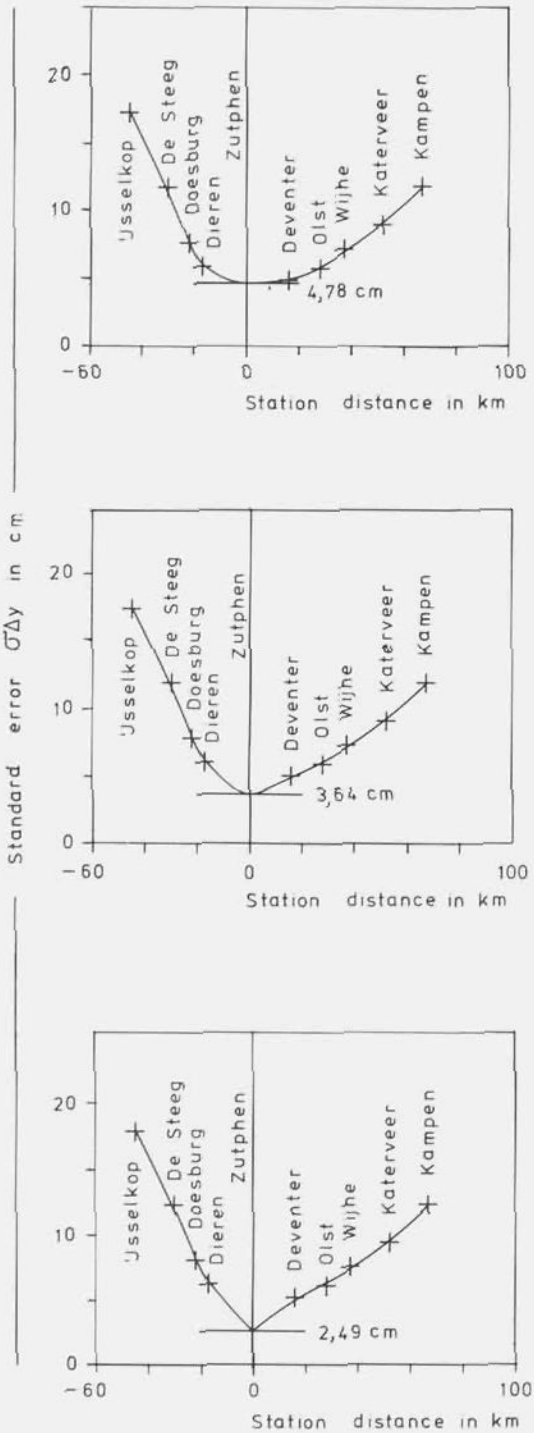


Fig 3-12 A
no dummy points

Fig. 3-12 B
dummy points at
5 km from Zutphen

Fig 3-12 C
dummy points at
100 m from Zutphen

Fig. 3-12 Standard error σΔy of station Zutphen vs station distance.

2) RIO MAGDALENA STATIONS

The Rio Magdalena is a river in Colombia, flowing into the Caribean Sea. A situation map is shown in Fig. 3-13. To be examined is the standard error of measurement of the station Purificación, using daily water level data from the other stations for the year 1974.

The nearest station downstream is Girardot 2 (57 km), and the nearest upstream is Puente Santander (130 km). The distances are much larger than those between the stations in the first example. No nearby information is available. When carrying out the same procedure as in Fig. 3-12, the curve of Fig. 3-14 is obtained. This yields at distance 'zero' a standard error of $\sigma\Delta y$ = 13,62 cm. Undoubtedly this value is much larger than the actual measurement accuracy. Local circumstances between Purificación and Girardot-2 will cause the standard error obtained from the relation between these two stations, to have a relatively high value (i.e. 17,6 cm) which dominates the interpolation result at 'zero'. Apparently for the approach to yield satisfactory results, stations located closer to the station under examination should be taken into account. In this case the curve would move down to more realistic values. Even when measurements at other times are used too, no considerable improvement is attained. This is shown in Fig. 3-15. At the network stations also water levels of 2 days and 1 day before and 1 day and 2 days afterwards are accounted for in the relation, yielding the water level at the station under examination.

A better result can be obtained using the upstream data only. The one-sided approach leads to $\sigma\Delta y$ = 4,49 cm as is also shown in Fig. 3-14. The value may correspond with a standard error of measurement of 3,17 cm.But the distance between the station under examination and the nearest upstream station is as much as 130 km, which makes such an extrapolation at least questionable. For a more reliable result, information from gauging stations located more closely to the station under examination is necessary.

The conclusion drawn from this example is that for the determination of the standard error of measurement, a series of network stations should be used, the nearest stations of which should be located not too far from the station under examination. How far depends on the local conditions, such as the stream channel pattern and the topography of the area. Without knowledge of these features no good examination can be carried out.

3) RIVER RHINE STATIONS

This time a number of gauging stations is considered along the River Rhine in the Federal Republic of Germany together with the Dutch border station Lobith. This station was already discussed in Subsection 3.4.2. A map of the situation is given in

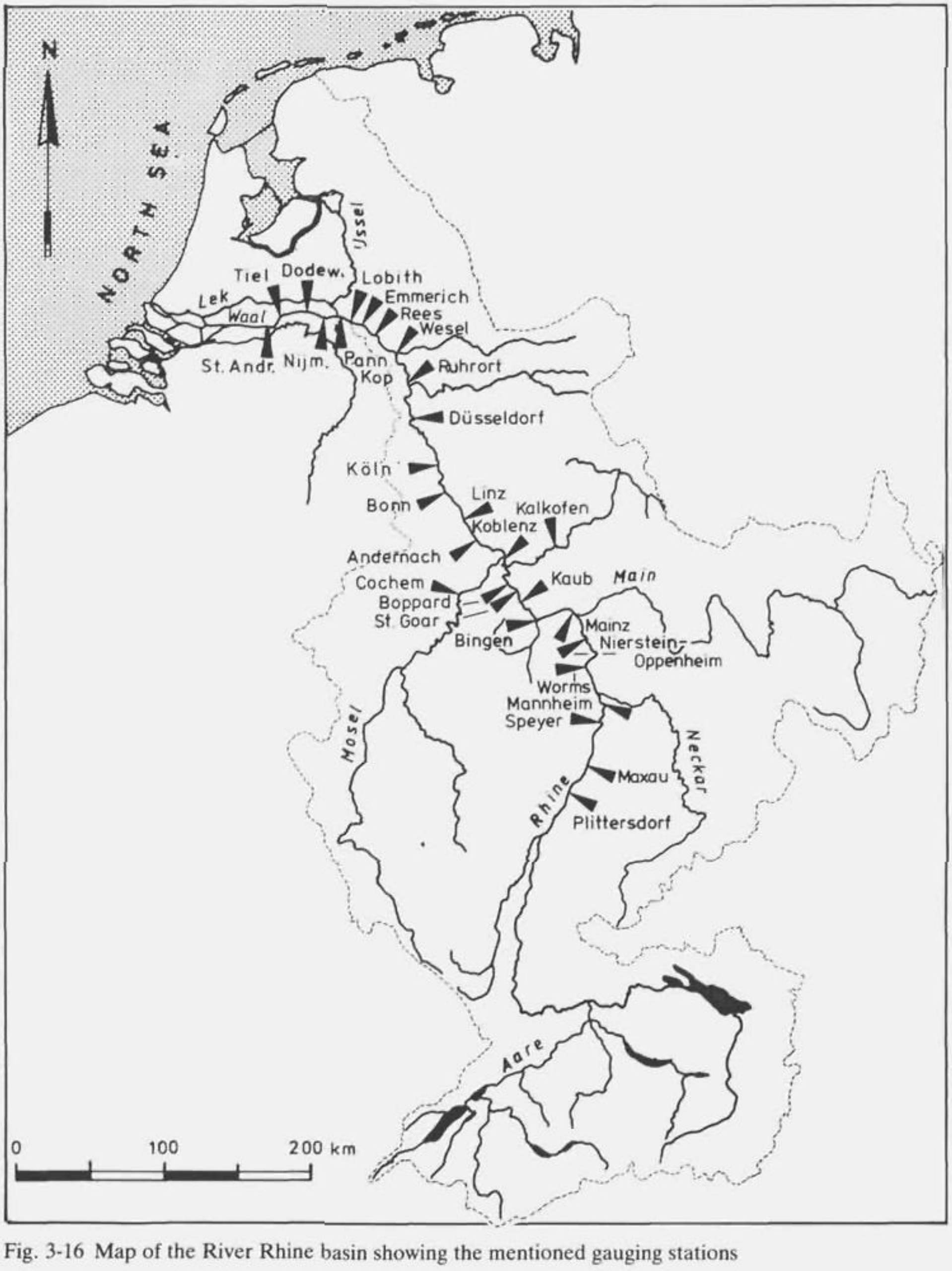


Fig. 3-16 Map of the River Rhine basin showing the mentioned gauging stations

Fig. 3-16. Use is made of simultaneous daily measurements, and also of measurements shifted in time. Like in the preceding case the latter include measurements made 2 days and 1 day before and 1 day and 2 days after the measurements at the station under examination.

The following stations were considered.

Lobith. Fig. 3-17 shows, for simultaneous measurements as well as for measurements shifted in time, the standard error σΔy vs the station distance. As might be expected the standard error increases with the distance, with remarkable jumps at the inflows of the tributaries Mosel (at Koblenz), Main (at Mainz) and Neckar (at Mannheim).

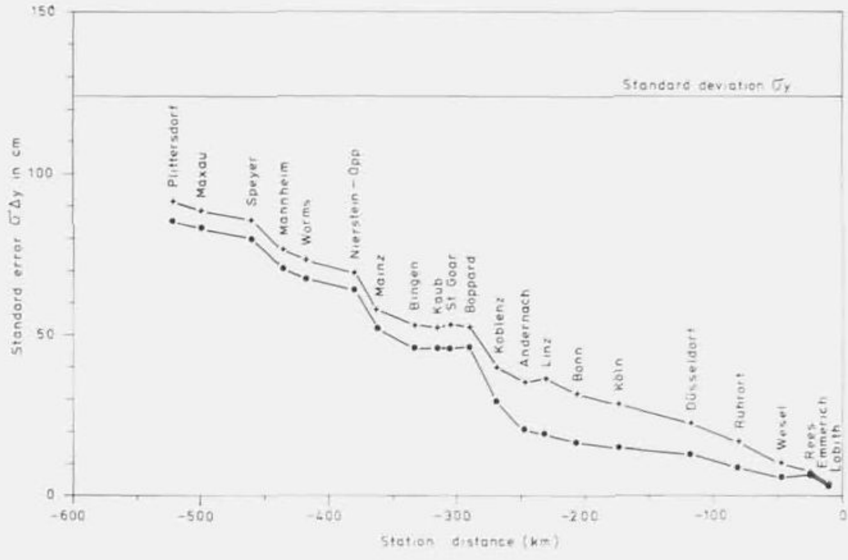


Fig. 3-17 Standard error σΔy of station Lobith vs. station distance.
+ simultaneous measurements
• 5 days measurements

Düsseldorf (Fig. 3-18). This station can feasibly be treated by a two sided approach, in view of the regular shape and the curve at both sides of this station.

Andernach (Fig. 3-19). This is an example where a one-sided approach is to be preferred. At the downstream side the curve gradually rises when receding from the station under examination; at the upstream side there is a jump to higher values due to the river Mosel inflow at Koblenz.

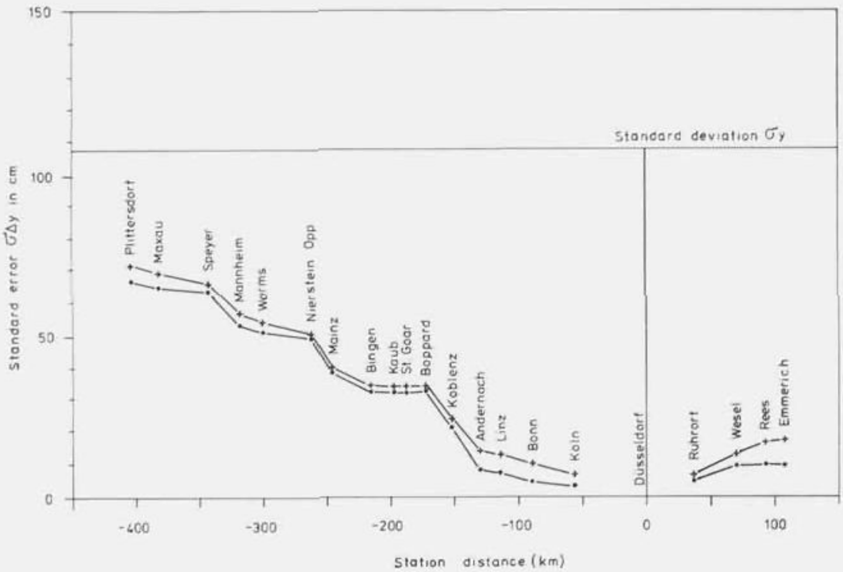


Fig. 3-18 Standard error σΔy of station Düsseldorf vs. station distance.
+ simultaneous measurements
• 5 days measurements

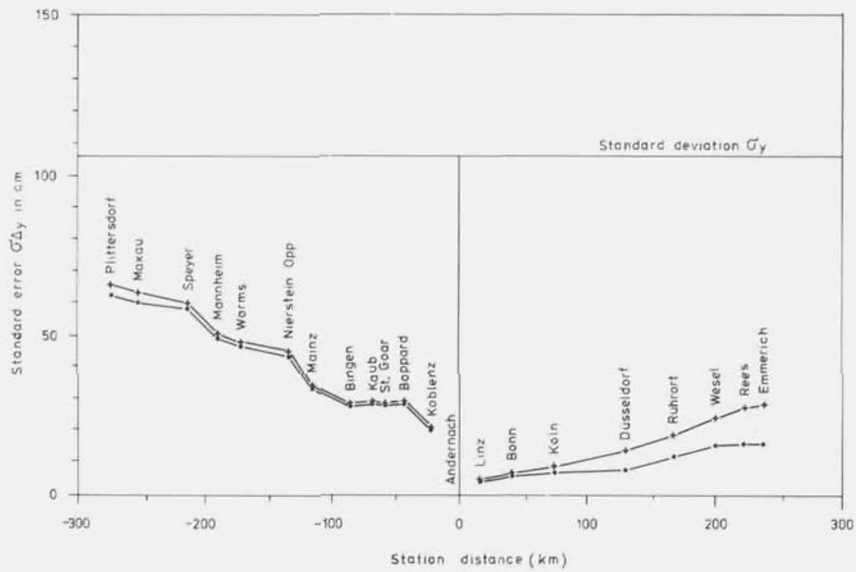


Fig. 3-19 Standard error σΔy of station Andernach vs. station distance.
+ Simultaneous measurements
• 5 days measurements

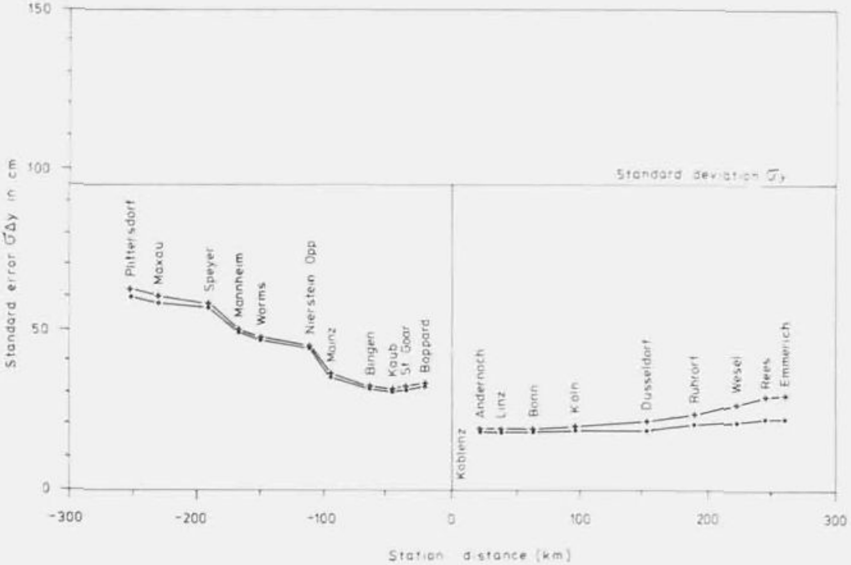


Fig. 3-20 Standard error σΔy of station Koblenz vs. station distance.
+ simultaneous measurements
• 5 days measurements

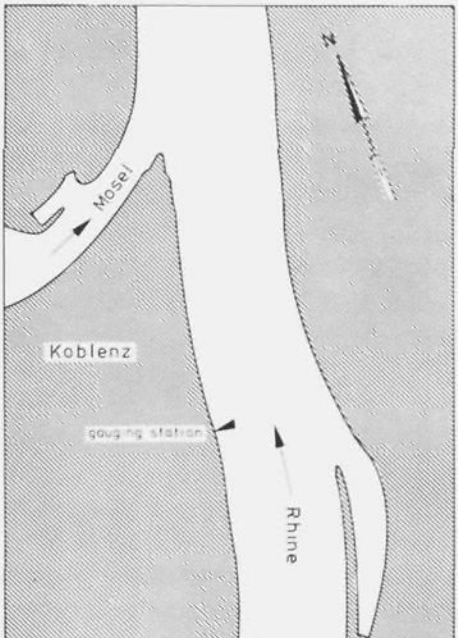


Fig. 3-21 Location of gauging station at Koblenz.

Koblenz (Fig. 3-20). The standard error σΔy takes rather high values at both sides, already at the nearest upstream and downstream network stations, although the downstream side yields somewhat lower values than the upstream side. Apparently the River Mosel inflow, and the location of gauging stations, being located nearly 2 km upstream (Fig. 3-21), prevents unique relations even at relatively short distances. With the information available, no value of the standard error of measurement can be detected. For this to be possible would require a very dense local micro network.

Worms (Fig. 3-22). Close to this station the situation appears to be acceptable, but at some distance, upstream as well as downstream, influences of tributary inflows (upstream the river Neckar, downstream the river Main) the standard errors increase considerably. For a satisfactery detection of the standard error of measurement some additional data would be desirable at intermediate sites between the stations Mannheim and Nierstein-Oppenheim.

In the following an attempt will be made to estimate the standard error of measurement for the stations considered above.

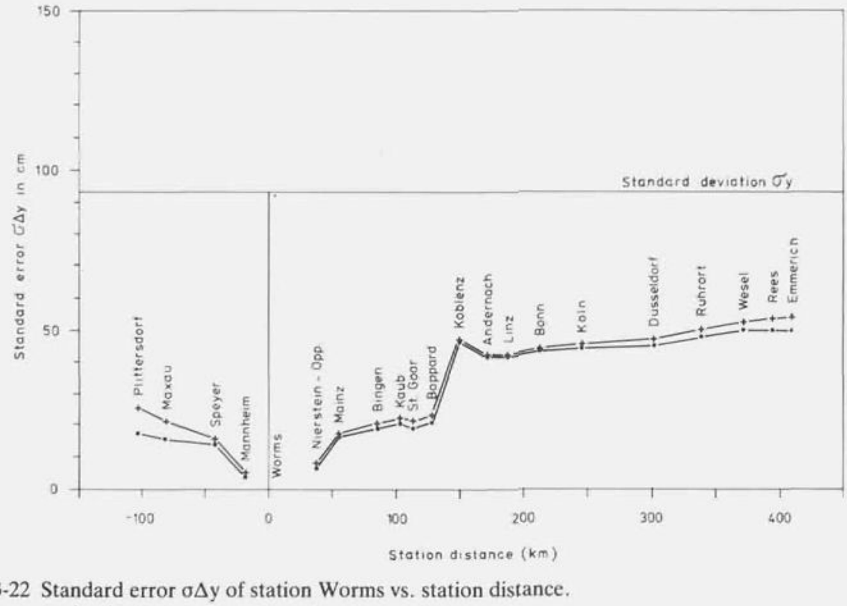


Fig. 3-22 Standard error σΔy of station Worms vs. station distance.
+ simultaneous measurements
• 5 days measurements

Lobith. In Fig. 3-23 the interpolation is carried out for the case of simultaneous measurements, as shown in Fig. 3-14, using also data from a number of stations on the Dutch, i.e. the downstream reach of the rivers Rhine and Waal. This led to a value of $\sigma\Delta y$=2,01 cm, corresponding with $\varepsilon=2{,}01/\sqrt{2}=1{,}42$ cm. This result was compared with those of the Subsections 3.4.1 and 3.4.2. If the value of ε is assessed at 1,4 cm, then there appears to be an autocorrelation coefficient for $\tau = 0$ of

$$R_{\Delta} = 0{,}362,$$

as can be derived from eq (3-34). There is also, according to eq (3-57) a spatial correlation coefficient between the old and the new station (i.e. for z=0) of

$$\varrho_{\Delta} = 0{,}757.$$

The above combination of ε, R_{Δ} and ϱ_{Δ} is plotted in Fig. 3-5, 3-9 and 3-10.

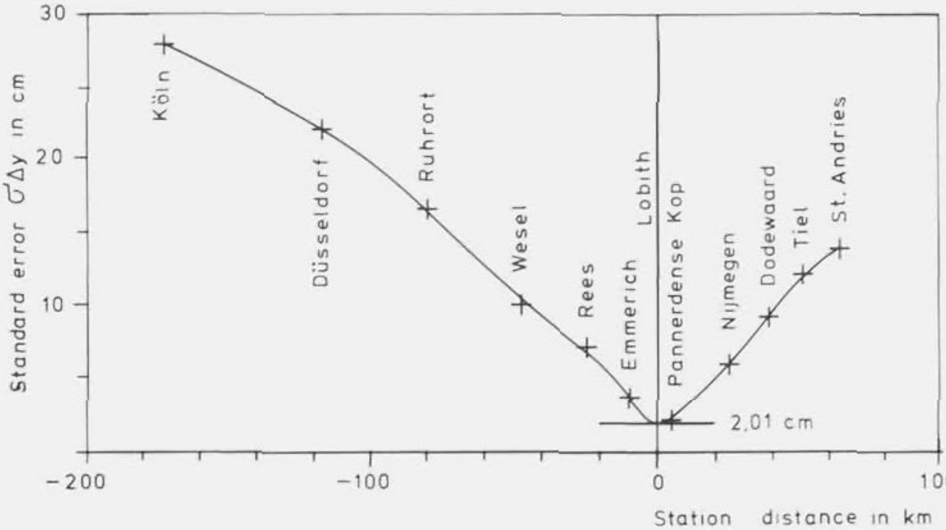


Fig. 3-23 Standard error $\sigma\Delta y$ of station Lobith vs. station distance, with interpolation (simultaneous measurements).

Düsseldorf (Fig. 3-24, for the simultaneous measurements). Here the simultaneous measurements yield a standard error of measurement $\varepsilon=1{,}45/\sqrt{2}=1{,}03$ cm.

Andernach (Fig. 3-25, for the simultaneous measurements). Here only the downstream side is considered, in view of the relation shown in Fig. 3-19. It produces a value of the standard error of measurement $\varepsilon=4{,}35/\sqrt{2}=3{,}0$ cm. This is higher than the values of Lobith and Düsseldorf, but not impossible, in view of the more variable local character of the river.

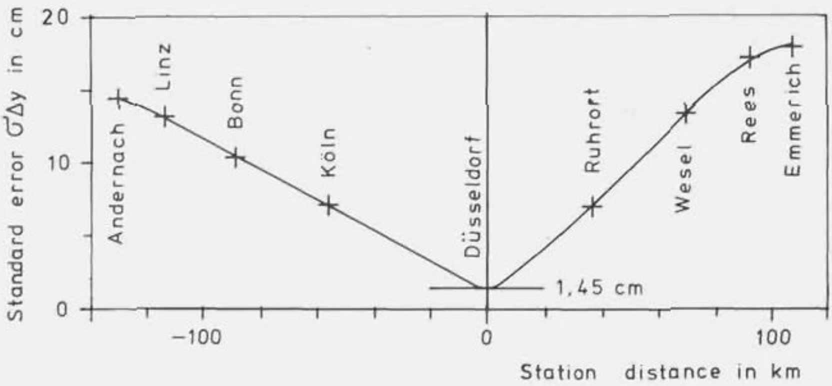


Fig. 3-24 Standard error $\sigma\Delta y$ of station Düsseldorf vs. station distance, with interpolation (simultaneous measurements).

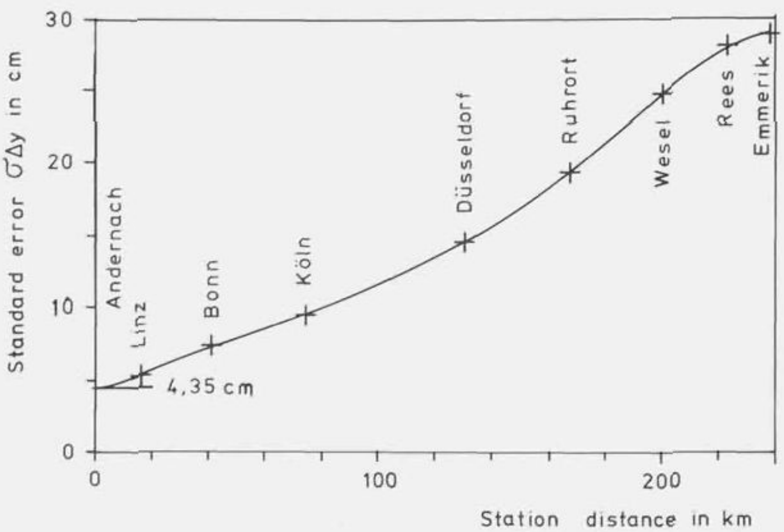


Fig. 3-25 Standard error $\sigma\Delta f$ station Andernach vs. station distance, with one sided interpolation (simultaneous measurements).

Worms (Fig. 3-26 for the simultaneous measurements). Although the positions of the plots of $\sigma\Delta y$ are rather peculiar, as appeared in the discussion of Fig. 3-16, it is possible to obtain an estimate of ε by a special approach. In the first place the adjacent stations (i.e. Mannheim and Nierstein-Oppenheim) were disregarded and an interpolation curve was constructed through the plots of the other stations. This curve is the upper curve of Fig 3-26. Subsequently the part of that curve between the stations Speyer and Mainz was lowered to a level, which fitted the plots of Mannheim and Nierstein-Oppenheim. This became the lower curve of Fig 3-26. This curve was assumed to indicate the behaviour of $\sigma\Delta y$ in the vicinity of the station Worms. In this way, the influences of the tributaries Neckar and Main, which introduce considerable disturbances, were considered to be eliminated.

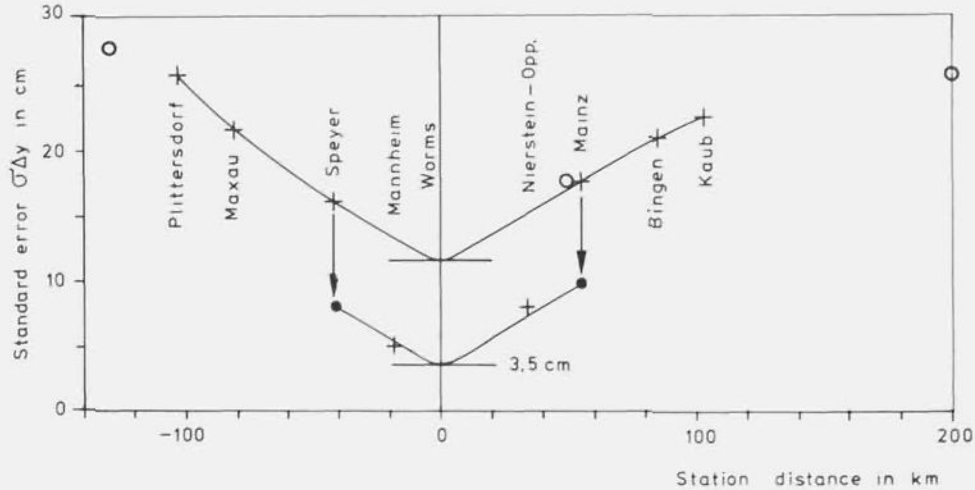


Fig. 3-26 Standard error $\sigma\Delta y$ of station Worms vs. station distance, with interpolation and central part shifted down (simultaneous measurements).
o: plots of Fig. 3-14 (Purificación)

The above procedure led to an interpolated value of about $\sigma\Delta y = 3{,}5$ cm, corresponding to a standard error of measurement of $\varepsilon = 3{,}5/\sqrt{2} = 2{,}5$ cm.

Although no specific information is available, it is possible that the conditions of the station Worms are comparable with those of the Rio Magdalena station Purificación. The standard errors $\sigma\Delta y$, related to sites further than 50 km away from the station under examination have values of the same magnitude for both cases. This is shown in Fig 3-26 where plots for the station Purificación (o) are shown together with the plots for the station Worms.

A summary of the results, obtained for the River Rhine stations considered is given in Table 3.4.

This table also gives some results for the cases where time shifted measurements are considered too. It appears that in the examples of the stations Lobith and Düsseldorf the latter lead to higher values than if only simultaneous measurements are con-

station under examination	simultaneous measurements		time shifted measurements	
	$\sigma\Delta y$ (cm)	ε (cm)	$\sigma\Delta y$ (cm)	ε (cm)
Lobith (2 sides	2,01	1,42	2,14	1,51
Düsseldorf (2 sides)	1,45	1,03	1,89	1,33
Andernach (1 side; downstream)	4,35	3,08	4,26	3,01
Worms (2 sides; shifted)	3,5	2,5	–	–

Table 3-4 Standard errors $\sigma\Delta y$ and standard errors of measurement ε, calculated for some river Rhine gauging stations.

sidered. This may seem a remarkable result, since additional information of time shifted measurements will reduce the standard errors, but not increase them. This is indeed the case, but the reduction is much stronger for remote stations than for nearby stations, which leads to a rotation of the interpolation curve and to larger values at the station that is examined. This is shown in Fig. 3-27.

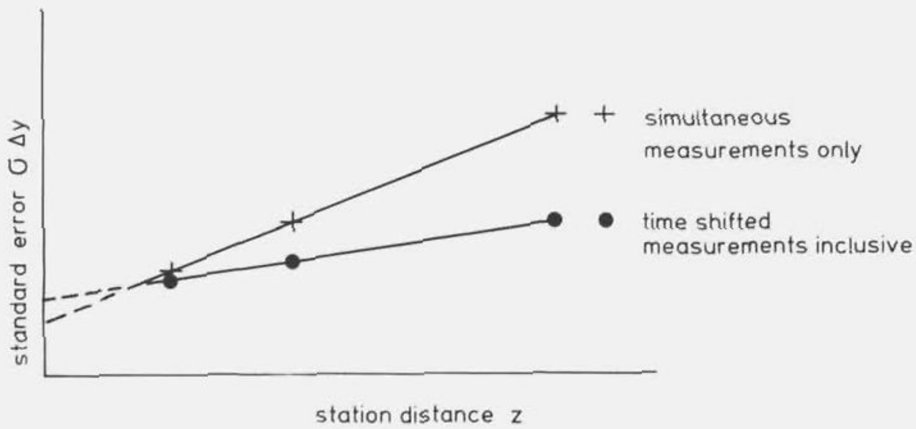


Fig. 3-27 Rotation of the interpolation curve for time shifted measurements.

In such cases the use of simultaneous measurements only is to be preferred. In Subsection 6.4.5 such an example will be discussed in more detail for a tidal river.

General discussion of Section 3.4.3

The method described in Section 3.4.3., where a series of standard errors $\sigma\Delta y$ were obtained from differences between the measurements at the station under examination and the values derived from each of the network stations used, and plotted against the station distance, and subsequently interpolated or extrapolated to the site of the station under examination, can be a useful technique to determine the standard error of measurement. It will be particularly useful where there are no important disturbances in the vicinity, so that the $\sigma\Delta y$-values increase gradually with increasing distance from the station under examination (e.g. Düsseldorf, Lobith, Zutphen). The method can be used two sided as well as one sided. If large disturbances are absent from one side only, then the one sided approach should be applied (e.g. Andernach).

If even the nearest stations are too remote, a value of the standard error of measurement can not be calculated in the way described above. A thus found value will be unacceptably large in particular when important error sources exist within the reach to the nearest network station. This would be the case for Worms, if the nearest stations Mannheim and Nierstein-Oppenheim did not exist, and probably also in the case of Purificación. Then special care should be given to the interpolation procedure in such cases. However in such cases one cannot always be sure that a satisfactory result can be obtained. The inclusion of measurements made at different times must surely improve the result. However it is possible, under certain conditions, that the standard error of measurement derived in this way is still too high. This is the case when the standard errors related to the remote stations are reduced much more than those of the nearby stations, which causes a rotation of the interpolation curve (Düsseldorf). In such cases the use of simultaneous measurements only is to be preferred.

3.5 Summary

The standard error of measurement is of essential importance for the network design procedure. It is defined as the standard deviation of the differences between the measured water levels and the 'true' water levels. By 'true' water level is to be understood the water level, which is representative of the area around the station over a certain time span. Generally this will not correspond to the water level, as it occurs in reality, where it is affected by a number of movements, which, as a rule, are of no importance for hydrology. The data finally assessed as the 'measured water levels', are the result of a sequence of actions, aimed at achieving an optimum approximation of what is looked for. However differences will remain.

These differences can be characterized by the standard error of measurement ε and the systematic error. The first includes the random variations in the measurement errors, the second implies a constant deviation. For the detection of the standard error of measurement three approaches are discussed. The systematic error is rather difficult to determine. It might be detected, and possibly be reduced by a careful execution of the various acquisition and processing activities.

Methods for the detection of the standard error of measurement comprise:
1) comparison of measurements at the same station at different times;
2) comparison of measurements at the station under examination with measurements at a nearby station;
3) comparison of measurements at the station under examination with measurements at other stations at different distances from the station under examination.

In approach 1) the influences of local phenomena are hidden from view, but if frequent measurements are available it surely can lead to an insight into the order of magnitude of ε. Approaches 2) en 3) reveal local influences, but circumstances can be envisaged where their application might encounter difficulties. In general the best procedure is to make use of all possible approaches and to try to encapsulate the problem in this way.

Examples were discussed of stations along the rivers IJssel,Magdalena and Rhine. The standard error of measurement seems to be of the order of 1 to 3 cm. For the gauging stations in the Netherlands it was finnally assessed at a fixed value of ε=2,5 cm. This value was used for the design of the future water level measurement network.

4 Mathematical interpolation methods

4.1 Interpolation methods in general

As was stated in Section 2.1 an interpolation method is to be used to derive the required data at ungauged sites from data at adjacent gauging stations. This may concern actual data, like water levels or discharges, but also derived data such as their averages and standard errors, frequencies and several types of correlation coefficients, e.g. spatial correlation coefficients between various sites, serial correlation (or autocorrelation) coefficients between data at one site and cross correlation coefficients over distance and time.

For interpolation a certain relation (model) between the values of the examined phenomenon (e.g. water levels) at various sites is assumed. This model is fitted using available measurements.

The most simple fitting procedure is the visual one. Measured values of the phenomenon are plotted on a graph and a curve drawn through, or close to, the plotted points. This can be done by hand or using a rule, a drawing curve or a spline. Before the advent of the computer this was the most commonly used method. Although the result is somewhat subjective, as a rule the approximation will be reasonable when carried out by experienced people. The main advantage is that certain specific and local phenomena, which cannot be expressed mathematically, can easily be taken into account. Therefore this method will always remain of some value. Curves can be drawn exactly through the plotted points, which can, in some cases, lead to some enforced shape, or follow approximately the location of the points. It is not possible to give strict rules for this type of interpolation.

In the following, more objective types of interpolation method will be discussed. Depending on the model adopted as the basis of the relation, the following approaches can be distinguished:
1. the mathematical approach
2. the statistical approach
3. the physical approach.

The present chapter will focus on the mathematical approach. The interpolation curve will be given the shape of the curve of a specified mathematical function. Only power functions will be considered, i.e. polynomials of a given degree n, according to:

$$\hat{y} = C_o + C_1 z \ldots + C_n z^n \qquad (4\text{-}1)$$

where:
z = the horizontal coordinate (abcis) of the site considered
y = the value of the phenomenon examined.

The symbol x is not used here for the horizontal distance in order to avoid confusion with the symbols used in the preceding chapters. As was done there, $\underline{x}$ indicates measured levels at the network stations and y measured ($\underline{y}$) and calculated ($\hat{y}$) levels at the examined sites, so x and y are used for the same type of variable.

Other types of mathematical expression, like implicit power functions or transcendental functions will not be considered here in order to limit the material to be dealt with, although such functions may lead to acceptable solutions.

The mathematical relations to be discussed in this chapter do not account for the extent of coherence between phenomena in time and place. The statistical approach comes closest to meeting this objection, and will be discussed in Chapters 5 and 6. In general the statistical methods are based on the correlation structure in space and time of the phenomenon examined.

If neither the mathematical nor the statistical approach are able to account for the behaviour of the phenomenon under all conditions, then use can be made of knowledge of the physical conditions. This approach will be discussed in Chapters 7 and 8. It should be noted however that it is never possible to explain fully the behaviour of the phenomenon on the base of physical considerations alone. These will always be approximations of the reality, and some stochastic component will always remain.

Although the physical approach might seem the most appropriate, it is recommended that the other approaches be examined first. This may save a lot of effort since the physical approach, as a rule is extremely time consuming and requires an intensive preparation in modelling and calibration. It should be noted that the examples of Chapters 7 and 8 concern only a very simple application.

Within the field of the mathematical approach a number of different approaches are possible:

1. By a polynomial, drawn exactly through the plotted points of a given set of data

i.e. by an n-degree curve through (n+1) points. A longer series of points may be subdivided into subgroups of (n+1) points each, which are interpolated independently, regardless of the adjacent subgroups (Section 4.2).

2. By a polynomial, drawn approximately along the plotted points according to a certain criterion which minimizes the departures from the points to the curve. This method accounts for possible inaccuracies in the plotted points and, as a rule assures a more smoothed shape than if the enforced interpolation of Section 4.2 were used. (Section 4.3).
3. By a series of coupled, but different polynomials which are assessed such that discontinuities at their transition points are avoided. The curves may either pass exactly through the plotted points or pass approximately along them. Criteria of assessment may take account of the shape of the entire curve (the coupled partial curves) and the departures of the points from the curve. Functions of this type are called spline functions, after the carpenter's splines (Section 4.4).

4.2 Exact interpolation using a single polynomial or separated polynomials of degree n

For interpolation using a polynomial of degree n, as is given by eq (4-1), at least (n+1) combinations (x_i, z_i) are required. If less combinations are available no unique solution can be found. If this is the case then either the degree of the polynomial must be decreased or an additional condition must be introduced, such as values of the slope of the curve at certain points. This case will not be dealt with here.

Now consider the case where only (n+1) combinations are known. For this case the derivation of the interpolation curve, which corresponds with the calculation of the coefficients C_o ... C_n of eq (4-1), can be accomplished by substitution of the (n+1) combinations (x_i, z_i) into eq (4-1) and by solving the system of linear equations:

$$\begin{bmatrix} \underline{x}_1 \\ . \\ . \\ . \\ \underline{x}_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & z_1 & \cdots & z_1^n \\ . & & & \\ . & & & \\ . & & & \\ 1 & z_{n+1} & \cdots & z_{n+1}^n \end{bmatrix} \cdot \begin{bmatrix} C_o \\ . \\ . \\ . \\ C_n \end{bmatrix} \tag{4-2}$$

or, when replacing these matrices by corresponding symbols:

$$\underline{X} = ZC. \tag{4-3}$$

The solution is

$$C = Z^{-1}.\underline{X} \tag{4-4}$$

which requires a matrix inversion of Z.

The quantities $\underline{x}_i$ (i = 1...(n+1)) are affected by errors which carry through to affect the value $\hat{y}$ when calculated according to eq (4-1).In fact this is the contribution of Var $\underline{m}$ to Var Δ y seen in eq (2-17). In the following it will be shown that a linear relation like eq (2-19) exists between y and x_i(i=1...n), so that Var $\underline{m}$ can be calculated according to eq (2-20).

Eq (4-1) will be written as follows:

$$\hat{y} = [1\ z\ \ldots\ z^n]\ C \tag{4-5}$$

Substitution of eq (4-4) yields

$$\hat{y} = [1\ z\ \ldots\ z^n]\ Z^{-1}\underline{X}. \tag{4-6}$$

Defining

$$A = [1\ z\ \ldots\ z^n]\ Z^{-1}, \tag{4-7}$$

then

$$\hat{y} = A\underline{X} \tag{4-8}$$

where A is a (n+1) row vector with elements $A_1 \ldots A_{n+1}$.

This implies a linear relation between the measured values, $\underline{x}$ ... $\underline{x}_{n+1}$, and the calculated value, $\hat{y}$, so that eq (2-19) is indeed true, but with $A_o = 0$. The coefficients of A are n^{th} degree functions of the location z.

If it can be assumed that the standard errors ε_i of eq (2-20) are equal i.e.

$$\varepsilon_i = \varepsilon\ (i = 1 \ldots (n+1)) \tag{4-9}$$

then

$$\text{Var m} = \varepsilon^2 \sum_{i=1}^{n+1} A_i^2 \tag{4-10}$$

In this case it is also useful to calculate the value of $\sum_{i=1}^{n+1} A_i^2$, which can be written as

$$\sum_{i=1}^{n+1} A_i^2 = AA^T \tag{4-11}$$

The coefficients, $A_i=f\ (z_i)$, are in fact the functions of the so called Lagrange interpolation (*Lagrange functions*). The function AA^T, being the sum of squares of the Lagrange functions, gives an impression of the propagation of the measurement errors in the calculated value $\hat{y}$. Therefore this function will be called here the *propagation function*.

Examples of error propagation

1. Linear interpolation

If the degree n equals unity the interpolation is linear, and n+1=2 measured values are required to carry out the interpolation. Let these be the combinations ($\underline{x}_1$, z_1) and ($\underline{x}_2$, z_2). Now the coefficients C_o and C_1 are calculated according to eq (4-4):

$$\begin{bmatrix} C_o \\ C_1 \end{bmatrix} = \begin{bmatrix} 1 & z_1 \\ 1 & z_2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix}$$

$$\begin{bmatrix} -\dfrac{z_2}{z_1-z_2} & \dfrac{z_1}{z_1-z_2} \\ \dfrac{1}{z_1-z_2} & -\dfrac{1}{z_1-z_2} \end{bmatrix} \cdot \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix}$$

According to eq (4-6):

$$\hat{y} = [1 \quad z] \begin{bmatrix} -\dfrac{z_2}{z_1-z_2} & \dfrac{z_1}{z_1-z_2} \\ \dfrac{1}{z_1-z_2} & -\dfrac{1}{z_1-z_2} \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{-z_2+z}{z_1-z_2} & \dfrac{z_1-z}{z_1-z_1} \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix}$$

$$= \frac{-z_2+z}{z_1-z_2}\underline{x}_1 + \frac{z_1-z}{z_1-z_2}\underline{x}_2$$

If $z_1=0$ and $z_2=l$ this becomes

$$\hat{y} = (1-z/l)\underline{x}_1 + (z/l)\underline{x}_2$$

The Lagrange functions are in this case:

$$A_1 = 1-z/l$$
$$A_2 = z/l$$

whereas the propagation function reads:

$$AA^T = (1-z/l)^2 + (z/l)^2$$
$$= 1-2z/l + 2(z/l)^2$$

In Fig. 4-1 the functions A_1, A_2 and AA^T are represented graphically. It appears that the propagation function is less than unity everywhere within the reach considered and that it takes a minimum value of $AA^T = 0,5$ for $z/l = 0,5$. This means that, with respect to the propagation of measurement errors, the linear interpolation scheme appears attractive.

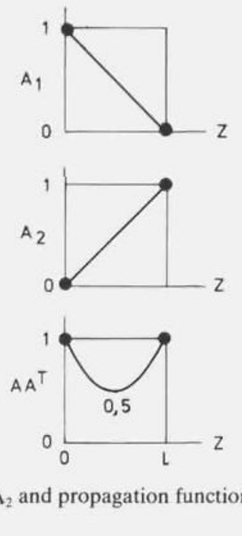


Fig. 4-1 Lagrange functions A_1 and A_2 and propagation function AA^T for linear interpolation.

2. Square interpolation

In this case the degree of the interpolation curve is n=2, so n+1=3 measured values have to be available. Consider, for reasons of simplicity the ordinates z_1=-l, z_2=0, z_3=l. Using eq (4-4) yields

$$\begin{bmatrix} C_0 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 & z_1 & z_1^2 \\ 1 & z_2 & z_2^2 \\ 1 & z_3 & z_3^2 \end{bmatrix}^{-1} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \underline{x}_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -l & l^2 \\ 1 & 0 & 0 \\ 1 & l & l^2 \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \underline{x}_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -0,5/l & 0 & 0,5/l \\ 0,5/l^2 & -1/l^2 & 0,5/l^2 \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \underline{x}_3 \end{bmatrix}$$

Thus:

$$\begin{aligned} C_o &= & \underline{x}_2 & \\ C_1 &= -0,5\underline{x}_1/l & & +0,5\underline{x}_3/l \\ C_2 &= \ \ 0,5\underline{x}_1/l^2 & -\underline{x}_2/l^2 & +0,5\underline{x}_3/l^2 \end{aligned}$$

According to eq (4-1) the interpolation curve is given by:

$$\hat{y} = \underline{x}_2 + (-0,5\underline{x}_1 + 0,5\underline{x}_3)\, z/l + (0,5\underline{x}_1 - \underline{x}_2 + 0,5\underline{x}_3)\, (z/l)^2.$$

The Lagrange functions follow from eq (4-7) as

$$\begin{bmatrix} A_1 \ A_2 \ A_3 \end{bmatrix} = \begin{bmatrix} 1 \ z \ z^2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 14 \\ -0,5/l & 0 & 0,5/l \\ 0,5/l^2 & -1/l^2 & 0,5/l^2 \end{bmatrix}$$

Thus:

$$\begin{aligned} A_1 &= \quad -0,5z/l + 0,5(z/l)^2 \\ A_2 &= 1 \qquad\qquad - \ (z/l)^2 \\ A_3 &= \quad\ \ 0,5z/l + 0,5(z/l)^2. \end{aligned}$$

The propagation function, AA^T, follows from the sum of the squares of A_1, A_2 and A_3 and reads

$$AA^T = 1 - 1{,}5(z/l)^2 + 1{,}5(z/l)^4$$

In Fig. 4-2 the Lagrange and propagation functions are shown. The propagation function has values, everywhere smaller than 1 and has a minimum value of 0,625 for $z/l = \pm \frac{1}{2}\sqrt{2}$. Compared with the corresponding linear case this is a somewhat higher value, which means that measurement errors are now reduced by a smaller amount.

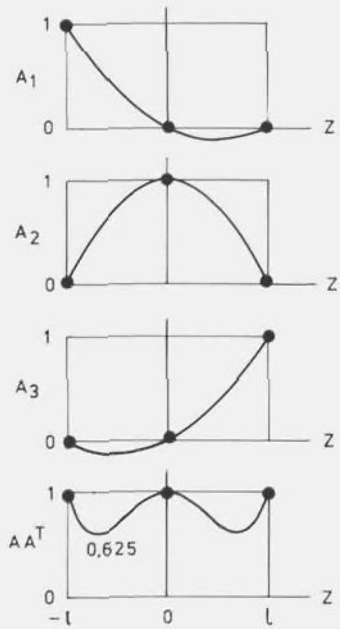


Fig. 4-2 Lagrange functions A_1, A_2 and A_3 and propagation function AA^T for square interpolation.

The example considered here was for two stretches of equal length. If the lengths differ, then the propagation of the errors becomes unfavourable. In Fig. 4-3 some examples of AA^T functions are shown. In particular if one stretch is very short compared with the other one, error propagation can develop to very high values.

3 Cubic interpolation

in this case the interpolation curve is of degree n=3, and n+1=4 measured values are required. Now consider the symmetrical case where z_1=-1,5 l; z_2=-0,5 l; z_3=0,5 l; z_4=1,5 l

As for the other examples the Lagrange functions are calculated as follows

$$A_1 = -\frac{1}{16} + \frac{1}{24}\; z/l + \frac{1}{4}(z/l)^2 - \frac{1}{6}\;(z/l)^3$$

$$A_2 = \frac{9}{16} - \frac{9}{8}\; z/l - \frac{1}{4}(z/l)^2 + \frac{1}{2}\;(z/l)^3$$

$$A_3 = \frac{9}{16} + \frac{9}{8}\; z/l - \frac{1}{4}(z/l)^2 - \frac{1}{2}\;(z/l)^3$$

$$A_4 = -\frac{1}{16} - \frac{1}{24}\; z/l + \frac{1}{4}(z/l)^2 + \frac{1}{6}\;(z/l)^3$$

and

$$AA^T = \frac{41}{64} + \frac{275}{144}(z/l)^2 - \frac{73}{36}(z/l)^4 + \frac{5}{9}\;(z/l)^6$$

The corresponding curves are given in Fig. 4-4.

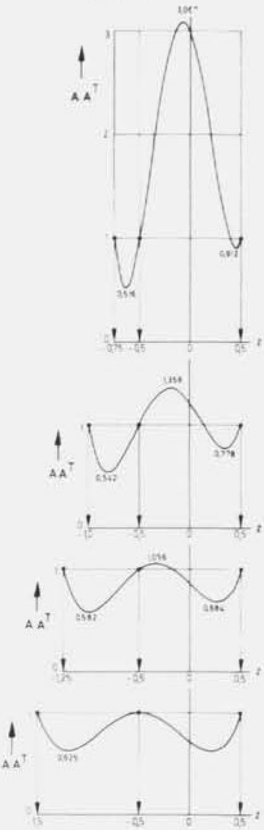


Fig. 4-3 Propagation functions, AA^T, for some square interpolations

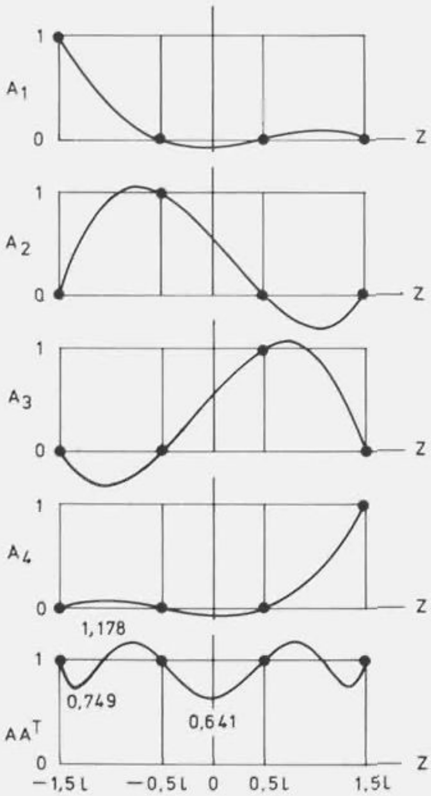


Fig. 4-4 Lagrange functions A_1, A_2, A_3 and A_4 and the propagation functions AA^T for cubic interpolation

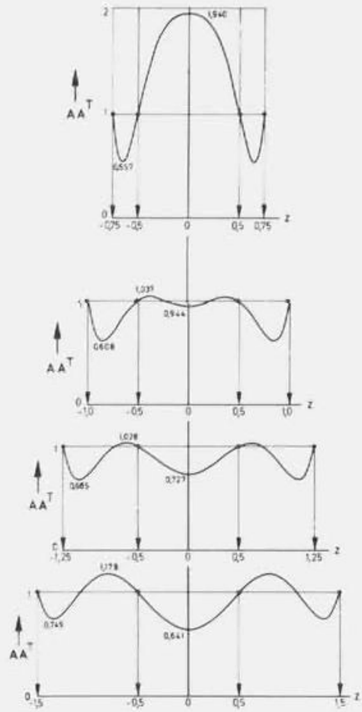


Fig. 4-5 Propagation functions, AA^T, for some cubic interpolations

Now values of the propagation function, AA^T, larger than unity occur, with a maximum of 1,178. A minimum value of 0,641 is attained in the centre of the reaches. Compared with the earlier examples, the minimum as well as the maximum values are larger. When using more measured values, propagation of the measurement errors leads to higher values of the errors due to the measurement errors.

When the two symmetrical side stretches become progressively smaller relative to the central stretch, error propagation decreases in the side stretches but increases in the middle stretch. This is shown in Fig. 4-5 which shows AA^T curves for a number of cases. If the side stretches are very small, then the values in the centre can become unacceptably high.

Example of interpolation

Consider the following five measured values (Fig. 4-6):
$(z_1, x_2) = (1,2)$
$(z_2, x_2) = (2,1)$
$(z_3, x_3) = (3,4)$
$(z_4, x_4) = (4,3)$
$(z_5, x_5) = (5,1)$

An interpolation has to be made between the corresponding points. This can be done by means of one continuous curve, which, as it concerns 5 points, should be of the 4th degree. It is also possible to split up the set in subsets along which separate interpolations can be carried out. These are, without overlap, two 2nd-degree interpolations or four linear interpolations*). The results are given in Fig. 4-7 for linear interpolation, in Fig. 4-8 for 2nd-degree (square) interpolation, and in Fig. 4-9 for 4th degree interpolation.

The objection to using separate interpolations is that there are discontinuities in the gradient in the transition points as can be seen for the linear and square interpolations in this case. Whether or not the continuous 4th-degree interpolation is to be preferred depends also on the accuracy of the resulting curve. If the measured values are affected by large measurement errors, their position may be scattered and the interpolation curve may take on an unrealistic and distorted shape. In such cases, simpler lower degree interpolations be preferred. For that reason also, the propagation functions AA^T should be considered. For the cases concerned the graphs of these functions are drawn in Figs. 4-10A, B and C. These curves show that the result is worst for the 4th-degree interpolation (Fig. 4-10C). If the underlying standard errors of measurement, ε, are large, this can be a reason to reject the 4th-degree interpolation, and to use one of a lower degree, perhaps even the linear interpolation (Fig. 4-10A). The square interpolation gives a reasonable result in the present example (Fig. 4-10B), but as was shown in example 2, this type can also produce bad results, in particular when stretches of different lengths are considered.

The conclusion of this section is that, although a polynomial interpolation of the highest possible degree gives a continuous curve, it is surely not always the most appropriate one from the point of view of the propagation of measurement errors. The extent of propagation of the measurement errors should be analysed in any case and a decision about the type of interpolation to be applied should be based on the result of such an analysis.

*) Not to be confused with the spline functions of Section 4-4.

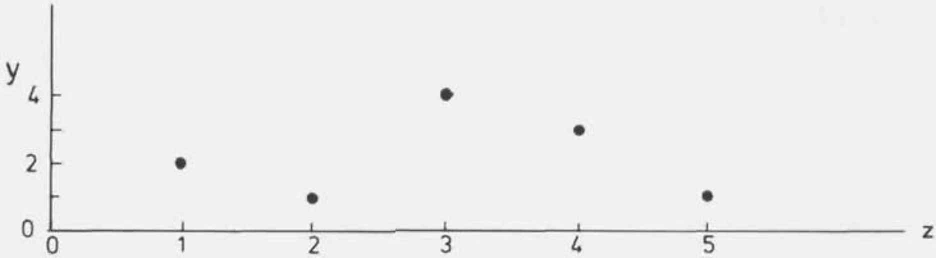


Fig. 4-6 Five measured values between which to interpolate

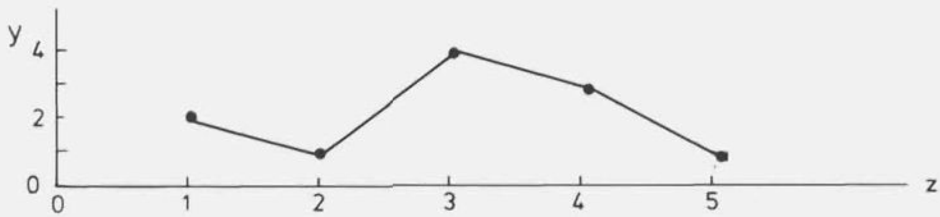


Fig. 4-7 Linear interpolation between the 5 points of Fig. 4-6 (4 separate interpolations).

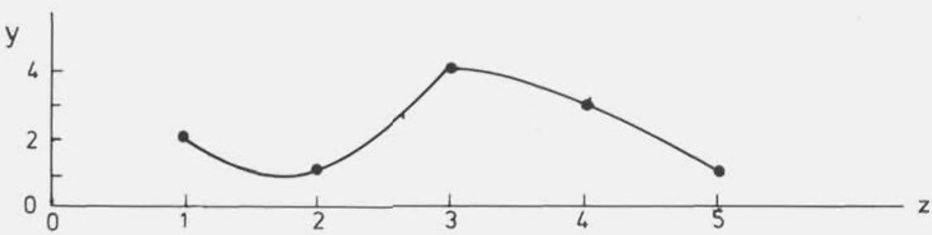


Fig. 4-8 2nd-degree interpolation between the 5 points of Fig. 4-6 (2 separate interpolations).

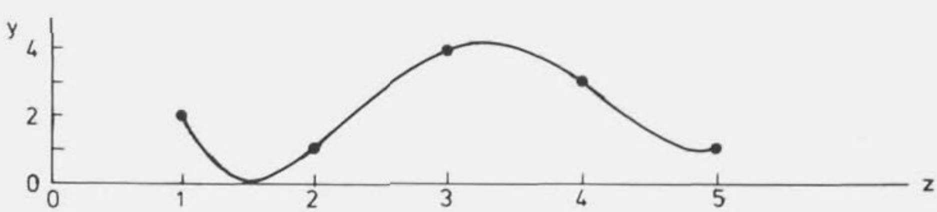


Fig. 4-9 4th-degree interpolation between the 5 points of Fig. 4-6.

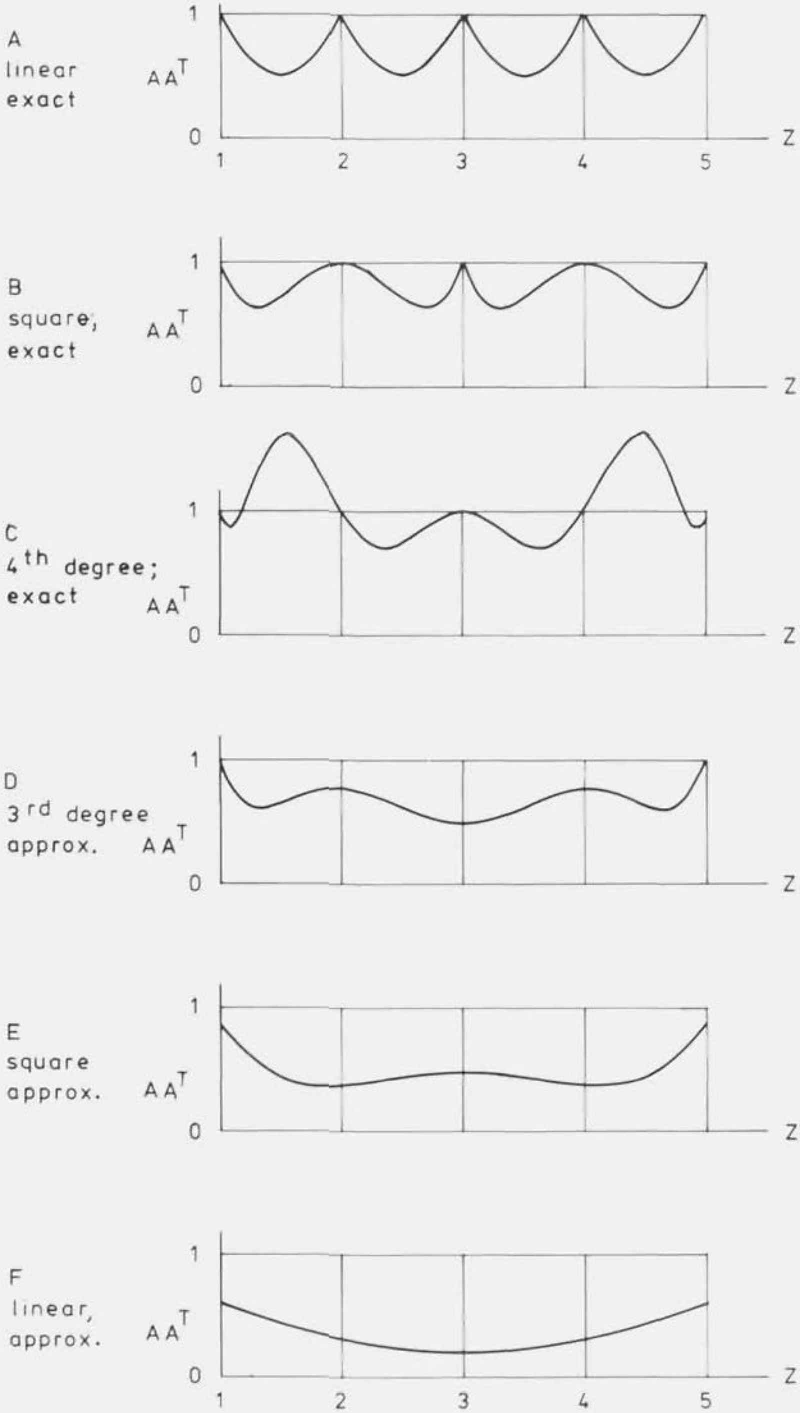


Fig. 4-10 Propagation functions, AA^T, of the measurement errors of six types of interpolation.

4.3 Approximated interpolation by one polynomial of degree n

In this approach the curve of the polynomial should follow the positions of m given points ($m > n+1$) as close as possible, according to a certain criterion which minimizes the departures of the resulting curve from the plotted points. Here the 'least sum of squares criterion' will be applied, which minimizes the sum of the squares of the above mentioned departures.

The polynomial is expressed again by eq (4-1):

$$\hat{y} = C_o + C_1 z \ldots + C_n z^n,$$

and the m given values, $(z_i, \underline{x}_i)$ $(i = 1 \ldots m)$, are available. For the sites z_i, eq (4-1) can be used to calculate the x_i values as

$$\hat{x}_i = C_o + C_1 z_i \ldots + C_n z_i^n \qquad (4\text{-}12)$$

Now consider the departure

$$\Delta x_i = \underline{x}_i - \hat{x}_i \qquad (4\text{-}13)$$

Substitution of eq (4-12) into eq (4-13) gives

$$\Delta x_i = \underline{x}_i - C_o - C_i z_i \ldots - C_n z_i^n \qquad (4\text{-}14)$$

with a square of

$$(\Delta x_i)^2 = (\underline{x}_i - C_o - C_1 z_i \ldots - C_n z_i^n)^2 \qquad (4\text{-}15)$$

The sum of these values for all m points is

$$\sum_{i=1}^{m} (\Delta x_i)^2 = \sum_{i=1}^{m} (\underline{x}_i - C_o - C_i z_i \ldots - C_n z_i^n)^2 \qquad (4\text{-}16)$$

Values of $C_o \ldots C_n$ are chosen so as to minimize this sum. These can be derived by differentiating eq (4-16) with respect to each of these coefficients. Thus

$$\frac{\partial \sum_{i=1}^{m} (\Delta x_i)^2}{\partial Cj} = 0 \qquad \text{for } j = 0 \ldots n \qquad (4\text{-}17)$$

This leads to n+1 linear equations with n unknowns, $C_o \ldots C_n$. These equations can be expressed as follows*)

$$\begin{bmatrix} m & \Sigma z_i & \ldots & \Sigma z_i^n \\ \Sigma z_i & \Sigma z_i^2 & & \Sigma z_i^{n+1} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \Sigma z_i^n & \Sigma z_i^{n+1} & \ldots & \Sigma z_i^{2n} \end{bmatrix} \begin{bmatrix} C_o \\ C_1 \\ \cdot \\ \cdot \\ \cdot \\ C_n \end{bmatrix} = \begin{bmatrix} \Sigma \underline{x}_i \\ \Sigma z_i \underline{x}_i \\ \cdot \\ \cdot \\ \cdot \\ \Sigma z_i^n \underline{x}_i \end{bmatrix} \tag{4-18}$$

or in matrix notation

$$\Sigma.C. = [ZX]. \tag{4-19}$$

The solution is:

$$C = \Sigma^{-1}. [ZX]. \tag{4-20}$$

Now the interpolation curve can be derived by substituting eq (4-20) into eq (4-1), so:

$$\hat{y} = [1\ z \ldots z^n].\ C$$

$$= [1\ z \ldots z^n].\ \Sigma^{-1}.\ [ZX] \tag{4-21}$$

Now elaborate the matrix [ZX] as follows:

$$[ZX] = \begin{bmatrix} \Sigma \underline{x}_i \\ \Sigma z_i \underline{x}_1 \\ \cdot \\ \cdot \\ \cdot \\ \Sigma z_i^n \underline{x}_i \end{bmatrix} = \begin{bmatrix} 1 & \ldots & 1 \\ z_1 & & z_m \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ z_1^n & & z_m^n \end{bmatrix} . \begin{bmatrix} \underline{x}_1 \\ \cdot \\ \cdot \\ \cdot \\ \underline{x}_m \end{bmatrix} \tag{4-22}$$

Express this in the following symbols:

$$[ZX] = Z^T.\underline{X}. \tag{4-23}$$

*) Σ stands for $\sum_{i=1}^{m}$

Substitution of eq (4-23) into eq (4-21) yields:

$$\hat{y} = [1\ z\ \dots\ z^n].\ \Sigma^{-1}.Z^T.\underline{X} \tag{4-24}$$

Introducing the $(1\times m)$ matrix

$$A = [A_1\ \dots\ A_m] = [1\ z\ \dots\ z^n].\ \Sigma^{-1}.Z^T \tag{4-25}$$

and substituting this into eq (4-24) gives

$$\hat{y} = A\underline{X} \tag{4-26}$$

which corresponds to

$$\hat{y} = A_1\underline{x}_1 + \dots + A_m\underline{x}_m \tag{4-27}$$

Now, as in Section 4.2, the values $\hat{y}$ of the interpolation curve are expressed linearly in terms of the measured values $\underline{x}_i\ \dots\ \underline{x}_m$. The factors $A_1\ \dots\ A_m$ are the Lagrange functions of the approximate interpolation.

Examples

Consider again the five plotted points in fig. 4-6. Interpolation curves of 1st-, 2nd- and 3rd-degree can be derived according to the above description. Only the 3rd-degree case will be described here in extenso, since the 1st- and 2nd-degree cases can be treated in a similar way.

Starting with the measured values

$$(z_1, \underline{x}_1) = (1,2)$$
$$(z_2, \underline{x}_2) = (2,1)$$
$$(z_3, \underline{x}_3) = (3,4)$$
$$(z_4, \underline{x}_4) = (4,3)$$
$$(z_5, \underline{x}_5) = (5,1)$$

the following matrices are calculated:

$$\Sigma = \begin{bmatrix} 5 & 15 & 55 & 225 \\ 15 & 55 & 225 & 979 \\ 55 & 225 & 979 & 4425 \\ 225 & 979 & 4425 & 20515 \end{bmatrix}$$

$$Z^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \\ 1 & 8 & 27 & 64 & 125 \end{bmatrix}$$

$$\underline{X} = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 3 \\ 1 \end{bmatrix} \qquad Z^T X = \begin{bmatrix} 11 \\ 33 \\ 115 \\ 435 \end{bmatrix}$$

These yield, according to eq (4-19), the coefficients of eq (4-1):

$$C = \begin{matrix} 6{,}2000 \\ -7{,}2619 \\ 3{,}3214 \\ -0{,}4167 \end{matrix}$$

The matrix $\Sigma^{-1}Z^T$ of eq (4-25) is derived:

$$\Sigma^{-1}Z^T = \begin{matrix} 3{,}2000 & -2{,}8000 & -0{,}8000 & 2{,}2000 & -0{,}8000 \\ -3{,}0238 & 4{,}2619 & 0{,}8571 & -3{,}4048 & 1{,}3095 \\ 0{,}8929 & -1{,}5714 & -0{,}1429 & 1{,}4286 & -0{,}6071 \\ -0{,}0833 & 0{,}1667 & 0{,}0000 & -0{,}1667 & 0{,}0833 \end{matrix}$$

and, according to eq (4-25), the five Lagrange functions are:

$$A_1 = 3{,}2000 - 3{,}0238\ z + 0{,}8929\ z^2 - 0{,}0833\ z^3$$
$$A_2 = -2{,}8000 + 4{,}2619\ z - 1{,}5714\ z^2 + 0{,}1667\ z^3$$
$$A_3 = -0{,}8000 + 0{,}8571\ z - 0{,}1429\ z^2$$
$$A_4 = 2{,}2000 - 3{,}4048\ z + 1{,}4286\ z^2 - 0{,}1667\ z^3$$
$$A_5 = -0{,}8000 + 1{,}3095\ z - 0{,}6071\ z^2 + 0{,}0833\ z^3$$

The curves of these functions are shown in Fig. 4-11, together with those of the exact interpolation obtained using a 4th-degree polynomial. It appears that, whereas the Lagrange curves for the latter type of interpolation show values 1 for the points to which they are related and values 0 for the other points, the Lagrange functions of the approximate interpolation deviate from these values. This means that, for this type of interpolation the calculated values at the measured points are weighted over all measured values and are not only related to the local measurements. This implies a kind of averaging of all measurements which yields a reduction of the error of estimate. This is demonstrated by the shape of the propagation function, AA^T, shown in Fig. 4-10D. Compared with Fig. 4-10C for the 4th-degree exact interpolation its values are considerably lower.

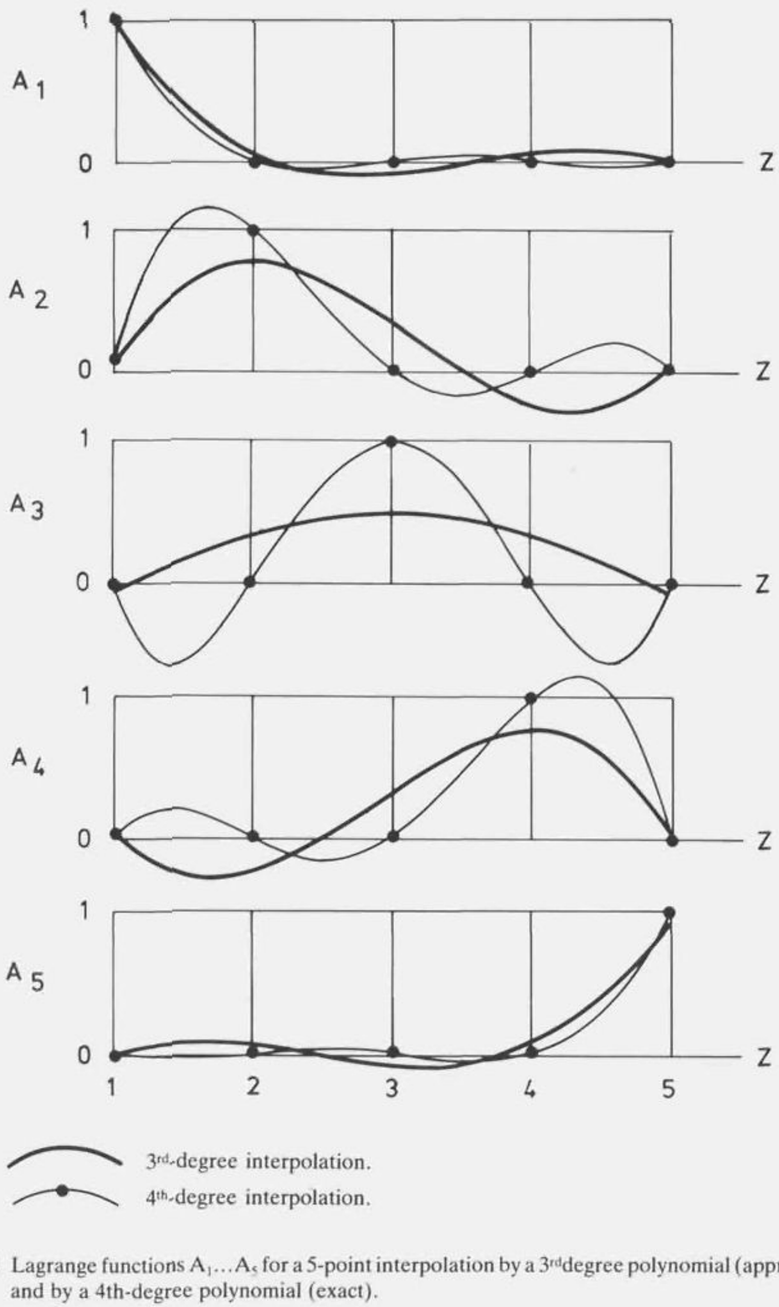


Fig. 4-11 Lagrange functions $A_1 \ldots A_5$ for a 5-point interpolation by a 3rd degree polynomial (approximate) and by a 4th-degree polynomial (exact).

Figures 4-10E and 4-10F show the propagation functions of a 2nd-degree approximate interpolation and of a linear approximate interpolation, respectively. These figures show that the lower the degree of interpolation, the more the propagated values of the measurement errors are reduced. In this sense the lowest degree gives the best result. However, the departures of the measured values from the interpolation curves will increase as the degree of the curves is reduced. These two aspects should be counterbalanced in the choice of the interpolation curve. If the measurement errors are large it is better to have low values of the propagation function, and a low degree interpolation is then preferable. As a rule the positions of the measured values will be considerably scattered because of the randomness of the measurement errors. This pattern will be smoothed by a low degree interpolation curve.

If the measurement errors are small then the positions of the plotted points will be more precisely. In that case a higher degree interpolation curve, following the positions more closely, can be justified.

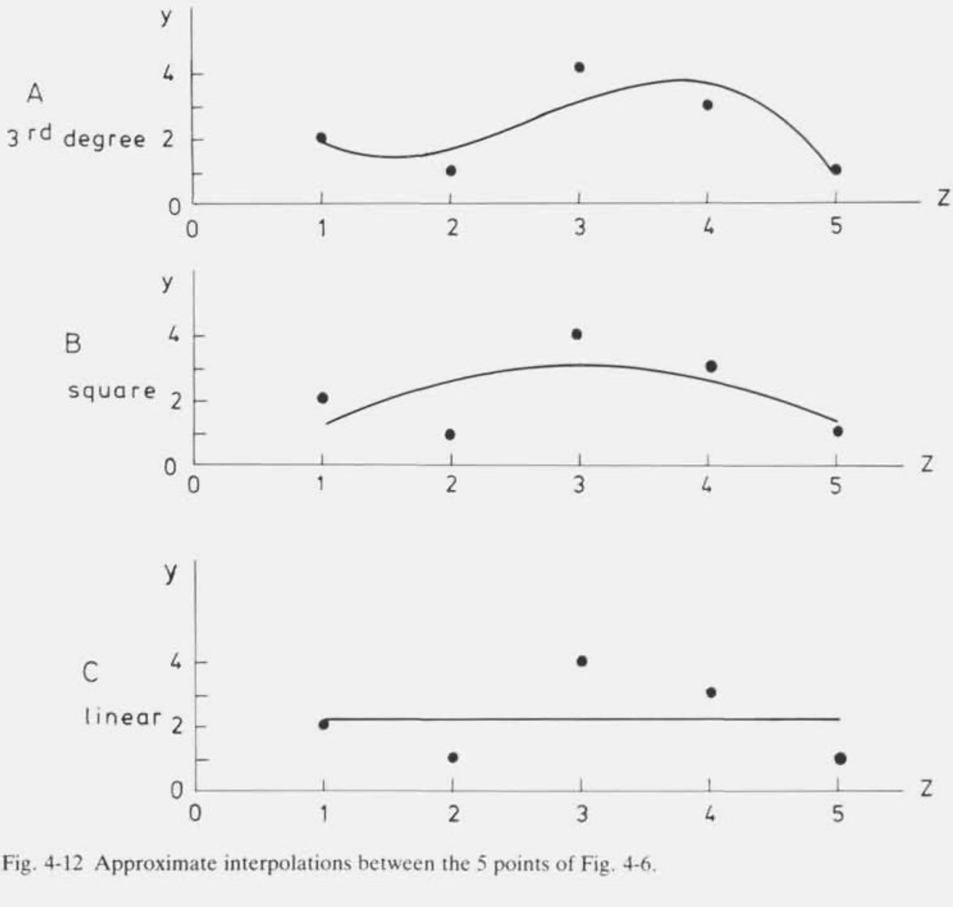


Fig. 4-12 Approximate interpolations between the 5 points of Fig. 4-6.

Fig. 4-12 shows the interpolation results for the plots of Fig. 4-6 using 3^{rd}, 2^{nd} and 1^{st}-degree curves. In this case the linear interpolation is horizontal, and thus yields a constant estimate $\hat{y}$.

The effect of different distances between the stations is shown in Fig. 4-13. This figure deals with the same cases as Fig. 4-5, but for approximate 2^{nd}-degree interpolations. As a rule the latter yield lower values for the propagated measurement errors. Only in the middle of the central reach the same values are found as for 3^{rd}-degree exact interpolation. So the effect of averaging, which generally reduces the propagated measurement errors, has, at least in this case, no influence in the middle of the central reach.

The conclusion is that approximate interpolation by polynomials of a lower degree than the number of measured points minus one for most sites generally leads to lower values of the propagated measurement errors.

An objection to interpolation using polynomials, either exact or approximate, is that there is only a limited choice concerning the type of curve, since this is related to the degree of the curve which must always be a natural number (1, 2, 3 etc). Intermediate curves are not available in this series. In the case of Figures 4-5 and 4-13, for instance, a 3^{rd}-degree or a 2^{nd}-degree curve can be selected but there exists no curve of intermediate degree. Consequently this type of interpolation is not very flexible.

4.4 Interpolation with spline functions

These functions are represented by curves consisting of a series of mathematically defined partial curves which join each other at transition points. The curves are fitted through or along plotted points, corresponding to measurements. In the most general form, the points where the partial curves join do not necessarily coincide with the positions of the measurements on the abcissa. In that case the transition points, as a rule, are arranged according to a fixed grid. When the curves represent power functions they are called 'piecewice polynomials' (Kubik, 1971).

In Subsections 4.4.1. and 4.4.2. the transition points coincide with the positions of the measurements. Subsection 4.4.1. concerns exact interpolation, following precisely the plotted points, and Subsection 4.4.2. deals with approximate interpolation. The measure of approximation is expressed by a weight factor P, which can be given all values between o and ∞.

In Subsection 4.4.3. some extensions of the method are discussed, namely

- the use of dummy points, i.e. transition points of the partial curves where no measurements are available (Subsection 4.4.3.1.)

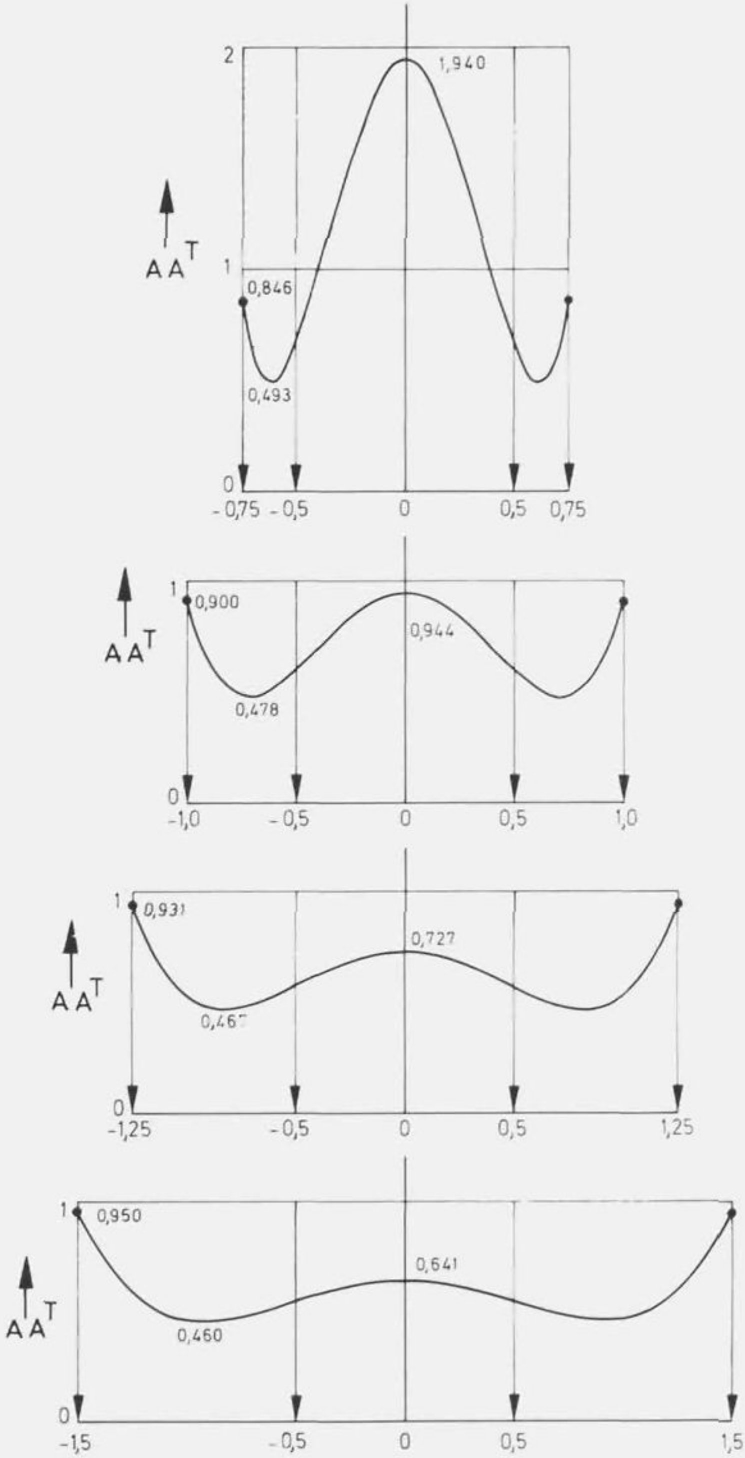


Fig. 4-13 Propagation functions, AA^T, for some approximate square interpolations.

- the use of measured values at positions which do not coincide with transition points of the partial curves (Subsection 4.4.3.2.)
- the fixation of the slope of the curve at a given value, at a transition point (Subsection 4.3.3.3.).

The above possibilities give the interpolation method using spline functions a great flexibility, which is an advantage with respect to interpolation using a single polynomial.

4.4.1 Exact interpolation

Now consider a series of m gauging stations along a river, measuring values of water levels for instance, which are denoted by $x_1 \ldots x_m$ (Fig. 4-14). The continuous profile is indicated by y.

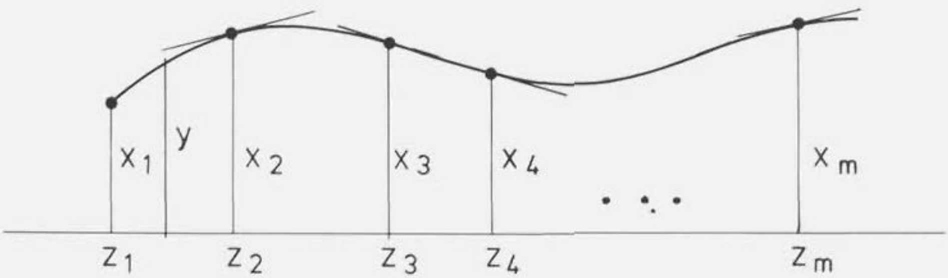


Fig. 4-14 Exact interpolation with spline functions.

Suppose that it is intended to construct the interpolation curves exactly through the measurement points.

For the interpolation curves along the intervals (z_i, z_{i+1}), 2^{nd}-degree parabolas are chosen, with the condition that two connected parabolas have the same slope at both sides of their transition point, thus assuring continuity of the gradient of the full interpolation curve. Once one of these parabolas is fixed (e.g. the one between z_i and z_{i+1}), its slope in z_{i+1} fixes the slope in z_{i+1} of the next parabola (i.c. between z_{i+1} and z_{i+2}). In this way the whole set of parabolas is fixed.

There is still one degree of freedom. One may choose for instance the initial slope of the first parabola in z_1. But one may also put a requirement concerning the general shape of the curve. It can be desired that the curve should show a minimum of oscillations, or, in other words, a shape as stretched as possible. Mathematically this has been realized here in such a way, that the local slopes have a minimum departure from the average slope between the gauging stations at both ends of the river reach considered. This can be attained by minimizing the variance of the slopes between the outer stations.

For the variance of the slopes along the i^{th} reach one can write:

$$\mathrm{Var}_i\left(\frac{dy}{dz}\right) = \frac{\displaystyle\int_{z_i}^{z_{i+1}} \left(\frac{dy}{dz}\right)^2 dz - \frac{(x_{i+1} - x_i)^2}{z_{i+1} - z_i}}{z_{i+1} - z_i} \tag{4-28}$$

or

$$\mathrm{Var}_i\left(\frac{dy}{dz}\right) = \frac{G_i(z)}{\Delta z_i} \tag{4-29}$$

where

$$G_i(z) = \int_{z_i}^{z_{i+1}} \left(\frac{dy}{dz}\right)^2 dz - \frac{(x_{i+1} - x_i)^2}{z_{i+1} - z_i} \tag{4-30}$$

and

$$\Delta z_i = z_{i+1} - z_i \tag{4-31}$$

The variance along all m-1 reaches together amounts to

$$\mathrm{Var}\left(\frac{dy}{dz}\right) = \frac{\sum_{i=1}^{m-1} G_i(z)}{z_n - z_1} \tag{4-32}$$

Now, the initial slope $(\frac{dy}{dz})_1 = x_1'$ is to be chosen such that the variance Var $(\frac{dy}{dz})$ is a minimum. This can be found by the solution of

$$\frac{d\,\mathrm{Var}\left(\frac{dy}{dz}\right)}{dx_i'} = \frac{1}{z_n - z_i} \sum_{i=1}^{m} \frac{d\,G_i(z)}{dx_i'} = 0 \tag{4-33}$$

or of

$$\sum_{i=1}^{m-1} \frac{dG_i(z)}{dx_i'} = 0 \tag{4-34}$$

Now consider the partial parabolas. An expression for such a parabola along a stretch z_i (Fig. 4-15) is

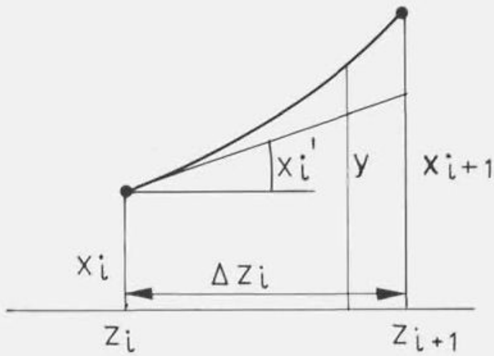


Fig. 4-15 Partial parabola along reach Δz_i

$$y_i = x_i + x_i' (z-z_i) + \left\{ \frac{x_{i+1}-x_i}{\Delta z_i^2} - \frac{x_i'}{\Delta z_i} \right\} (z-z_i)^2 \tag{4-35}$$

where y_i denotes the continuous value within the i^{th} stretch,

Thus for the slopes y_i' the following holds:

$$y_i' = \frac{dy}{dz} = x_i' + 2 \left\{ \frac{x_{i+1} - x_i}{\Delta z_i^2} - \frac{x_i'}{\Delta z_i} \right\} (z-z_i) \tag{4-36}$$

Now expressions are required for the following quantitites:

$$\left(\frac{dy}{dz}\right)^2, \; G_i(z), \quad \frac{d\,G_i(z)}{dx_i'} \text{ and } \frac{d\,G_i(z)}{dx_1'}$$

Eq (4-30) leads, after some algebra, to

$$G_i(z) = \frac{1}{3} (x_i')^2 \Delta z_i + \frac{1}{3} \frac{(x_{i+1}-x_i)^2}{\Delta z_i} - \frac{2}{3}(x_{i+1}-x_i)x_i' \tag{4-37}$$

and

$$\frac{dG_i(z)}{dx_i'} = \frac{2}{3}\{ x_i'\Delta z_i - (x_{i+1}-x_i)\} \qquad (4\text{-}38)$$

The derivative, after the initial slope x_1' of the first reach, follows from a consideration of eq (4-36) Because of this, the slope at z_{i+1} reads:

$$x_{i+1}' = - x_i' + 2 \quad \frac{x_{i+1} - x_i}{\Delta z_i} \qquad (4\text{-}39)$$

which implies

$$\frac{d\ x_{i+1}'}{dx_i'} = -1. \qquad (4\text{-}40)$$

Thus a change of slope at z_i will cause an equal change of opposite sign to the slope at z_{i+1}. When comparing the i^{th} slope with the first slope, it follows that

$$\frac{dx_i'}{dx_1'} = (-1)^{i-1} \qquad (4\text{-}41)$$

This implies that

$$\frac{dG_i(z)}{dx_1'} = (-1)^{i-1}\ \frac{dG_i(z)}{dx_i'}\ . \qquad (4\text{-}42)$$

Now according to eq (4-34) the values of $\frac{dG_i(z)}{dx_i'}$ for all m-1 reaches should be added and be set equal to zero, yielding a linear equation in x_i and x_i'. In this expression, being 'zero', the factor 2/3 can be deleted. The result is

$$x_1 - 2x_2 + 2x_3 \ldots -2(-1)^{m-1}x_{m-1} -(-1)^m x_m + x_1'\Delta z_1 - x_2'\Delta z_2 \ldots -(-1)^{m-1}x_{m-1}'\ \Delta z_{m-1} = 0 \qquad (4\text{-}43)$$

This can be developed to

$$\sum_{i=2}^{m} - (-1)^i \{x_i -(x_{i-1} + x_{i-1}'\Delta z_{i-1})\} = 0 \qquad (4\text{-}44)$$

The expression between accolades in eq (4-44) is the difference between the water level x_i at z_i and the linear extrapolation of the tangent at the previous point z_{i-1} (see Fig. 4-16). The sum of these differences for the reaches z_{i-1} ($i = 2 \ldots m$), if each difference is alternately multiplied by +1 and -1, should be zero.

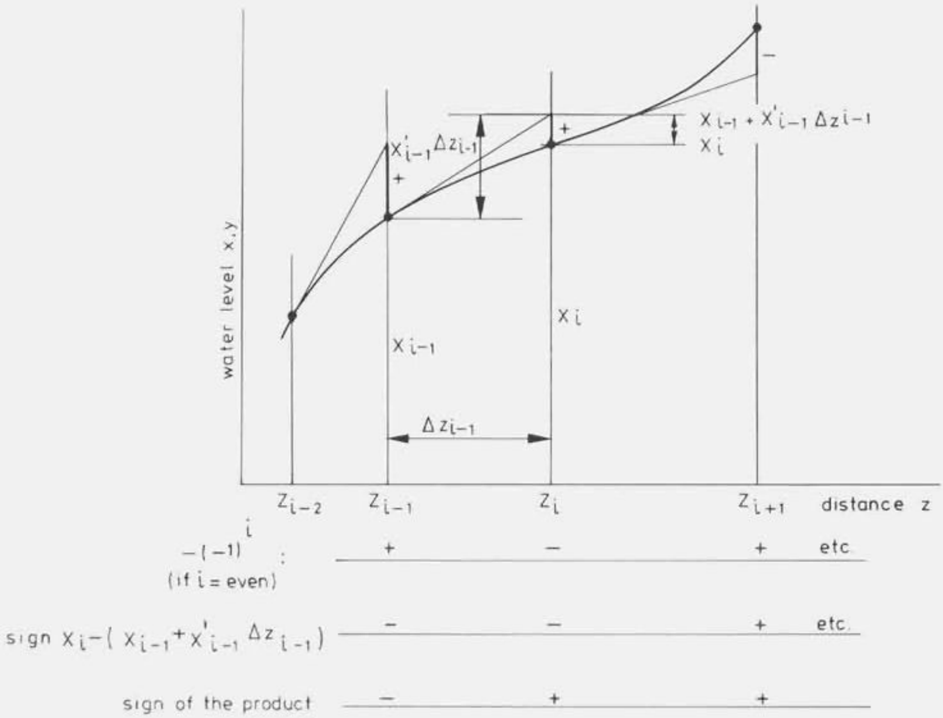


Fig. 4-16 Graphical representation of the meaning of dy minimization of Var $\left(\frac{dy}{dz}\right)$

If the intervals Δz_i become smaller and smaller then the interpolation approaches the real curve. In that case the relation of eq (4-34) is fulfilled automatically, since then:

$$\lim_{\Delta z_i \to 0} \frac{x_{i+1} - x_1}{\Delta z_i} = \frac{d\,x_i}{d\,z_i} = x_i' \tag{4-45}$$

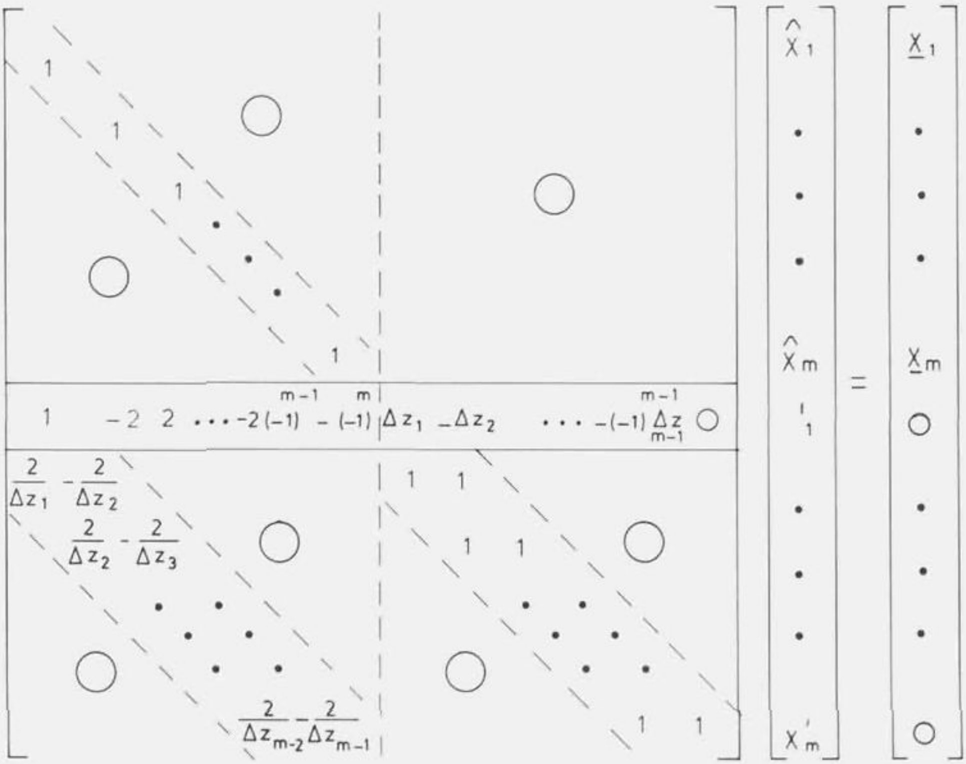
This means that in all cases eq (4-38) is zero, i.e. all terms of eq (4-34) are zero.

In eq (4-43) $x_1' \ldots x_{m-1}'$ are the unknowns. Eq (4-39) is used to obtain the relation between two adjacent slopes x_i' and x_{i+1}'.

Now consider the values x_i. A distinction is made between measured values $\underline{x}_i$ and calculated values $\hat{x}_i$. In the case of exact interpolation these are the same and therefore

$$\hat{x}_i = \underline{x}_i \tag{4-46}$$

Considering all $\hat{x}_i$ and x_i' values as unknowns, the following matrix equation can be set up by combining eqs (4-46), (4-43) and (4-39).



(4-47)

or in symbols

$$M\hat{X} = \underline{X} \tag{4-48}$$

The solution is

$$\hat{X} = M^{-1}\underline{X} \tag{4-49}$$

The reason for using this relatively complicated approach is that it can easily be extended when discussing the approximate interpolation (Subsection 4.4.2.).

Example

In case of four reaches of unit length the matrix M is as follows:

$$M = \left[\begin{array}{c|c} \begin{matrix}1 & & & & \\ & 1 & & \bigcirc & \\ & & 1 & & \\ & \bigcirc & & 1 & \\ & & & & 1\end{matrix} & \bigcirc \\ \hline \begin{matrix} 1 & -2 & 2 & -2 & 1 \\ 2 & -2 & & & \\ & 2 & -2 & \bigcirc & \\ & & 2 & -2 & \\ \bigcirc & & & 2 & -2 \end{matrix} & \begin{matrix} 1 & -1 & 1 & -1 & 0 \\ 1 & 1 & & & \\ & 1 & 1 & \bigcirc & \\ & & 1 & 1 & \\ \bigcirc & & & 1 & 1\end{matrix}\end{array}\right]$$

with an inverse:

$$M^{-1} = \left[\begin{array}{ccccc|ccccc} 1 & & & & & & & & & \\ & 1 & & \bigcirc & & & & \bigcirc & & \\ & & 1 & & & & & & & \\ & \bigcirc & & 1 & & & & & & \\ & & & & 1 & & & & & \\ \hline -1{,}75 & 3 & -2 & 1 & -0{,}25 & 0{,}25 & 0{,}75 & -0{,}5 & 0{,}25 & 0 \\ -0{,}25 & -1 & 2 & -1 & 0{,}25 & -0{,}25 & 0{,}25 & 0{,}5 & -0{,}25 & 0 \\ 0{,}25 & -1 & 0 & 1 & -0{,}25 & 0{,}25 & -0{,}25 & 0{,}5 & 0{,}25 & 0 \\ -0{,}25 & 1 & -2 & 1 & 0{,}25 & -0{,}25 & 0{,}25 & -0{,}5 & 0{,}75 & 0 \\ 0{,}25 & -1 & 2 & -3 & 1{,}75 & 0{,}25 & -0{,}25 & 0{,}5 & -0{,}75 & 1 \end{array}\right]$$

Only the left hand half of this matrix is of use since the elements of the second (lower) half $\underline{X}$ are zero.

In the example given in Fig. 4-6 the following points were plotted

$(z_1, \underline{x}_1) = (1,2)$
$(z_2, \underline{x}_2) = (2,1)$
$(z_3, \underline{x}_3) = (3,4)$
$(z_4, \underline{x}_4) = (4,3)$
$(z_5, \underline{x}_5) = (5,1)$

For this case it can be calculated:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_1' \\ x_2' \\ x_3' \\ x_4' \\ x_5' \end{bmatrix} = M^{-1} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \underline{x}_3 \\ \underline{x}_4 \\ \underline{x}_5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = M^{-1} \begin{bmatrix} 2 \\ 1 \\ 4 \\ 3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 3 \\ 1 \\ -5{,}75 \\ 3{,}75 \\ 2{,}25 \\ -4{,}25 \\ 0{,}25 \end{bmatrix}$$

The last five numbers in the last vector give the slopes at the measurement points. Now for all four reaches the curves can be constructed according to eq (4-35). This gives the following equations:

reach 1-2: $y = 2 - 5{,}75\,(z-1) + 4{,}75\,(z-1)$
reach 2-3: $y = 1 + 3{,}75\,(z-2) - 0{,}75\,(z-2)$
reach 3-4: $y = 4 + 2{,}25\,(z-3) - 3{,}25\,(z-3)$
reach 4-5: $y = 3 - 4{,}25\,(z-4) + 2{,}25\,(z-4)$

The interpolation is shown in Fig. 4-17. The image is more or less similar to that of the 4th-degree interpolation of Fig. 4-9, although there are minor differences.

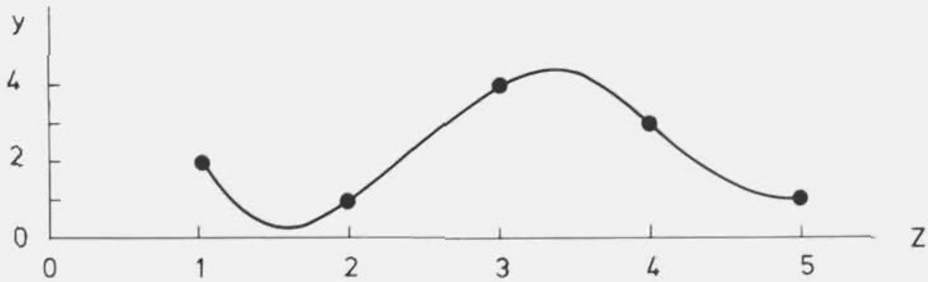


Fig. 4-17 Exact spline function interpolation between the five points of Fig. 4-6.

Lagrange functions

It is possible to derive Lagrange functions for this type of interpolation, as well as the propagation function. The continuous value of y of eq (4-33) has to be expressed first in terms of $\hat{x}_i$, $\hat{x}_{i+1}$ and x_i' and finally in terms of $\underline{x}_1 \ldots \underline{x}_m$. From eq (4-35) it can be derived that:

$$y_i = \hat{x}_i \left\{ 1 - \frac{(z-z_i)^2}{\Delta z_i^2} \right\} + \hat{x}_{i+1} \frac{(z-z_i)^2}{\Delta z_i^2} + x_i' (z-z_i) \left\{ 1 - \frac{z-z_i}{\Delta z_i} \right\} \tag{4-50}$$

Introducing

$$B_i = \frac{(z-z_i)^2}{\Delta z_i^2}$$

$$C_i = (z-z_i) \left\{ 1 - \frac{z-z_i}{\Delta z_i} \right\} \tag{4-51}$$

this yields:

$$y_i = (1-B_i)\hat{x}_i + B_i\hat{x}_{i+1} + C_i\, x_i' \tag{4-52}$$

In matrix notation:

$$y_i = [0 \ldots 0\ (1-B_i)\ B_i\ 0 \ldots 0\ 0 \ldots 0\ C_i\ 0 \ldots 0]\hat{X} \tag{4-53}$$

where $\hat{x}$ has the meaning of eqs (4-47/48).

In symbols:

$$y_i = [B_iC_i]\hat{X} \tag{4-54}$$

where $[B_iC_i]$ denotes the corresponding matrix of eq (4-53).

Now, with regard to eq (4-49):

$$y_i = [B_iC_i]M^{-1}\underline{X} \tag{4-55}$$

From this the matrix of the Lagrange functions for the i^{th} stretch can be separated:

$$A_i = [A_{i1} \ldots A_{im}] = [B_iC_i]\, M^{-1} \tag{4-56}$$

Thus:

$$y_i = A_i\underline{X} \tag{4-57}$$

For each stretch a separate set of Lagrange curves holds. This is expressed in the matrix $[B_iC_i]$. The differences between these matrices of the separate reaches only consist of the place of the values $(1-B_i)$, B_i and C_{ij} in the row of elements. Of course it is possible to combine the matrices $[B_iC_i]$ to one overall matrix [BC] given by

$$[BC]=\left[\begin{array}{ccccccc|cccc} (1-B_1) & B_1 & & & & & & C_1 & & & \\ & (1-B_2) & B_2 & & & & & & C_2 & & \\ & & \cdot & & & & & & & \cdot & \\ & & & \cdot & & & & & & & \cdot \\ & & & & \cdot & & & & & & \\ & & & & & (1-B_{m-1}) & B_{m-1} & & & & C_{m-1} \end{array}\right] \tag{4-58}$$

Now all m Lagrange functions for all (m-1) reaches can be included in one matrix A:

$$A = [BC]\, M^{-1} \tag{4-59}$$

Example

For reasons of convenience the following notations are used for the horizontal dimensions:

$$\begin{aligned} &\text{reach 1-2: } z(1) = z-z_1 = z-1 \\ &\text{reach 2-3: } z(2) = z-z_2 = z-2 \\ &\text{reach 3-4: } z(3) = z-z_3 = z-3 \\ &\text{reach 4-5: } z(4) = z-z_4 = z-4. \end{aligned} \tag{4-60}$$

The matrix [BC] is given in eq (4-61) for the example discussed previously in this section. In this example for all reaches is $z_i = 1$.

$$[BC] = \left[\begin{array}{ccccc|cccc} 1-z(1)^2 & z(1)^2 & & & & z(1)\{1-z(1)\} & & & \\ & 1-z(2)^2 & z(2)^2 & & & & z(2)\{1-z(2)\} & & \\ & & 1-z(3)^2 & z(3)^2 & & & & z(3)\{1-z(3)\} & \\ & & & 1-z(4)^2 & z(4)^2 & & & & z(4)\{1-z(4) \end{array}\right] \tag{4-61}$$

Multiplication by M^{-1} yields the following set of equations:

reach 1-2

$$\begin{aligned} A_{11} &= 1-z(1)^2-1{,}75z(1)\ \{1-z(1)\} \\ A_{12} &= z(1)^2+3\quad z(1)\ \{1-z(1)\} \\ A_{13} &= -2\quad z(1)\ \{1-z(1)\} \\ A_{14} &= z(1)\ \{1-z(1)\} \\ A_{15} &= -0{,}25z(1)\ \{1-z(1)\} \end{aligned}$$

reach 2-3

$$A_{21} = -0{,}25z(2)\,\{1-z(2)\}$$
$$A_{22} = 1-z(2)^2- \quad z(2)\,\{1-z(2)\}$$
$$A_{23} = z(2)^2+2 \quad z(2)\,\{1-z(2)\}$$
$$A_{24} = - \quad z(2)\,\{1-z(2)\}$$
$$A_{25} = 0{,}25z(2)\,\{1-z(2)\}$$

reach 3-4

$$A_{31} = 0{,}25z(3)\,\{1-z(3)\}$$
$$A_{32} = - \quad z(3)\,\{1-z(3)\}$$
$$A_{33} = 1-z(3)^2$$
$$A_{34} = z(3)^2+ \quad z(3)\,\{1-z(3)\}$$
$$A_{35} = -0{,}25z(3)\,\{1-z(3)\}$$

reach 4-5

$$A_{41} = -0{,}25z(4)\,\{1-z(4)\}$$
$$A_{42} = z(4)\,\{1-z(4)\}$$
$$A_{43} = -2 \quad z(4)\,\{1-z(4)\}$$
$$A_{44} = 1-z(4)^2+ \quad z(4)\,\{1-z(4)\}$$
$$A_{45} = z(4)^2+0{,}25z(4)\,\{1-z(4)\}$$

In Fig. 4-18 the Lagrange functions as well as the propagation function are shown. The curves resemble those of the 4th-degree exact interpolation (Fig. 4-10C and Fig. 4-11), although there are minor differences, in particular with respect to the extremes (both smaller and larger values). It depends on the actual situation what method is to be preferred with respect to the propagated measurement errors.

Fig. 4-19 demonstrates the influence of unequal length reaches using a three reach example. For reaches of equal length, the curves are more or less the same as for 3rd-degree (cubic) interpolation (see Fig. 4-5). If the side reaches are reduced the propagated values of the measurement errors become considerably smaller for spline interpolation than for 'one polynomial'-interpolation, even if approximate interpolation is considered (see Fig. 4-13). In the case of small side reaches the spline function interpolation is much to be preferred to the 4th-degree interpolation.

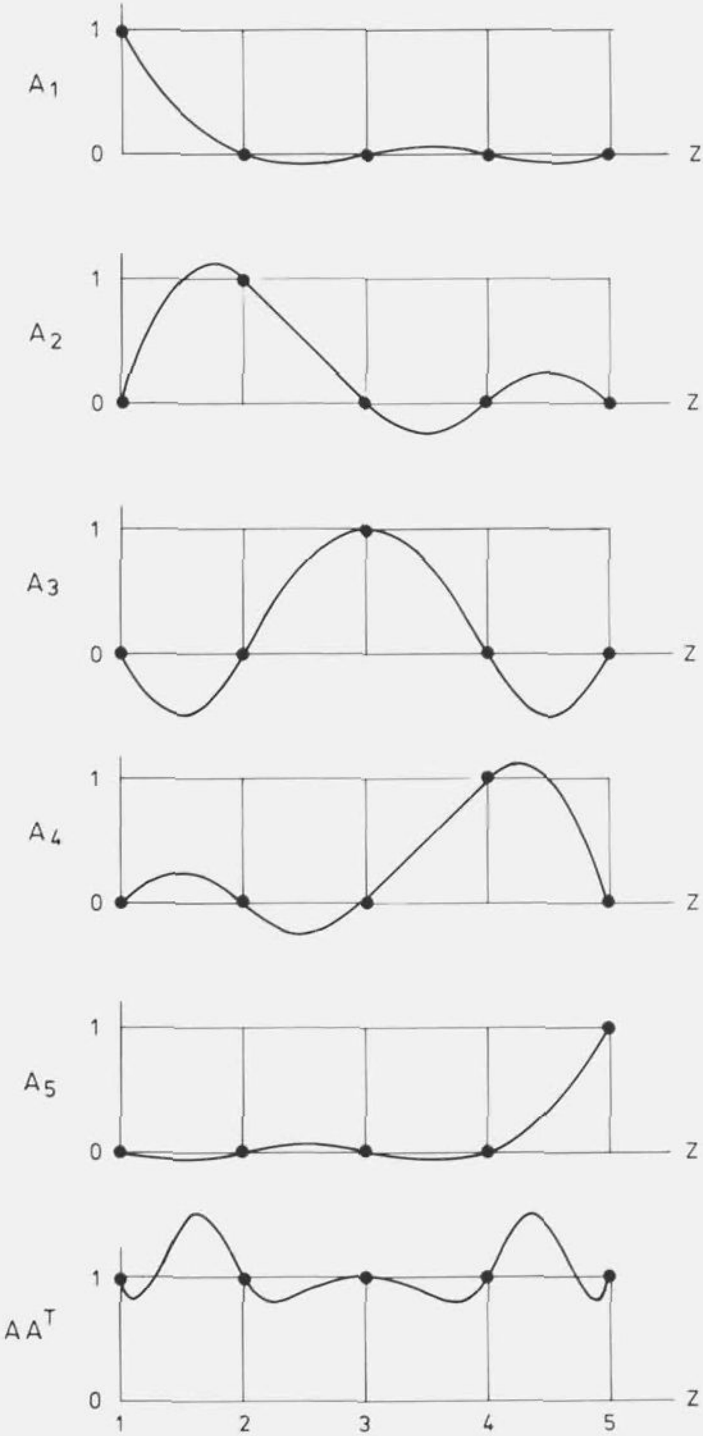


Fig. 4-18 Lagrange functions $A_1 \ldots A_5$ and propagation function AA^T for spline function interpolation (exact).

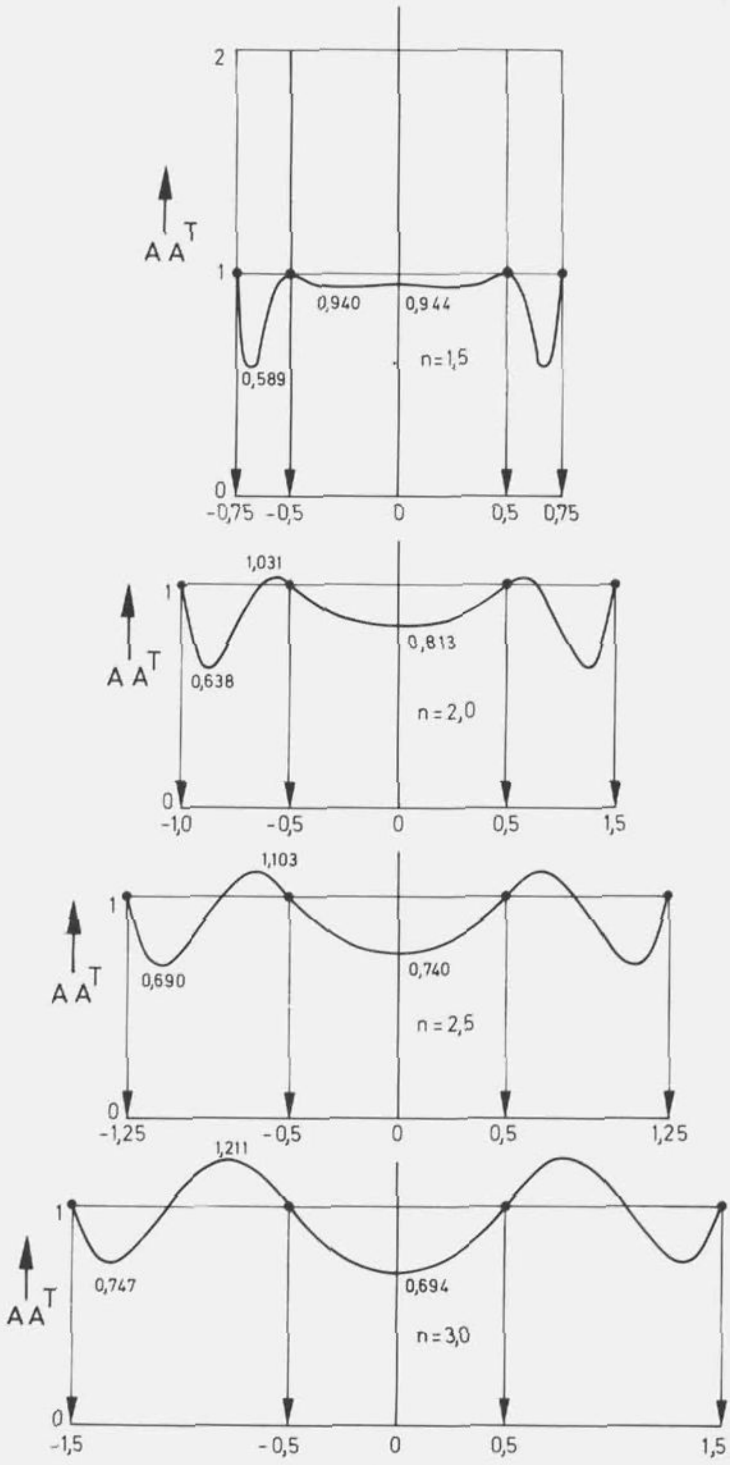


Fig. 4-19 Propagation functions AA^T for some spline interpolations

4.4.2 Approximate interpolation using spline functions

In this case the interpolation curve is no longer bound to the plotted points of the measured data, but a certain departure is allowed. These departures are minimized, as a rule according to the least sum of squares criterion. Thus

$$\sum_{i=1}^{m} (\underline{x}_i - \hat{x}_i)^2 \rightarrow \min \qquad (4\text{-}62)$$

where:

$\hat{x}_i$ = the calculated value of the interpolation curve at the measurement site;
$\underline{x}_i$ = the corresponding measured value;
m = the number of measurement sites.

The condition of eq (4-62) is contrary to that of the stretched shape, which minimizes the variance of the slopes, as expressed by eq (4-28). The more the shape is stretched, the more it tends to deviate from the points, whereas the more it fits the points the more the shape will be curved. Because of this contradiction both requirements have to be counterbalanced.

Weight factors p_i can be introduced, which indicate the importance that is required to be attached to the requirement of eq (4-62). The function to be minimized, denoted by S, is now

$$S = \mathrm{Var}\left(\frac{dy}{dz}\right) + \sum_{i=1}^{m} p_i (\underline{x}_i - \hat{x}_i)^2 \rightarrow \min \qquad (4\text{-}63)$$

This procedure must yield 2m values, namely the values $\hat{x}_i$ (i=1...m) and the corresponding slopes x_i'. However, only m+1 equations can be set up in order to meet eq (4-63), namely the derivatives with respect to the x_i-values and with respect to one of the slopes, e.g. x_i'. These derivatives have to be equal to zero. Once these values are determined the whole interpolation curve is fixed. The slopes at the other measurement points are related by eq (4-39), giving m-1 other equations, thus completing the number of 2m.

The following equations have to be solved:

$$\frac{\partial S}{\partial \hat{x}_j} = 0 \qquad (j = 1...m) \qquad (4\text{-}64)$$

$$\frac{\partial S}{\partial x_i'} = 0 \tag{4-65}$$

(m-1) equations of type of eq (4-39)

Eq (4-65) corresponds to eqs (4-33) and (4-43) since the second right hand term of eq (4-63) disappears when differentiating with respect to x_i'. However this time this equation relates to the calculated values $\hat{x}_i$ instead of the measured values $\underline{x}_i$.

In eq (4-64) the second right hand term of eq (4-63) remains, although in much simplier form. The following holds

$$\frac{\partial \sum_{i=1}^{m} p_i \quad (\underline{x}_i - \hat{x}_i)^2}{\partial \hat{x}_j} = - 2p_j \, (\underline{x}_j - \hat{x}_j) \tag{4-66}$$

Thus only one term of the series under the Σ-sign remains, namely the one for which i=j.

To calculate the derivatives of the first term of eq (4-63) the following values should be computed, starting with $\frac{dy}{dz}$ of eq (4-36):

$\left(\frac{dy}{dz}\right)^2$, $G_i(z)$ of eq (4-30) and $\frac{\partial G_i(z)}{\partial \hat{x}_j}$

leading to, respectively:

– for i=j:

$$\frac{\partial G_i(z)}{\partial \hat{x}_j} = \frac{2}{3}\left\{- \frac{\hat{x}_{j+1} - \hat{x}_j}{\Delta z_j} + x_j' - (\hat{x}_{j+1} - \hat{x}_j - x_j'\Delta z_j)\frac{\partial x_j'}{\partial \hat{x}_j}\right\} \tag{4-67}$$

– for i=j-1:

$$\frac{\partial G_{j-1}(z)}{\partial \hat{x}_j} = \frac{2}{3}\left\{\frac{\hat{x}_j - \hat{x}_{j-1}}{\Delta z_{j-1}} - x_{j-1}' - (\hat{x}_j - \hat{x}_{j-1} - x_{j-1}'\Delta z_{j-1})\frac{\partial x_{j-1}'}{\partial \hat{x}_j}\right\} \tag{4-68}$$

and for all others i ≠ j ≠ j-1:

$$\frac{\partial G_i(z)}{\partial \hat{x}_j} = - \frac{2}{3} \left(\hat{x}_{i+1} - \hat{x}_i - x_i'\Delta z_i \right) \frac{\partial x_i'}{\partial \hat{x}_j} \, . \tag{4-69}$$

The sum of these eqs (4-67), (4-68) and (4-69) for i=1... (m-1) yields the contribution of Var $(\frac{dy}{dz})$ to eq (4-64) (compare eq (4-32)):

$$\frac{1}{z_n - z_1} \sum_{i=1}^{m-1} \frac{\partial G_i(z)}{\partial \hat{x}_j} =$$

$$\frac{1}{z_n - z_1} \cdot \frac{2}{3} \left\{ \underbrace{- \frac{\hat{x}_{j-1}}{\Delta z_{j-1}} + \hat{x}_j \left(\frac{1}{\Delta z_{j-1}} + \frac{1}{\Delta z_j} \right) - \frac{\hat{x}_{j+1}}{\Delta z_j} x'_{j-1} + x'_j}_{①} \right.$$

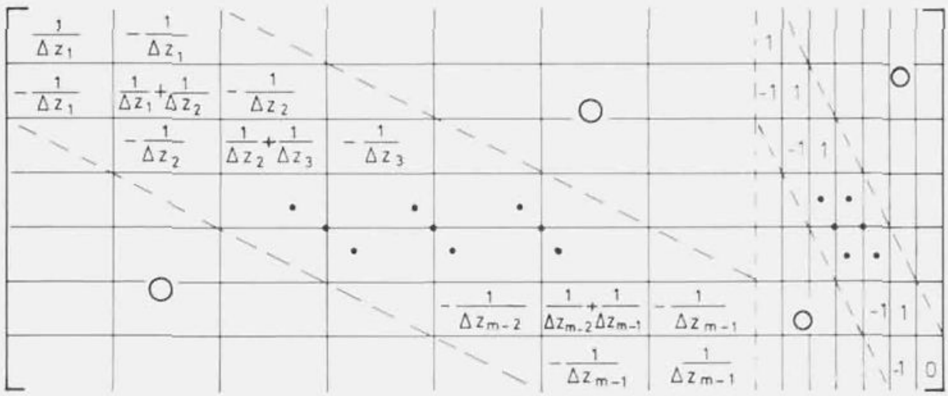
$$\left. \underbrace{- \sum_{i=1}^{m-1} (- \hat{x}_i + \hat{x}_{i+1} - x'_i \Delta z_i) \frac{\partial x'_i}{\partial \hat{x}_j}}_{②} \right\} \qquad (4\text{-}70)$$

Part 1 can be generalized in a matrix multiplication, covering all cases j=1...m, as follows:

$$I^* = M_1 \hat{X} \; ^{*)} \qquad (4\text{-}71)$$

where

$M_1 =$



(4-72)

whereas $\hat{X}$ is defined by eqs (4-47) and (4-48).

*) The matrix I^* should not be confused with the unit matrix I.

Note in matrix M_1 the reduced number of elements in the first and in the last row: for the first and for the last point of measurement only one stretch length can be taken into account.

Part 2 of eq (4-70) can also be written as a matrix multiplication covering the derivatives after all $\hat{x}_j$ (j=1...m). The sequence of the multiplication is changed.

The multiplication is now

$$II = \begin{bmatrix} \frac{\partial x_1'}{\partial \hat{x}_1} & \frac{\partial x_2'}{\partial \hat{x}_1} & \cdots & \frac{\partial x_{m-1}'}{\partial \hat{x}_1} \\ \frac{\partial x_1'}{\partial \hat{x}_2} & \frac{\partial x_2'}{\partial \hat{x}_2} & & \frac{\partial x_{m-1}'}{\partial \hat{x}_2} \\ \vdots & & & \\ \frac{\partial x_1'}{\partial \hat{x}_m} & \frac{\partial x_2'}{\partial \hat{x}_m} & \cdots & \frac{\partial x_{m-1}'}{\partial \hat{x}_m} \end{bmatrix} \cdot \begin{bmatrix} -\hat{x}_1 + \hat{x}_2 - x_1'\Delta z_2 \\ -\hat{x}_2 + \hat{x}_3 - x_2'\Delta z_2 \\ \vdots \\ -\hat{x}_{m-1} + \hat{x}_m - x_{m-1}'\Delta z_{m-1} \end{bmatrix} \tag{4-73}$$

or in symbols:

$$II = DV \tag{4-74}$$

where D and V represent the matrices of (4-70).

Now consider the elements of D. For the calculation subsequent application of eq (4-39) will be carried out:

$$x_1' = x_1' \tag{4-75}$$

$$x_2' = -x_1' - 2\frac{\hat{x}_1}{\Delta z_1} + 2\frac{\hat{x}_2}{\Delta z_1} \tag{4-76}$$

$$x_3' = x_1' + 2\frac{\hat{x}_1}{\Delta z_1} - 2\hat{x}_2\left(\frac{1}{\Delta z_1} + \frac{1}{\Delta z_2}\right) + 2\frac{\hat{x}_3}{\Delta z_2} \tag{4-77}$$

etc., until:

$$x'_{m-1} = -(-1)^{m-1} \left\{ x'_1 + 2 \frac{\hat{x}_1}{\Delta z_1} - 2\hat{x}_2 \left(\frac{1}{\Delta z_1} + \frac{1}{\Delta z_2} \right) \dots \right\}$$

$$- 2\hat{x}_{m-2} \left(\frac{1}{\Delta z_{m-3}} + \frac{1}{\Delta z_{m-2}} \right) + 2 \frac{\hat{x}_{m-1}}{\Delta z_{m-2}} \tag{4-78}$$

This yields:

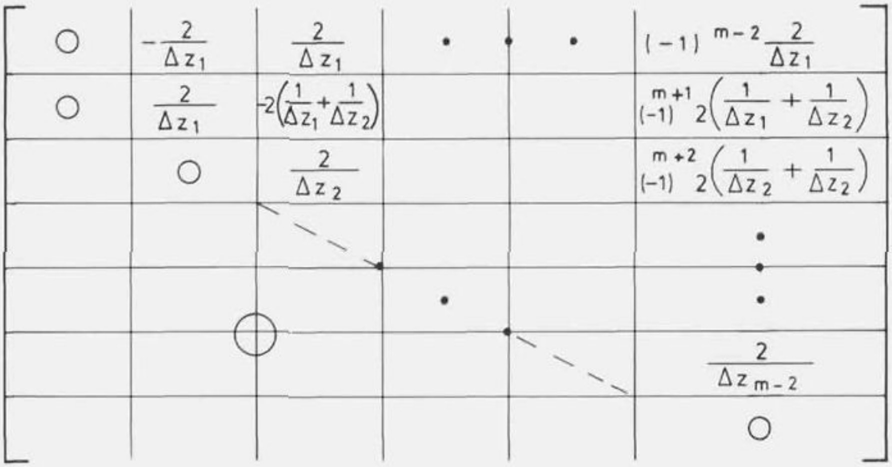
$$\left.\begin{aligned}
&\frac{\partial x'_1}{\partial \hat{x}_j} = 0 \quad (j = i \dots m) \\
&\frac{\partial x'_2}{\partial \hat{x}_1} = - \frac{2}{\Delta z_1}; \quad \frac{\partial x'_2}{\partial \hat{x}_2} = \frac{2}{\Delta z_1}; \quad \frac{\partial x'_2}{\partial \hat{x}_j} = 0 \ (j = 3 \dots m) \\
&\frac{\partial x'_3}{\partial \hat{x}_1} = \frac{2}{\Delta z_1}; \quad \frac{\partial x_3}{\partial \hat{x}_2} = -2\left(\frac{1}{\Delta z_1} + \frac{1}{\Delta z_2}\right); \quad \frac{\partial x'_3}{\partial \hat{x}_3} = \frac{2}{\Delta z_2} \\
&\frac{\partial x'_3}{\partial \hat{x}_j} = 0 \ (j = 4 \dots m)
\end{aligned}\right\} \tag{4-79}$$

In general:

$$\left.\begin{aligned}
&\frac{\partial x'_i}{\partial \hat{x}_j} = (-1)^{i+j}\, 2\left(\frac{1}{\Delta z_{j-1}} + \frac{1}{\Delta z_j}\right) && (i > 1 \text{ and } j<i) \\
&\frac{\partial x'_i}{\partial \hat{x}_j} = \frac{2}{\Delta z_{j-1}} && (i>1 \text{ and } j = i) \\
&\frac{\partial x'_i}{\partial \hat{x}_j} = (-1)^{i-1} \frac{2}{\Delta z_1} && (i > \text{ and } j = 1) \\
&\frac{\partial x'_i}{\partial \hat{x}_j} = 0 && (i = 1 \text{ or } j > i)
\end{aligned}\right\} \tag{4-80}$$

In this way the matrix D can be filled in. This leads to

$D =$



(4-81)

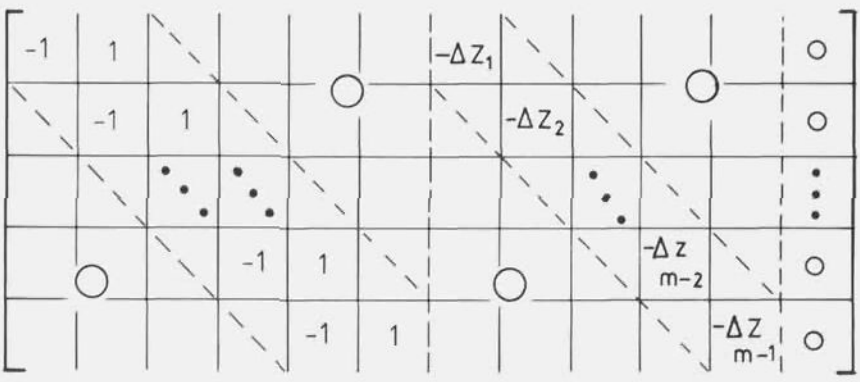
The matrix D is of the order $m\times(m\text{-}1)$.

The matrix V of eq (4-74) will be written as the product of a matrix M_2 of order $(m\text{-}1)\times 2m$ and the vector $\hat{X}$ of eqs (4-47) and (4-48), so

$$V = M_2\hat{X} \tag{4-82}$$

According to eq (4-73) M_2 is

$M_2 =$



(4-83)

Now all terms of eq (4-70) have been analyzed. Summarizing one can write (see also eqs (4-71) and (4-74)):

$$\left[\frac{\partial \mathrm{Var}\left(\frac{dy}{dz}\right)}{\partial \hat{x}_j}\right]_{(j=1\ldots m)} =$$

$$\left[\frac{1}{z_m - z_1} \sum_{i=1}^{m-1} \frac{\partial G_i(z)}{\partial \hat{x}_j}\right] (j = 1\ldots m) =$$

$$\frac{1}{z_m - z_1} \cdot \frac{2}{3} (M_1 - DM_2)\hat{X} \qquad (4\text{-}84)$$

Returning now to eq (4-66) concerning the deviations between the interpolation curve and the measured values $\underline{x}_j$. This equation can be written in matrix notation as follows, where, for reasons of simplicity the same weight factor p is adopted for all measured values $\hat{x}_j$:

$$[2p(\hat{x}_j - x_j)]_{(j=1\ldots m)} =$$

$$2p \left[\begin{array}{ccccc|c} 1 & & & & & \\ & 1 & & & & \\ & & \cdot & & & 0 \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & 1 & \end{array}\right] \hat{X} - 2p \begin{bmatrix} \underline{x}_1 \\ \cdot \\ \underline{x}_m \\ 0 \\ \cdot \\ 0 \end{bmatrix}$$

$$= 2pI_o\hat{X} - 2p\underline{X} \qquad (4\text{-}85)$$

where I_o and $\underline{X}$ denote the corresponding matrices.

Combination of eqs (4-84) and (4-85) yields a set of m equations

$$\left\{ \frac{1}{z_n - z_1} \cdot \frac{2}{3} (M_1 - DM_2) + 2pI_o \right\} \hat{X} - 2p\underline{X} = 0, \qquad (4\text{-}86)$$

Which fulfil the condition of eq (4-64)

Introducing:

$$P = p(z_n - z_1) \qquad (4\text{-}87)$$

eq (4-86) can be rewritten as follows:

$$\left\{\frac{1}{3}(M_1 - DM_2) + PI_o\right\} \hat{X} = P\underline{X} \tag{4-88}$$

or

$$\left\{\frac{1}{3P}(M_1 - DM_2) + I_o\right\} \hat{X} = \underline{X} \tag{4-89}$$

These m equations have to be considered together with those of eq (4-65), i.e. eq (4-43) and the (m-1) equations of eq (4-39). Now again a general matrix M, similar to those of eqs (4-74) or (4-48) can be composed. This time the upper m rows are given by eq (4-89), whereas the lower m rows remain as they were in eq (4-47). Indicating the lower set of m rows by M↓ , the matrix M can be written as

$$M = \begin{bmatrix} \frac{1}{3P} \quad (M_1 - DM_2) + I_o \\ \hline M\downarrow \end{bmatrix} \tag{4-90}$$

This time eq (4-48) also holds, so

$$M\hat{X} = \underline{X}$$

and eq (4-49) gives

$$\hat{X} = M^{-1}\underline{X}$$

Example

Consider once again the example of four equal reaches. Assess the weight factor at P=1. Now the following matrices can be derived.

– according to eq (4-72):

$$M_1 = \left[\begin{array}{ccccc|ccccc} 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

– according to eq (4-81):

$$D = \begin{bmatrix} 0 & -2 & 2 & -2 \\ 0 & 2 & -4 & 4 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

– according to eq (4-83):

$$M_2 = \left[\begin{array}{ccccc|ccccc} -1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

This yields:

$$M_1\text{-}DM_2 = \left[\begin{array}{ccccc|ccccc} 1 & -3 & 4 & -4 & 2 & 1 & -2 & 2 & -2 & 0 \\ -1 & 4 & -7 & 8 & -4 & -1 & 3 & -4 & 4 & 0 \\ 0 & -1 & 4 & -7 & 4 & 0 & -1 & 3 & -4 & 0 \\ 0 & 0 & -1 & 4 & -3 & 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

Further, by application of eq (4-90) it follows that

$$M = \left[\begin{array}{ccccc|ccccc} 4/3 & -1 & 4/3 & -4/3 & 2/3 & 1/3 & -2/3 & 2/3 & -2/3 & 0 \\ -1/3 & 7/3 & -7/3 & 8/3 & -4/3 & -1/3 & 1 & -4/3 & 4/3 & 0 \\ 0 & -1/3 & 7/3 & -7/3 & 4/3 & 0 & -1/3 & 1 & -4/3 & 0 \\ 0 & 0 & -1/3 & 7/3 & -1 & 0 & 0 & -1/3 & 1 & 0 \\ 0 & 0 & 0 & -1/3 & 4/3 & 0 & 0 & 0 & -1/3 & 0 \\ \hline 1 & -2 & 2 & -2 & 1 & 1 & -1 & 1 & -1 & 0 \\ 2 & -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 & -2 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Only the left hand half of the inverse matrix M^{-1} is of importance because the second half of $\underline{X}$ includes 'zero' values only. The first (i.e. the left hand) half of M^{-1} is denoted by "M^{-1} (1/2)". Inversion of M yields

$$
M^{-1}(1/2) = \left[\begin{array}{ccccc}
0,9156 & 0,1623 & -0,0649 & -0,0195 & 0,0065 \\
0,1623 & 0,6039 & 0,2857 & -0,0325 & -0,0195 \\
-0,0649 & 0,2857 & 0,5584 & 0,2857 & -0,0649 \\
-0,0195 & -0,0325 & 0,2857 & 0,6039 & 0,1623 \\
0,0065 & -0,0195 & -0,0649 & 0,1623 & 0,9156 \\
\hdashline
-1,0065 & 0,9286 & 0,1558 & -0,0714 & -0,0065 \\
-0,5000 & -0,0455 & 0,5455 & 0,0455 & -0,0455 \\
-0,0455 & -0,5909 & 0 & 0,5909 & -0,0455 \\
0,0455 & -0,0455 & -0,5455 & 0,0455 & 0,5000 \\
0,0065 & 0,0714 & -0,1558 & -0,9286 & 1,0065
\end{array}\right]
$$

Now consider again the example shown in Fig. 4-6. For a weight factor P=1, and also for weight factors P=0,1 and P=10 the results are given in Table 4-1. These values were calculated according to

$$
\hat{X} = M^{-1}\,(1/2) \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \underline{x}_3 \\ \underline{x}_4 \\ \underline{x}_5 \end{bmatrix} \tag{4-91}
$$

which gives a result comparable with that of eq (4-49)

$\hat{X}$	z_i	$\underline{x}_i$	Values in $\hat{X}$		
			P=0,1	P=1	P=10
1	2	3	4	5	6
$\hat{x}_1$	1	2	1,7687	1,6818	1,8954
$\hat{x}_2$	2	1	2,3099	1,9545	1,3219
$\hat{x}_3$	3	4	2,7357	3,1818	3,6423
$\hat{x}_4$	4	3	2,5242	3,0455	3,1680
$\hat{x}_5$	5	1	1,6615	1,1364	0,9723
x_1'	1	–	0,4718	-0,6818	-3,7111
x_2'	2	–	0,6106	1,2273	2,5641
x_3'	3	–	0,2410	1,2273	2,0767
x_4'	4	–	-0,6640	-1,4999	-3,0253
x_5'	5	–	-1,0614	-2,3183	-1,3661

Table 4-1. Calculated values of $\hat{x}$ and x' for spline function interpolation ofthe points of Fig. 4-6, whilst P=0,1; 1; 10.

In Fig. 4-20A, B and C the resulting interpolation curves are shown. As might be expected a high value of P allows the curve to adapt to the points (e.g. P=10), whereas a low value of P yields a relatively stretched shaped curve (e.g. P=0.1). Any value of P between 0 and ∞ may be chosen. For P=0 the result corresponds with that of the (approximate) linear interpolation; for P=∞ the curve of the exact spline function interpolation is obtained (Fig. 4-15). This freedom of choice implies a great flexibility for interpolation by spline functions, which is really the great advantage of this method compared with interpolation by single polynomials, where one can select only between a limited number of curves.

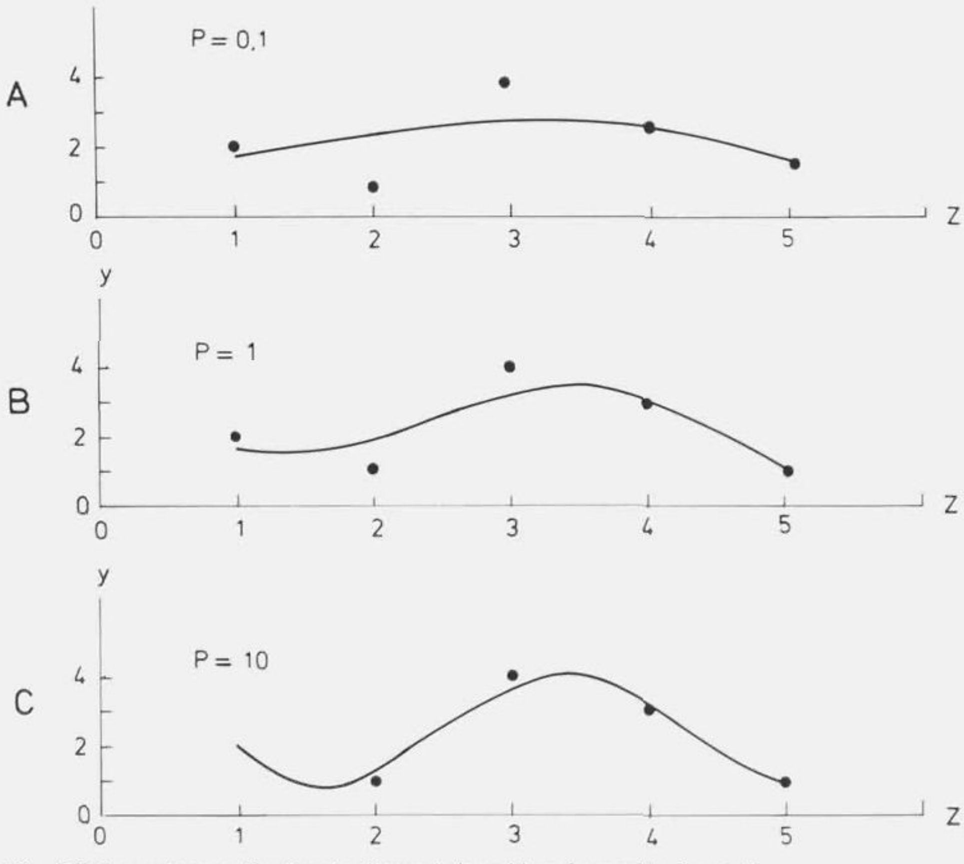


Fig. 4-20 Approximate spline function interpolation with various weight factors P.

The value of P to be chosen depends on the reliability that can be attached to the measured values. If their standard error of measurement is large, a low P-value is feasible. If the measurements are very accurate a high P value may be more appropriate. Also the shape of the curve may be a decisive factor: when the curve looks very distorted a decrease of P might be considered.

This approach might seem rather subjective. However, the single polynomial interpolation is also subjective in as far as the choice of the degree of the curve is concerned.

An objective method is the following. The squared sum of the departures of the measured values from the interpolation curve $\Sigma (x_i - x_i)^2$ are calculated. Let ε be the standard error of measurement. The curve which produces the lowest value of

$$\sum_{i=1}^{n.} (x_i - \hat{x}_i)^2 + AA^T\varepsilon^2$$

might be chosen as the best approximation. Here the propagation AA^T is to be taken into consideration also.

This approach yields the best approximation, according to the least-sum-of-squares criterion for the measurement sites. For intermediate sites between the measurements points nothing is known. It is therefore questionable whether a curve so derived is indeed the most appropriate. If any knowledge of the statistical or physical behaviour of the phenomenon is available, application of statistical or physically-based interpolation methods might be considered (see Chapter 5 and 6, respectively Chapter 7 and 8).

Lagrange and propagation functions.

The Lagrange functions, and also the propagation functions of the measurement errors, for the approximate spline function interpolation can be derived in the same way as for the exact interpolation. This means that eq (4-59) can be used, this time using the matrix M given by eq (4-90).

For the case of four reaches of equal length the results are shown in Fig. 4-21. The influence of not exactly fitting to the plotted points is clearly illustrated when this figure is compared with Fig. 4-18 of the same case, but with exact spline interpolation.

The Lagrange functions now have smaller values whereas the propagation function also has a favourable shape, similar to that of Fig. 4-10E obtained for approximate square interpolation. However the interpolation curve itself (Fig. 4-18B) shows smaller departures from the points than was the case for the interpolation curve of approximate square interpolation (Fig. 4-12B). Thus, from this point of view, the spline function interpolation is to be preferred to the square interpolation.

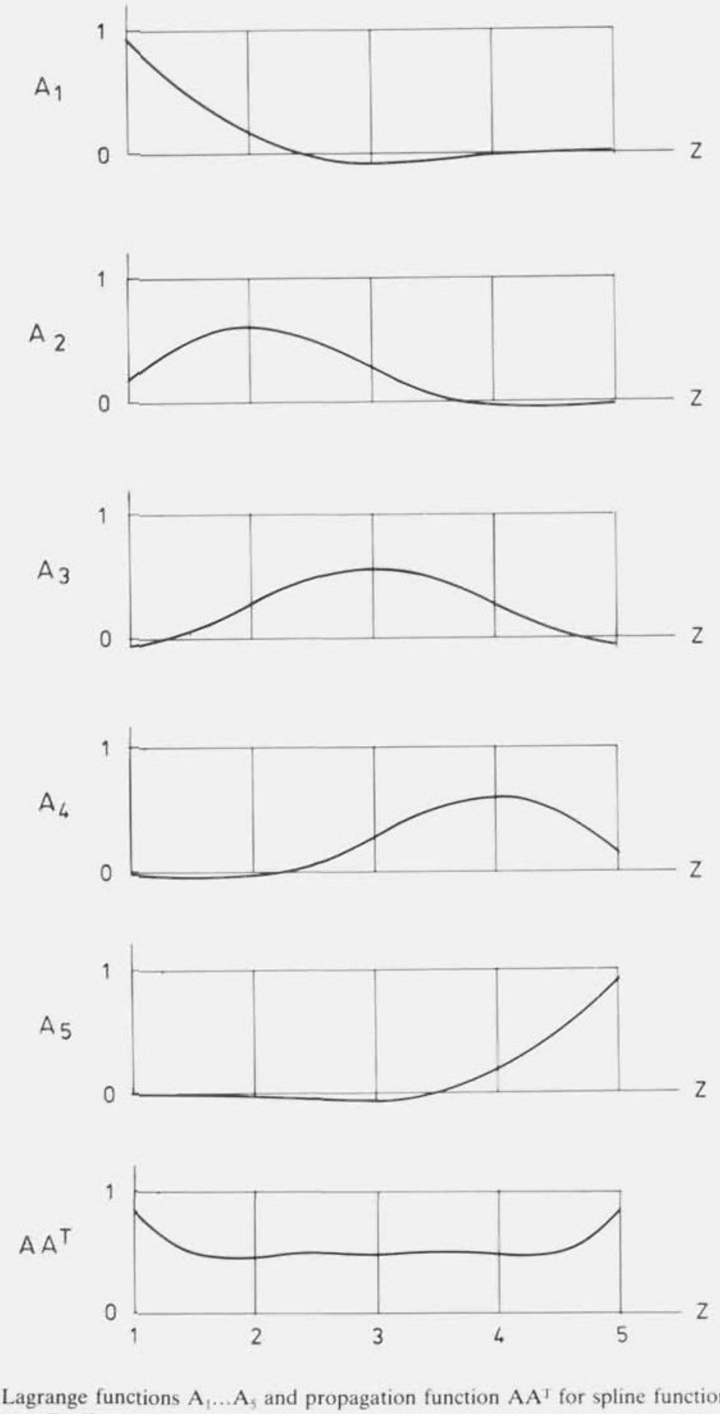


Fig. 4-21 Lagrange functions $A_1 \ldots A_5$ and propagation function AA^T for spline function interpolation (approximate; P=1)

In Fig. 4-22 the curves of the propagation functions for P = 0; 1; 10 and ∞ are given. Note that P = 0 corresponds with approximate linear interpolation and P = ∞ with exact spline function interpolation. Fig. 4-22 clearly shows for each case which parts of the reaches are most sensitive to measurement errors: those parts have the highest values of AA^T.

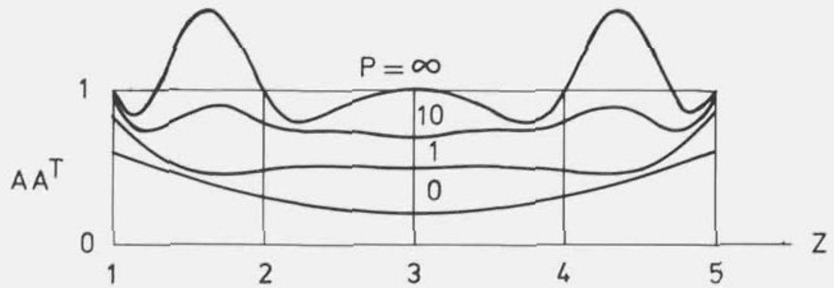


Fig. 4-22 Propagation function AA^T for spline function interpolation with various P-values.

4.4.3 Extensions

Extensions of the spline function interpolation method are possible, such as:
1. the introduction of dummy points (transition points without measurements);
2. the use of measured values between two transition points;
3. the fixation of a slope at a given value at one of the transition points.

4.4.3.1 DUMMY POINTS

Dummy points are transition points between the partial curves where no measurements are available. Sometimes it is desirable to change from one partial curves to another if important changes in the curvature are expected e.g. possible inflexion points. Those points do not contribute to the fulfilment of the requirement of minimum departure of the measurement points to the interpolation curve. This means that no contribution to eq (4-85) is available for those points. This implies that in the matrices I_0 and $\underline{X}$, at the positions corresponding to the dummy point concerned, a value 0 has to be placed instead of 1.

Using the example of the preceding section it will now be assumed that at point 3 no measurements are available. In this case the affected matrices become:

$$
I_o = \left[\begin{array}{ccccc|c}
1 & & & & & \\
 & 1 & & & & \\
 & & 0 & & & 0 \\
 & & & 1 & & \\
 & & & & 1 &
\end{array}\right]
$$

$$
\underline{X} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ -0 \\ \underline{x}_4 \\ \underline{x}_5 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}
$$

The left hand part of the inverse matrix, for the case that P = 1, is as follows:

$$
M^{-1}(1/2) = \left[\begin{array}{ccccc}
0{,}9251 & 0{,}1203 & -0{,}1471 & -0{,}0615 & 0{,}0160 \\
0{,}1203 & 0{,}7888 & 0{,}6471 & 0{,}1524 & -0{,}0615 \\
-0{,}1471 & 0{,}6471 & 1{,}2647 & 0{,}6471 & -0{,}1471 \\
-0{,}0615 & 0{,}1524 & 0{,}6471 & 0{,}7888 & 0{,}1203 \\
0{,}0160 & -0{,}0615 & -0{,}1471 & 0{,}1203 & 0{,}9251 \\
\hdashline
-1{,}0294 & 1{,}0294 & 0{,}3529 & 0{,}0294 & -0{,}0294 \\
-0{,}5802 & 0{,}3075 & 1{,}2353 & 0{,}3984 & -0{,}1257 \\
0{,}0455 & -0{,}5909 & 0 & 0{,}5909 & -0{,}0455 \\
0{,}1257 & -0{,}3984 & -1{,}2353 & -0{,}3075 & 0{,}5802 \\
0{,}0294 & -0{,}0294 & -0{,}3529 & -1{,}0294 & 1{,}0294
\end{array}\right]
$$

The corresponding interpolation curve is given in Fig. 4-23A. Fig. 4-23B shows a similar curve for P = 10.

In the matrix $M^{-1}(1/2)$ the 3^{rd} columm is of no importance now because the 3^{rd} element of $\underline{X}$ is zero.

In this case there are only four Lagrange functions instead of five, as in the preceding examples. These functions again are derived using eq (4-60). In Fig. 4-24 their curves are given together with that of the propagation function AA^T.

4.4.3.2 MEASURED VALUES BETWEEN TWO TRANSITION POINTS

At the measurements locations there is no change in the parabolas defining the interpolation curve.

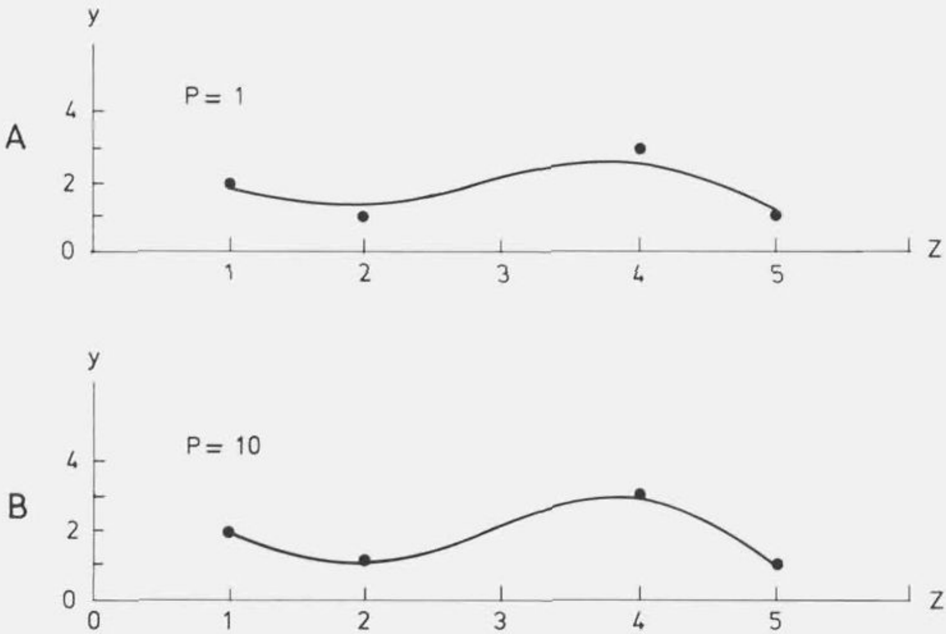


Fig. 4-23 Approximate spline function interpolation with one dummy point at z=3 (for P=1 and P=10)

In Fig 4-25 there are three such measured values between the transition points z_i and z_{i+1}.

Intermediate measurement points affect the curve. Here the same requirements as for the curves without intermediate points should be applied, that is

- the departure of the interpolation curve from the points (intermediate as well as at the transition points) should be a minimum according to the least squares criterion.
- the shape of the curve should be as stretched as possible.

The second requirement concerns the minimization of Var $(\frac{dy}{dz})$. As can be found from eq (4-36) and eqs(4-28...32) only the values at the transition points x_i and one of the slopes e.g. x'_i play a role in this variance. The intermediate points do not contribute.

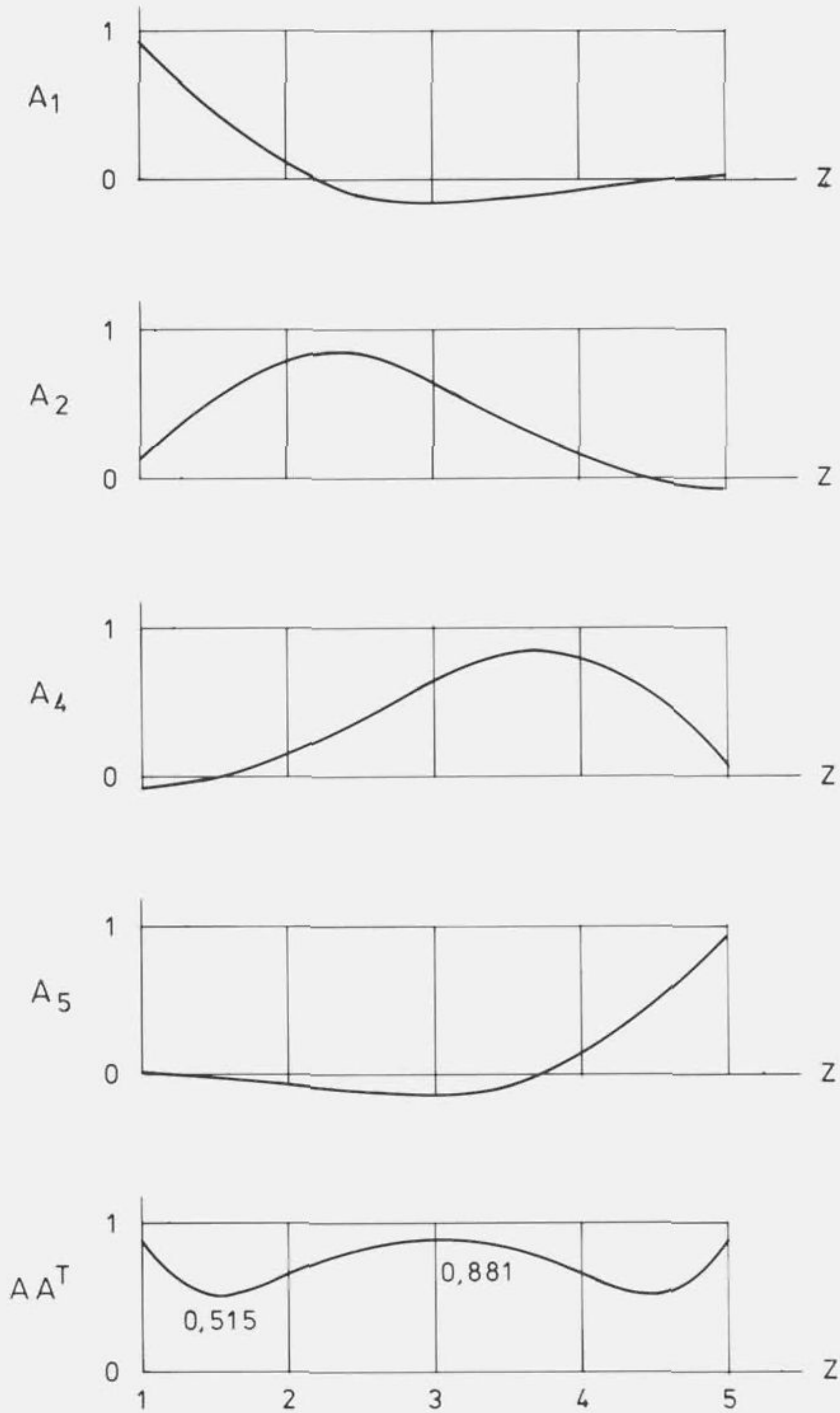


Fig. 4-24 Lagrange functions A_1, A_2, A_4, A_5 and propagation function AA^T for spline function interpolation with a dummy point at z=3 (approximate; P=1)

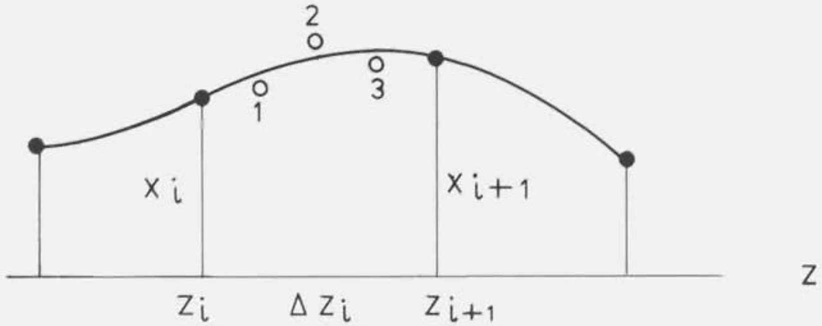


Fig. 4-25 Measured values 1, 2 and 3 between transition points z_i and z_{i+1}.

The effect of the intermediate points lies on the first mentioned requirement, i.e. the minimization of the departures of the interpolation curve from the measured points. This will be examined now.

The shape of the interpolation curve along a certain stretch can be expressed by eq (4-52),

$$y_i = (1-B_i)\hat{x}_i + B_i\hat{x}_{i+1} + C_i x_i'$$

Let $\underline{x}_k$ be a measured value at an intermediate point. The value at that point can be calculated using eq (4-52):

$$\hat{x}_k = (1-B_i)\,\hat{x}_i + B_i\,\hat{x}_{i+1} + C_i\,x_i' \tag{4-92}$$

Now consider the value:

$$\Delta x_k = \underline{x}_k - \hat{x}_k \tag{4-93}$$

The sum of its squares is included in the minimization, together with the other requirements, figuring in eq (4-63). Assuming one single weight factor P for all points the requirement reads

$$S = \mathrm{Var}\left(\frac{dy}{dz}\right) + P\left\{\sum_{i=1}^{n}(\underline{x}_i - \hat{x}_i)^2 + \sum_{k=1}^{m}(\underline{x}_k - \hat{x}_k)^2\right\} \rightarrow \min \tag{4-94}$$

assuming there to be $\underline{m}$ intermediate points.

This minimization can be done by differentiation of eq (4-94) with respect to $\hat{x}_i$, $\hat{x}_{i+1}$

and x_i', the latter after conversion into x_1' according to successive application of eq (4-40).

This procedure yields three expressions which have to be added to the other expressions, derived for the transition points and which are summarized in the matrix M. The three expressions are, when replacing $\sum_{k=1}^{m}$ by Σ, the following

$$\frac{\partial \Sigma \Delta x_k^2}{\partial \hat{x}_i} = -2\Sigma(1-B_i)\,\underline{x}_k + 2\hat{x}_i\Sigma(1-B_i)^2 + 2\hat{x}_{i+1}\Sigma(1-B_i)B_i + 2x_i'\Sigma(1-B_i)C_i$$

$$\frac{\partial \Sigma \Delta x_k^2}{\partial \hat{x}_{i+1}} = -2\Sigma B_i\underline{x}_k + 2\,\hat{x}_i\Sigma(1-B_i)B_i + 2\hat{x}_{i+1}\Sigma B_i^2 + 2x_i'\Sigma B_i C_i$$

$$\frac{\partial \Sigma \Delta x_k^2}{\partial x_i'} = (-1)^{i-1}\left\{-2\Sigma C_i\underline{x}_k + 2\hat{x}\Sigma(1-B_i)C_i + 2\hat{x}_{i+1}\Sigma B_i C_i + 2x_i'\Sigma C_i^2\right\} \qquad (4\text{-}95)$$

Multiplication of the last equation by $(-1)^{i-1}$ arises because of the conversion of x_i into x_1 in the differential quotient.

A weight factor P will be applied, as was done for data at the transition points. The right hand parts of the eqs (4-95) are now written in a matrix form:

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & (-1)^{i-1} \end{bmatrix} \cdot \begin{bmatrix} 2\Sigma(1-B_i)^2 & 2\Sigma(1-B_i)B_i & 2\Sigma(1-B_i)C_i \\ 2\Sigma(1-B_i)B_i & 2\Sigma B_i^2 & 2\Sigma B_i C_i \\ 2\Sigma(1-B_i)C_i & 2\Sigma B_i C_i & 2\Sigma C_i^2 \end{bmatrix} \cdot \begin{bmatrix} \hat{x}_i \\ \hat{x}_{i+1} \\ x_1' \end{bmatrix} - \begin{bmatrix} 2\Sigma(1-B_i)x_k \\ 2\Sigma B_i\underline{x}_k \\ 2(-1)^{i-1}\Sigma C_i\underline{x}_k \end{bmatrix} \qquad (4\text{-}96)$$

This expression can, after multiplication of both terms by P, be included in eq (4-90). The elements of the matrix, resulting from the multiplication of the first two matrices of eq (4-96), should be placed on the relevant positions in the matrix M of eq (4-90). The last row resulting from this multiplication is to be combined with the first row in the lowest half M $\downarrow$ of the matrix M. This is the row corresponding with eq (4-43). It should be remembered that in the earlier derivation the factors 2/3 and $1/(z_n-z_1)$ were deleted, because these were constant factors which did not influence the equality to zero of eq (4-43). This time, this is no longer the case: in the right hand matrix $\underline{X}$ the values $2\Sigma C_i\underline{x}_k$ occur. So the reduction mentioned is no longer allowed. Consideration of eqs (4-86...90) indicates that this objection can be removed by dividing the term of eq(4-43) by the value 3P. The factors 2 of eq (4-96) disappear.

To the matrix $\underline{X}$, the 'matrix of the knowns', the 2nd term of eq (4-96) is added. This yields

$$\underline{X} = \begin{bmatrix} \underline{x}_1 \\ \cdot \\ \cdot \\ \cdot \\ \underline{x}_i + \Sigma(1-B_i)\underline{x}_k \\ \underline{x}_{i+1} + \Sigma \quad B_i\underline{x}_k \\ \cdot \\ \cdot \\ \cdot \\ \underline{x}_m + (-1)^{i-1}\Sigma C_i\underline{x}_k \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \tag{4-97}$$

Now introduce a matrix K and a matrix X^+ so that:

$$\underline{X} = K\underline{X}^+ \tag{4-98}$$

or

$$\underline{X} = \left[\begin{array}{ccccc|ccc} 1 & & & & & & \bigcirc & \\ & 1 & & \bigcirc & & & & \\ & & \cdot & & & (1-B_i)_1 & \dots & (1-B_i)_{\underline{m}} \\ & & & \cdot & & B_{i1} & \dots & B_{i\,\underline{m}} \\ & \bigcirc & & & \cdot & & \bigcirc & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ \hline & & \bigcirc & & & (-1)^{i-1}C_{i1} & \dots & (-1)^{i-1}C_{i\underline{m}} \\ \hline & & & & \bigcirc & & & \\ & & & & & & & \end{array}\right] \begin{bmatrix} \underline{x}_1 \\ \cdot \\ \cdot \\ \cdot \\ \underline{x}_m \\ \underline{x}_{k1} \\ \cdot \\ \cdot \\ \cdot \\ \underline{x}_{k\,\underline{m}} \end{bmatrix} \tag{4-99}$$

Now all measured data are included in one $(m+\underline{m})\times 1$ matrix X^+.

Example

Now add a further point ($z=3{,}5$; $\underline{x}_k=6$) to the five plotted points of Fig. 4-6. At this location there is no transition of parabolas. As there is one additional point only, the Σ-signs of eq (4-95) need not to be used. The terms to be added to matrix M follow from (see eq 4-51):

$$B_i = \left(\frac{z-z_i}{\Delta z}\right)^2 = \left(\frac{3{,}5-3}{1}\right)^2 = 1/4$$

$$1-B_i = \frac{3}{4}$$

$$C_i = (z-z_i)\left(1-\frac{z-z_i}{\Delta z_i}\right) = \frac{1}{2}\left(1-\frac{1}{2}\right) = \frac{1}{4}.$$

Terms corresponding with these values, as indicated by eq (4-96), are added to the corresponding positions of M. Also the first row of the lower half has to be divided by 3P, i.e. by 3 if P=1. Now matrix M becomes

$$M = \left[\begin{array}{ccccc:ccccc}
4/3 & -1 & 4/3 & -4/3 & 2/3 & 1/3 & -2/3 & 2/3 & -2/3 & 0 \\
-1/3 & 7/3 & -7/3 & 8/3 & -4/3 & -1/3 & 1 & -4/3 & 4/3 & 0 \\
0 & -1/3 & 7/3+9/16 & -7/3+3/16 & 4/3 & 0 & -1/3 & 1+3/16 & -4/3 & 0 \\
0 & 0 & -1/3+3/16 & 7/3+1/16 & -1 & 0 & 0 & -1/3+1/16 & 1 & 0 \\
0 & 0 & 0 & -1/3 & 4/3 & 0 & 0 & 0 & -1/3 & 0 \\
\hdashline
1/3 & -2/3 & 2/3+3/16 & -2/3+1/16 & 1/3 & 1/3 & -1/3 & 1/3+1/16 & -1/3 & 0 \\
2 & -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 2 & -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 2 & -2 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 2 & -2 & 0 & 0 & 0 & 1 & 1
\end{array}\right]$$

The matrix K of eqs (4-98/99) has the form

$$K = \left[\begin{array}{ccccc:c}
1 & & & \bigcirc & & 0 \\
& 1 & & & & 0 \\
& & 1 & & & 3/4 \\
\bigcirc & & & 1 & & 1/4 \\
& & & & 1 & 0 \\
\hdashline
& & \bigcirc & & & 1/4 \\
\hdashline
0 & & & & & 0 \\
0 & & \bigcirc & & & 0 \\
0 & & & & & 0 \\
0 & & & & & 0
\end{array}\right]$$

Now from eq (4-49):

$$\hat{X} = M^{-1}\underline{X}$$

and taking $\underline{X}$ from eq (4-98), yields

$$\hat{X} = M^{-1}K.\underline{X}^{+} \qquad (4\text{-}100)$$

Calculation of $M^{-1}K$ yields

$$M^{-1}K = \left[\begin{array}{ccccc|c}
0{,}9069 & 0{,}1743 & 0{,}0355 & 0{,}0856 & 0{,}0025 & -0{,}2049 \\
0{,}1748 & 0{,}5867 & 0{,}1414 & -0{,}1834 & -0{,}0137 & 0{,}2942 \\
-0{,}0499 & 0{,}2649 & 0{,}3842 & 0{,}1034 & -0{,}0580 & 0{,}3553 \\
-0{,}0096 & -0{,}0461 & 0{,}1714 & 0{,}4842 & 0{,}1669 & 0{,}2333 \\
0{,}0062 & -0{,}0191 & -0{,}0617 & 0{,}1657 & 0{,}9155 & -0{,}0065 \\
\hline
-0{,}9689 & 0{,}8765 & -0{,}2812 & -0{,}5288 & 0{,}0109 & 0{,}8916 \\
-0{,}4955 & -0{,}0517 & 0{,}4931 & -0{,}0093 & -0{,}0434 & 0{,}1068 \\
0{,}0461 & -0{,}5918 & -0{,}0075 & 0{,}5831 & -0{,}0452 & 0{,}0153 \\
0{,}0345 & -0{,}0303 & -0{,}4183 & 0{,}1785 & 0{,}4949 & -0{,}2594 \\
-0{,}0029 & 0{,}0843 & -0{,}0479 & -0{,}8155 & 1{,}0023 & -0{,}2202
\end{array}\right]$$

This gives:

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \\ \hat{x}_5 \\ \\ x_1' \\ x_2' \\ x_3' \\ x_4' \\ x_5' \end{bmatrix} = M^{-1}K. \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \underline{x}_3 \\ \underline{x}_4 \\ \underline{x}_5 \\ \underline{x}_k \end{bmatrix} = M^{-1}K. \begin{bmatrix} 2 \\ 1 \\ 4 \\ 3 \\ 1 \\ 6 \end{bmatrix} = \begin{bmatrix} 1{,}1600 \\ 2.7032 \\ 4.0859 \\ 3{,}6369 \\ 1{,}1201 \\ \\ 1{,}5880 \\ 1{,}4992 \\ 1{,}2663 \\ -2{,}1606 \\ -2{,}8784 \end{bmatrix}$$

The shape of interpolation curve is given in Fig. 4-26.

The influence of the additional point is made clear when this figure is compared with Fig. 4-20B.

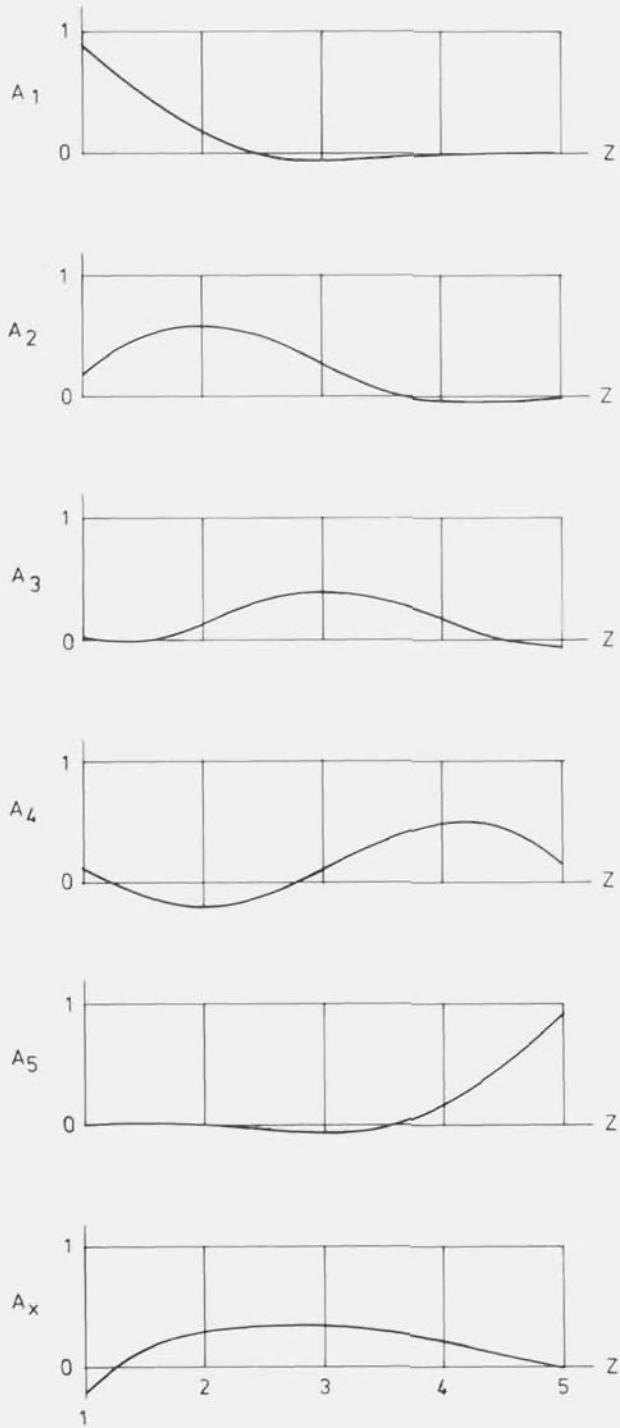


Fig. 4-27 Lagrange functions $A_1 \ldots A_5$ and A_x for spline function interpolation with one intermediate point in the reach 3-4 (P=1).

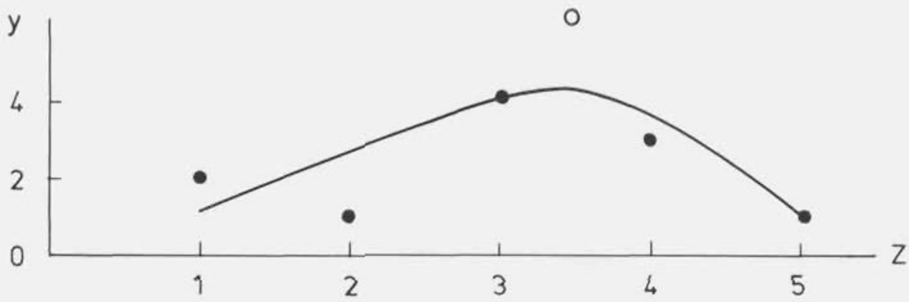


Fig. 4-26 Approximate spline function interpolation (P=1) between the five points of Fig. 4-6, with one additional point (z=3,5 ; x=6) in the reach 3-4.

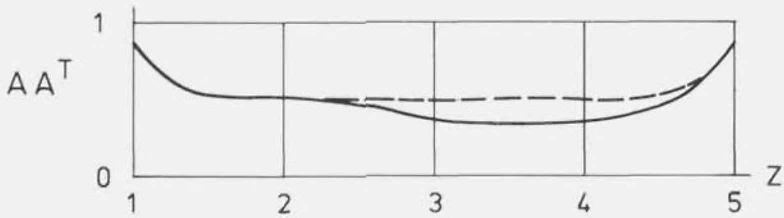


Fig. 4-28 Propagation function AA^T for spline function interpolation:
— with one intermediate point in the reach '3-4'
--- with no intermediate points (compare Fig. 4-21).

The Lagrange functions follow after multiplication of $M^{-1}K$ of eq (4-100) with $[B_iC_i]$ of eq (4-54):

$$A_i = [B_iC_i]\, M^{-1}K \qquad (4\text{-}101)$$

Note that this time there are six Lagrange functions: five for the original points and one for the additional point. The shape of these functions is shown in Fig. 4-27, the corresponding propagation function AA^T in Fig. 4-28.

The shapes of the Lagrange functions $A_1 \ldots A_5$ are similar to those of the corresponding curves of the earlier examples. However the shape of the curve of the additional point A_x is quite different. Its influence remains relatively important over all reaches.

The additional point causes a decrease of the propagation function AA^T in its own reach and, to a less extent, in the adjacent reaches. Apparently the use of more points reduces the propagated errors of measurement and thus has a decreasing effect on the error of estimate.

4.4.3.3 FIXATION OF A SLOPE

It is possible to fix the slope of the interpolation curve at one of the transition points (with or without measurements) to a given value, e.g. to zero if a horizontal tangent is required.

The calculation is carried out as follows. If it concerns the first point z_1, then an element 1 will be placed in the first row and in the $(m+1)^{th}$ columm of M of eq (4-90). All other elements become zero. This implies an equation:

$$1 \times x_1' = 0 \tag{4-102}$$

The rest of the matrix M of eq (4-90) remains unchanged. In the example following eq (4-90), where m=5, the first row of M ↓ concerns the 6th row of M in which the 6^{th} element becomes 1. The left hand half of the inversion is:

$$M^{-1}(1/2) = \left[\begin{array}{ccccc} 0{,}7095 & 0{,}3525 & -0{,}0330 & -0{,}0341 & -0{,}0052 \\ 0{,}3525 & 0{,}4285 & 0{,}2563 & -0{,}0190 & -0{,}0183 \\ -0{,}0330 & 0{,}2563 & 0{,}5535 & 0{,}2880 & -0{,}0647 \\ -0{,}0341 & -0{,}0190 & 0{,}2880 & 0{,}6029 & 0{,}1622 \\ 0{,}0052 & -0{,}0183 & -0{,}0647 & 0{,}1622 & 0{,}9156 \\ \hdashline 0 & 0 & 0 & 0 & 0 \\ -0{,}7141 & 0{,}1520 & 0{,}5786 & 0{,}0303 & -0{,}0468 \\ -0{,}0569 & -0{,}4965 & 0{,}0159 & 0{,}5836 & -0{,}0461 \\ 0{,}0548 & -0{,}0540 & -0{,}5469 & 0{,}0461 & 0{,}5001 \\ 0{,}0238 & 0{,}0554 & -0{,}1585 & -0{,}9275 & 1{,}0067 \end{array}\right]$$

This yields, according to eq (4-89) the values shown in Table 4-2, column 4:

			Values in $\hat{X}$		
$\hat{X}$	z_i	x_i	P=1	P=10	P=1; z_1=θ
1	2	3	4	5	6
$\hat{x}_1$	1	2	1,5422	1,6044	0,4241
$\hat{x}_2$	2	1	2,0833	1,7490	1,5277
$\hat{x}_3$	3	4	3,2034	3,4514	3,2558
$\hat{x}_4$	4	3	3,0355	3,2374	3,0895
$\hat{x}_5$	5	1	1,1355	0,9579	1,1273
x_1'	1	–	0	0	0
x_2'	2	–	1,0822	0,2892	2,2077
x_3'	3	–	1,1580	3,1156	1,2475
x_4'	4	–	-1,4938	-3,5436	-1,5799
x_5'	5	–	-2,3062	-1,0154	-2,3455

Table 4-2 Calculated values of $\hat{X}$ for various weightfactors P (horizontal gradient at z_1)

Table 4-2 also gives the results for P = 10 (column 5) and once again for P = 1 but this time with a dummy point at z_1 (column 6, indicated by z = ∅).

The corresponding interpolation curves are shown in Figs. 4-29A, 29B and 29C.

The Lagrange and propagation functions are calculated for P = 1 for both with and without a measured value at z_1. Fig. 4-30 shows the case with a measurement at z_1, Fig. 4-31 the case without such a measurement.

Comparing Fig. 4-30 with Fig. 4-21 (no fixed slopes) it appears that the fixation of slope has a smoothing effect on the Lagrange functions at and around the place of the fixed slope. This results in a reduction of the propagation function.

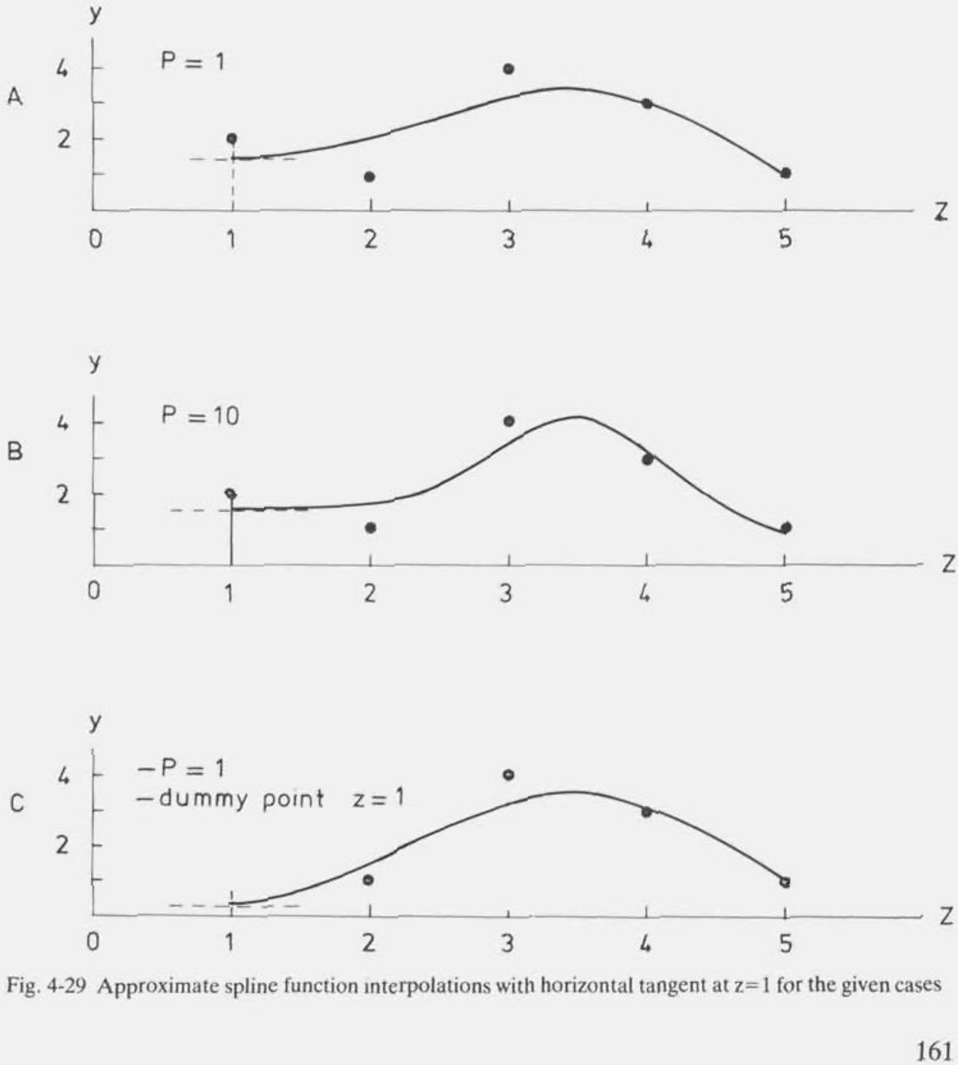


Fig. 4-29 Approximate spline function interpolations with horizontal tangent at z=1 for the given cases

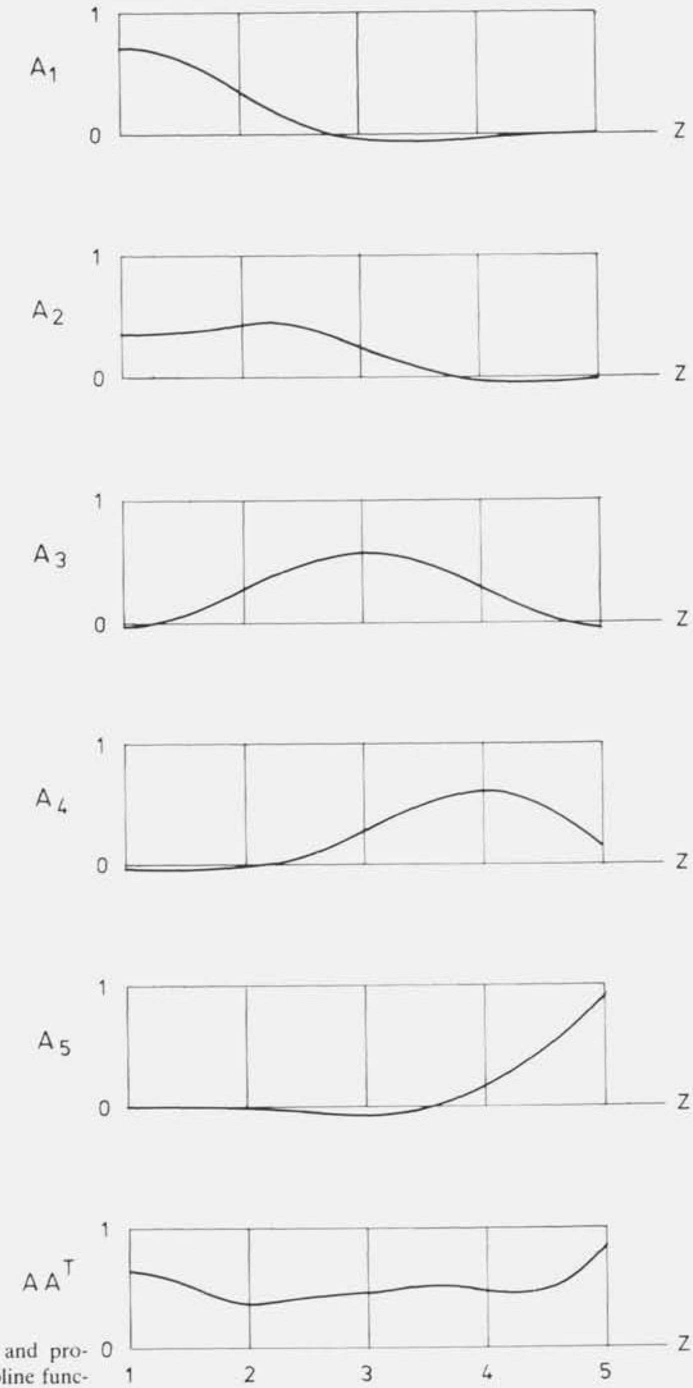


Fig. 4-30
Lagrange functions $A_1 \ldots A_5$ and propagation function AA^T for spline function interpolation with fixed initial slope $x_1' = 0$ (P = 1).

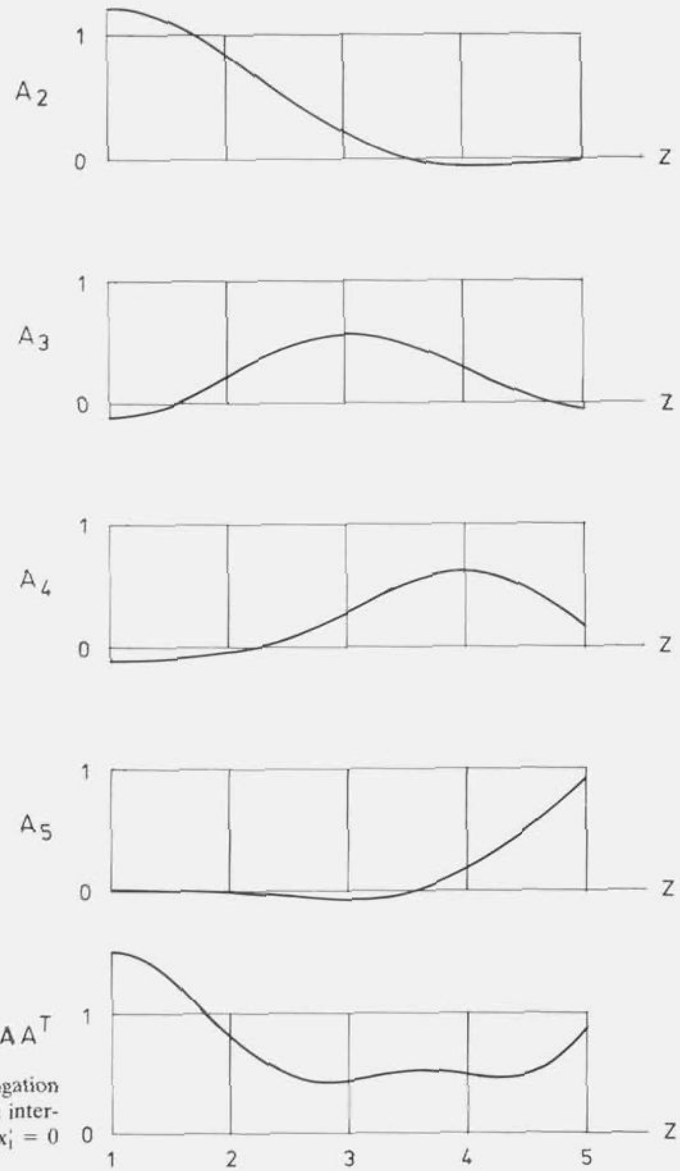


Fig. 4-31
Lagrange functions $A_2 \ldots A_5$ propagation function AA^T for spline function interpolation with fixed initial slope $x_1' = 0$ and dummy point at z_1 (P = 1).

If there is just a dummy point at the place z_1 of the initial (fixed) slope at the left hand side, the Lagrange functions concerned are strongly influenced, whereas the propagation function takes relatively high values at and near the dummy point which is located outside the measurement sites. This is not so surprising since locally it implies an extrapolation instead of an interpolation. However, this case can be very useful in practice if the propagated errors remain within acceptable limits.

Now consider the case of a fixed slope at a site other than z_1. If this is the k^{th} site z_k, a 1 is placed at the $(m+k)^{th}$ position in the $(m+1)^{st}$ row of the matrix M. This row becomes

$$
\begin{array}{ccc}
[0\ldots0 & | & 0\ldots1\ldots0] \\
\leftarrow m \rightarrow & | & \leftarrow m \rightarrow \\
 & & \leftarrow k \rightarrow
\end{array}
$$

This implies the equation:

$$1 \times x_k^{'} = 0 \tag{4-103}$$

The fact that this time the k^{th} slope is fixed instead of the 1^{st} one influences the composition of the matrix D of eqs (4-73/74).

This matrix is written now

$$
D = \begin{bmatrix}
\frac{\partial x_1^{'}}{\partial \hat{x}_1} & \cdots & \frac{\partial x_i^{'}}{\partial \hat{x}_i} & \cdots & \frac{\partial x_{m-1}^{'}}{\partial \hat{x}_1} \\
\vdots & & \vdots & & \vdots \\
\frac{\partial x_1^{'}}{\partial \hat{x}_j} & \cdots & \frac{\partial x_i^{'}}{\partial \hat{x}_j} & \cdots & \frac{\partial x_{m-1}^{'}}{\partial \hat{x}_j} \\
\vdots & & \vdots & & \vdots \\
\frac{\partial x_1^{'}}{\partial \hat{x}_m} & \cdots & \frac{\partial x_i^{'}}{\partial \hat{x}_m} & \cdots & \frac{\partial x_{m-1}^{'}}{\partial \hat{x}_m}
\end{bmatrix} \tag{4-104}
$$

→ k^{th}row

k^{th}column

All elements related to $x_k^{'}$, i.e. of the k^{th} columm become zero. The elements at the following matrix positions will be zero too, namely for $i > j$; $j < k$ and $i < j$; $j > k$. This can be explained as follows.

The slope, which is fixed for $j=k$ causes a fixation of the whole curve for all points between $j=k$ and the point where a variation is introduced. For the points beyond the point which is varied the slopes ofthe curve will change. If, for instance, the water level $\hat{x}_j$ at j=k+3 is varied (Fig. 4-32), the slopes at i=k+3, i=k+4 and i=k+5 will change, but the slopes at i=(k-4)...(k+2) remain as they are.

A variation at j=k-4 in Fig. 4-32 will leave all slopes at i> k-4 unchanged, but not at i=k-4.

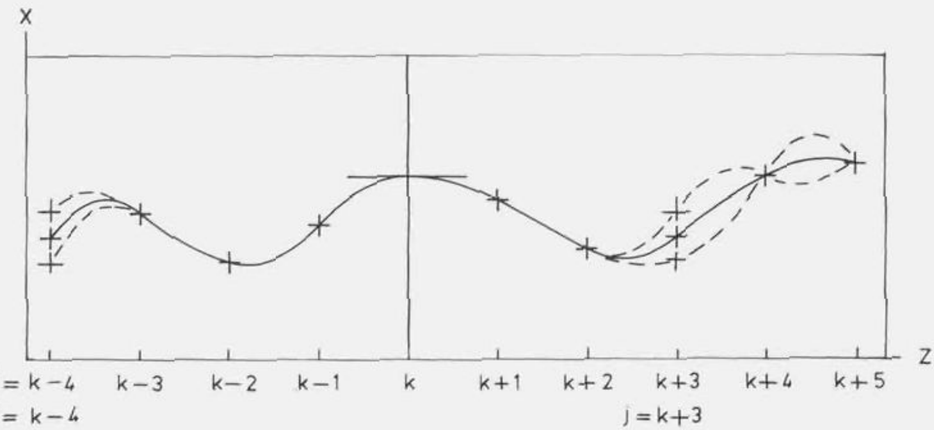


Fig. 4-32 Influence of variation of a value $\hat{x}_j$.

Fig. 4-33 shows a structural representation of the matrix D. The areas where the elements are zero are indicated by 0.

To calculate the other elements consider eq (4-38):

$$\dot{x}_{i+1} = -\dot{x}_i + 2 \frac{\hat{x}_{i+1} - \hat{x}_i}{\Delta z_i}$$

Starting from i=k then successive application of eq (4-38) up to i=k+p yields

$$x'_{k+p} = (-1)^p \left[x'_k + 2 \left\{ \frac{\hat{x}_k}{\Delta z_k} + \sum_{q=1}^{p-1} (-1)^q \, \hat{x}_{k+q} \left(\frac{1}{\Delta z_{k+q-1}} + \frac{1}{\Delta z_{k+q}} \right) \right\} \right]$$

$$+2 \frac{\hat{x}_{k+p}}{\Delta z_{k+p-1}} \qquad (4\text{-}105)$$

From this it follows that:

- for i=k+p>k, j=k : $$\frac{\partial x'_{k+p}}{\partial \hat{x}_k} = (-1)^p . \frac{2}{\Delta z_k} \qquad (4\text{-}106)$$

– for $i=k+p>k$
$j=i\ (=k+p)$: $\dfrac{\partial x'_{k+p}}{\partial \hat{x}_{k+p}} = \dfrac{2}{\Delta z_{k+p-1}}$ (4-107)

– for $i=k+p>k$
$k<j<i=k+p$: $\dfrac{\partial x'_{k+p}}{\partial \hat{x}_{k+q}} = (-1)^{p+q}.2\left(\dfrac{1}{\Delta z_{k+q-1}} + \dfrac{1}{\Delta z_{k+q}}\right)$ (4-108)
$(j=k+q)$

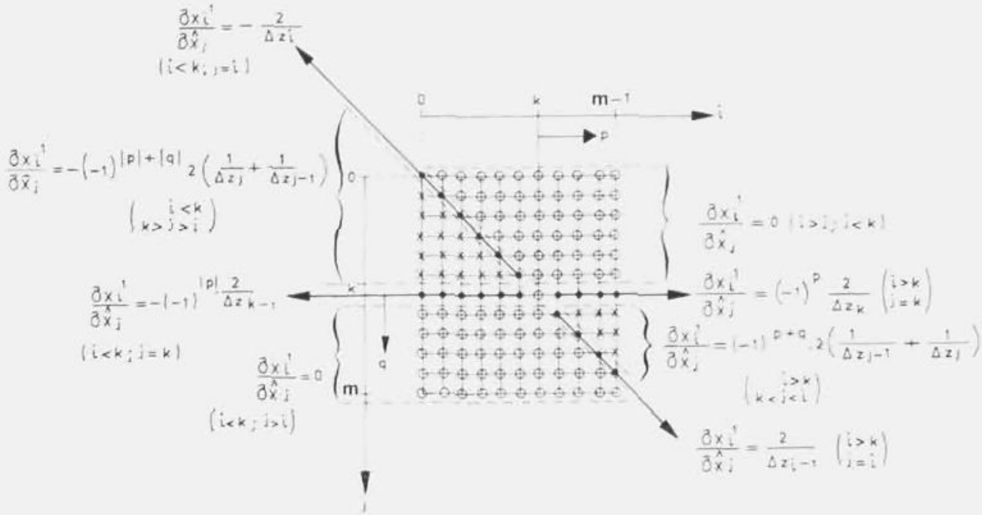


Fig. 4-33 Structure of matrix D

To calculate the part $i < k$ eq (4-38) is transformed by replacing i by (i-1) and changing positions of x'_i and x'_{i-1}, leading to:

$$x'_{i-1} = -x'_i+2\ \frac{\hat{x}-\hat{x}_{i-1}}{\Delta z_{i-1}} \tag{4-109}$$

Following the same procedure as for the part $i > k$ yields for $i = k-|p|$:

$$x'_{k-|p|} =(-1)^{|p|}\left[x'_k+2\left\{\frac{\hat{x}_k}{\Delta z_{k-1}} + \sum_{|q|=1}^{|p|-1}(-1)^{|q|}\hat{x}_{k-|q|}\left(\frac{1}{\Delta z_{k-|q|}} + \frac{1}{\Delta z_{k-|q|-1}}\right)\right\}\right] + 2\ \frac{\hat{x}_{k-|p|}}{\Delta z_{k-|p|}} \tag{4-110}$$

From this it can be derived that:

– for $i=k-|p|<k$, $j=k$:

$$\frac{\partial x'_{k-|p|}}{\partial \hat{x}_k} = (-1)^{|p|} . \frac{2}{\Delta z_{k-1}} \tag{4-111}$$

– for $i=k-|p|<k$, $j=i\ (=k-|p|)$:

$$\frac{\partial x'_{k-|p|}}{\partial \hat{x}_{k|p|}} = - \frac{2}{\Delta z_{k-|p|}} \tag{4-112}$$

– for $i=k-|p|<k$, $i=k-|p|<j<k$, $(j=k-|q|)$:

$$\frac{\partial x'_{k-|p|}}{\partial \hat{x}_{k-|q|}} = -(-1)^{|p|+|q|}\ 2\left(\frac{1}{\Delta z_{k-|q|}} + \frac{1}{\Delta z_{k-|q|-1}}\right) \tag{4-113}$$

In Fig. 4-33 alll different cases according to eqs (4-106, 4-107, 4-108, 4-111, 4-112 and 4-113) are indicated. In this way the full matrix can be completed. In this figure, for the reaches Δz the indices i and j are used instead of p and q.

For k = 0 only eqs (4-106, 4-107 and 4-108) hold: in this case the composition according to eq (4-81) is found.

Example

In the case of four equidistant stretches of unit length, if the slope of the central (= 3rd) point is fixed at 'zero' the matrix D will be:

$$D = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -2 & 2 & 0 & -2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrices M_1, M_2 and I_o, included in the matrix M of eq (4-90) remain unchanged. In the lower matrix $M\downarrow$ the first row includes an element 1 at the $(m+3)^{rd} = 8^{th}$ position; the other elements are zero.

For P = 1 eq (4-90) gives

$$
M=\left[\begin{array}{ccccc|ccccc}
2/3 & 1/3 & 0 & 0 & 0 & -1/3 & 0 & 0 & 0 & 0\\
1 & -1/3 & -1/3 & 0 & 0 & 1 & -1/3 & 0 & 0 & 0\\
-2/3 & 1 & 1 & -1 & 2/3 & -2/3 & 1/3 & 1/3 & -2/3 & 0\\
0 & 0 & -1/3 & 7/3 & -1 & 0 & 0 & -1/3 & 1 & 0\\
0 & 0 & 0 & -1/3 & 4/3 & 0 & 0 & 0 & -1/3 & 0\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
2 & -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0\\
0 & 2 & -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0\\
0 & 0 & 2 & -2 & 0 & 0 & 0 & 1 & 1 & 0\\
0 & 0 & 0 & 2 & -2 & 0 & 0 & 0 & 1 & 1
\end{array}\right]
$$

The first half of the inverse matrix is

$$
M^{-1}(1/2)=\left[\begin{array}{ccccc}
0,9145 & 0,1761 & -0,0649 & -0,0332 & 0,0076\\
0,1761 & 0,4252 & 0,2857 & 0,1462 & -0,0332\\
-0,0649 & 0,2857 & 0,5584 & 0,2857 & -0,0649\\
-0,0332 & 0,1462 & 0,2857 & 0,4252 & 0,1761\\
0,0076 & -0,0332 & -0,0649 & 0,1761 & 0,9145\\
\hline
-0,9949 & 0,7774 & 0,1558 & 0,0797 & -0,0181\\
-0,4820 & -0,2791 & 0,5455 & 0,2791 & -0,0634\\
0 & 0 & 0 & 0 & 0\\
0,0634 & -0,2791 & -0,5455 & 0,2791 & 0,4820\\
0,0181 & -0,0797 & -0,1558 & -0,7774 & 0,9949
\end{array}\right]
$$

This gives the following calculated values:

$$
\hat{X}=\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\\ \hat{x}_3\\ \hat{x}_4\\ \hat{x}_5\\ x_1'\\ x_2'\\ x_3'\\ x_4'\\ x_5'\end{bmatrix}= M^{-1}(1/2)\begin{bmatrix}2\\1\\4\\3\\1\end{bmatrix}=\begin{bmatrix}1,6535\\2,3256\\3,1817\\2,6743\\1,1652\\-0,3682\\1,7128\\0\\-1,0150\\-2,0032\end{bmatrix}
$$

The interpolation curve is shown in Fig. 4-34 and the Lagrange and propagation functions in Fig. 4-35.

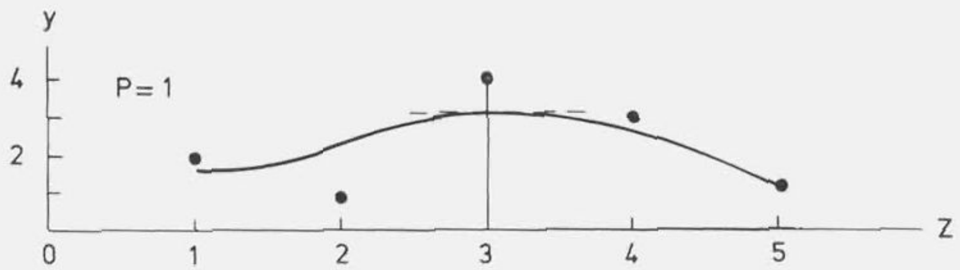


Fig. 4-34 Approximate spline function interpolation with horizontal tangent at z = 3.

4.5 Summary of mathematical interpolation methods

In this chapter interpolation methods were discussed which employ entire or partial polynomials passing exactly or approximately through the plotted points. Although the entire polynomials require less effort than the partial ones or spline functions, the latter appear to be more flexible.

It is possible to express the interpolation values as a linear equation of the measured values. The coefficients of this equation are the Lagrange functions. These provide a measure of the extent to which the measured values influence the final result. The influence of the measurement errors on the standard error of estimate is found by summing up the squared Lagrange functions, assumed that the measurement errors are inter-independent and have the same standard error.

For finding the variance of estimate as a whole, the variance of the model errors is also to be taken into account. The methods described in chapter 4 give no solution for this problem, but additional comparative measurements would be required. In Chapters 5 and 6 a more direct method is discussed, which makes use of the correlation structure.

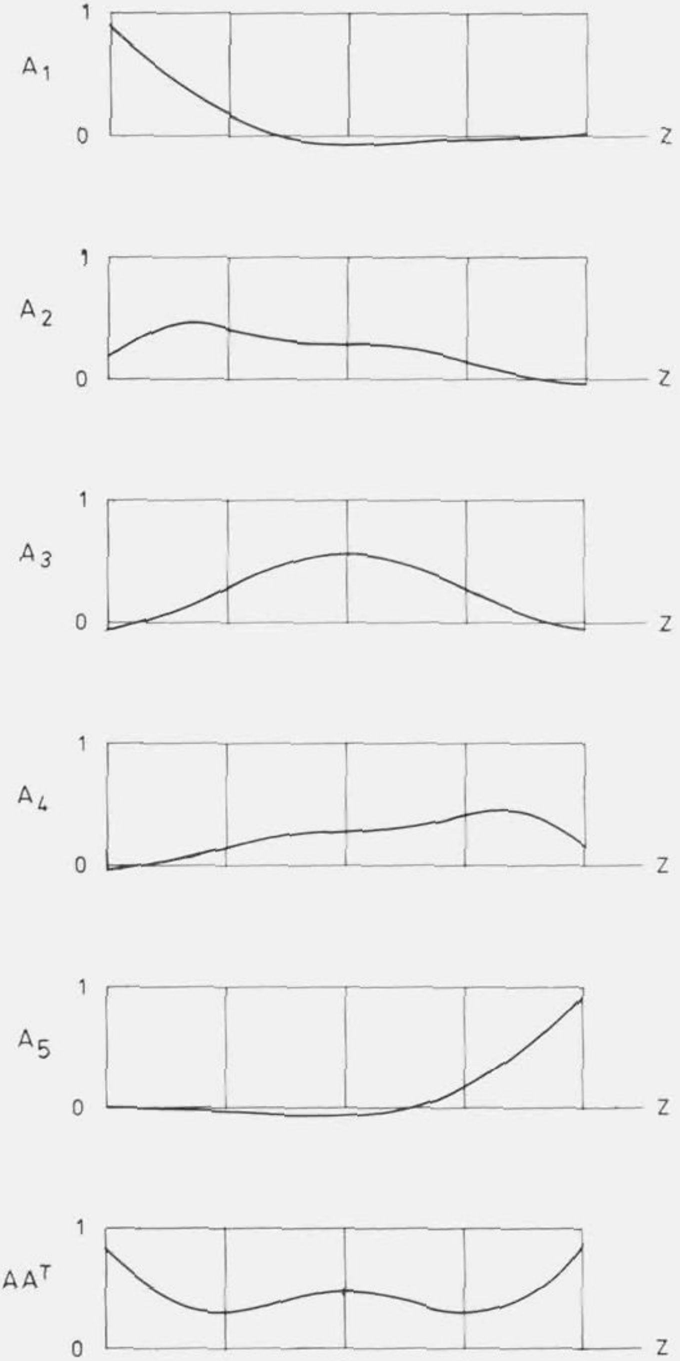


Fig. 4-35 Lagrange functions $A_1 \ldots A_5$ and propagation function AA^T for spline function interpolation with fixed slope $x_3'=0$ (P=1).

5. Interpolation by linear regression to sites with earlier measurements

5.1. General

The method described in this chapter is an interpolation method which makes use of the correlation structure between the variables (water levels) at the various sites along the river reach considered. It is a statistical method, since it is based on observations and not, like the methods described in Chapter 4, on assumptions such as the water surface profile being stretched as much as possible. Observations, made at an early stage at various existing and possible provisional stations, are used to examine the interrelations. Later on, a number of the stations might be discontinued. In the operational stage the levels at those sites can be calculated by using the interrelations previously established. Which and how many stations should be abandoned depends on whether the accuracy with which the water levels can be determined is sufficient to fulfil the assessed requirements.

In this connection three types of sites can be distinguished:

a. the sites where finally the network stations are established or maintained;
b. the sites where in the past or in the design stage measurements have been carried out for the assessment of the interrelations;
c. all other sites (and reaches) where measurements have never been or will be carried out.

The advantage of the method to be described is that, at least at the sites b, but to a certain extent also at the sites c, the derived water level data can be expected to correspond reasonably well with the true levels, since the relation is based on actual observations. This is not the case for the mathematical methods discussed in Chapter 4.

Another advantage is that, apart from an estimate of the water levels, an estimate can be made of their standard errors. In this way the feasibility of the network can be judged.

Since one cannot be sure that the interrelations will never change over the course of time, it is recommended that temporary measurements are made from time to time, in order to check the interrelations. However, it is not necessary to make continuous measurements using a high class network station. This policy can reduce the cost of *the measurement programme considerably. Sites b and c require a different approach*, and for this reason they will be discussed separately. The rest of Chapter 5 is devoted to sites b. Sites and reaches of category c will be dealt with in Chapter 6.

5.2. Determination of the values at the site under examination

For the relation between the levels at sites of interest and the levels at m network stations a linear model is assumed of the form

$$\hat{y} = A_0 + \sum_{i=1}^{m} A_i \underline{x}_i \qquad (5\text{-}1)$$

where:

$\hat{y}$	=	the calculated water level at the site under examination;
$\underline{x}_i$	=	the measured levels at the m network stations (i=1...m);
$A_0 \ldots A$	=	the regression coefficients.
m	=	the number of network stations.

In the first instance all levels y and x_i will be assumed to occur at the same time. Later *on, in Section 5.7, values of x_i at other times will be considered.*

The model of eq (5-1) should be such, that it is able to provide the best estimate of the values of y. Therefore the most feasible values should be given to the model parameters, which are in this case the regression coefficients $A_0 \ldots A_n$ These will be derived using historical measurements. Thus the model is in fact based on experience, whereby it is assumed that the former conditions will persist into the future.

As was discussed in Chapter 2 there are three kinds of data:

x_t,	y_t	=	the 'true' water levels;
$\underline{x}$,	$\underline{y}$	=	the measured water levels;
	$\hat{y}$	=	the calculated water levels.

If y is measured $\underline{y}$, a difference Δy can be expected with regard to $\hat{y}$. Compare eq (2-6)

$$\Delta y = \underline{y} - \hat{y} \qquad (5\text{-}2)$$

Substituting eq (5-1) into eq (5-2) yields:

$$\Delta y = \underline{y} - (A_o + \sum_{i=1}^{m} A_i \underline{x}_i) \qquad (5\text{-}3)$$

In order to derive the coefficients A_i (i = 0 ...m) this function is minimized according to the least squares principle. Therefore the following m+1 equations have to be solved:

$$\frac{\partial \sum_{j=1}^{n} (\Delta y_j)^2}{\partial A_i} = 0 \quad (i=0...m) \tag{5.4}$$

Let n sets of measurements $(\underline{x}_1...\underline{x}_m, y)_j$ (j=1...n) be available, where n>m.

Now introduce the following matrices:

$$X = \begin{bmatrix} 1\ x_{11}... & x_{m1} \\ \\ . & . \\ . & . \\ . & . \\ 1\ x_{1n}... & x_{mn} \end{bmatrix}; Y = \begin{bmatrix} y_1 \\ . \\ . \\ . \\ y_n \end{bmatrix}; A = \begin{bmatrix} A_0 \\ A_1 \\ . \\ . \\ . \\ A_m \end{bmatrix} \tag{5-5}$$

Thus for the generalized case, eq (5-1) can be written:

$$\hat{Y} = \underline{X}A \tag{5-6}$$

and eq (5-4) as:

$$\Delta Y = \underline{Y} - \hat{Y}, \tag{5-7}$$

which can, like eq (5-2), be transformed into:

$$\Delta Y = \underline{Y} - \underline{X}A \tag{5-8}$$

The sum of squares of Δy_j (j = 1...n) can be written as

$$\sum_{j=1}^{n} (\Delta y_j)^2 = \Delta Y^T.\Delta Y \tag{5-9}$$

This implies squaring of eq (5-8), so

$$\begin{aligned} \Delta Y^T.\Delta Y &= (\underline{Y} - \underline{X}A)^T (\underline{Y} - \underline{X}A) \\ &= \underline{Y}^T\underline{Y} - (\underline{X}A)^T\underline{Y} - Y^T(\underline{X}A) + (\underline{X}A)^T(\underline{X}A) \end{aligned} \tag{5-10}$$

This expression is a scalar. The 2nd and 3rd term have the same value and are both equal to $A^T\underline{X}^T\underline{Y}$. Thus

$$\Delta Y^T . \Delta Y = \underline{Y}^T\underline{Y} - 2A^T\underline{X}^T\underline{Y} + A^T\underline{X}^T\underline{X}A. \quad (5\text{-}11)$$

Differentiating $\Delta Y^T \, \Delta Y$ with regard to A and setting the result to zero yields

$$-2\,\underline{X}^T\underline{Y} + 2\,\underline{X}^T\underline{X}\,A = 0,$$

which gives

$$A = (\underline{X}^T\underline{X})^{-1}\,(\underline{X}^T\underline{Y}) \quad (5\text{-}12)$$

Using the result to obtain estimates of the regression coefficients A, the value $\hat{y}$ can easily be computed for future cases by using measured values $\underline{x}_i$ in eq (5-1). This equation can also be written in matrix notation as

$$\hat{y} = \underline{\vec{x}}\,A \quad (5\text{-}13)$$

where

$$\underline{\vec{x}} = [1\ \underline{x}_1 \ldots \underline{x}_m] \quad (5\text{-}14)$$

so that

$$\hat{y} = \underline{\vec{x}}\,(\underline{X}^T\underline{X})^{-1}\,(\underline{X}^T\underline{Y}). \quad (5\text{-}15)$$

However, due to model errors and measurement errors of $\underline{x}_i$, the resulting $\hat{y}$ will differ from the 'true' value y_t by an amount $\Delta\hat{y}$, called the error of estimate and defined by eq (2-5) as

$$\Delta y = \hat{y} - y_t. \quad (5\text{-}16)$$

Note that the errors of measurement were defined by eq (2-2) as

$$\left.\begin{aligned} \Delta\underline{x} &= \underline{x} - x_t \\ \Delta y &= \underline{y} - y_t \end{aligned}\right\} \quad (5\text{-}17)$$

Further it was defined by eq (2-6) that

$$\Delta y = \underline{y} - \hat{y},$$

from which it can be derived that (5-18)

$$\Delta y = (\underline{y} - y_t) - (\hat{y} - y_t) = \Delta \underline{y} - \Delta \hat{y} \tag{5-19}$$

A consequence of the use of matrix A, as calculated according to eq (5-12), is that the mean value of the elements Δy to be combined in a (n×1) matrix ΔY is 0, that is

$$\frac{1}{n} \sum_{j=1}^{n} \Delta y_j = \Delta y = 0 \tag{5-20}$$

This can be seen as follows. From eq (5-12) it can be derived that

$$\underline{X}^T \underline{X} A = \underline{X}^T \underline{Y} \tag{5-21}$$

or, when written in more detail *):

$$\begin{bmatrix} n & \Sigma x_1 & \dots & \Sigma \underline{x}_m \\ \Sigma \underline{x}_1 & \Sigma \underline{x}_1^2 & & \Sigma \underline{x}_1 \underline{x}_m \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \Sigma x_m & \Sigma \underline{x}_m x_1 & \dots & \Sigma \underline{x}_m^2 \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_m \end{bmatrix} = \begin{bmatrix} \Sigma \underline{y} \\ \Sigma x_1 \underline{y} \\ \cdot \\ \cdot \\ \cdot \\ \Sigma \underline{x}_m \underline{y} \end{bmatrix} \tag{5-22}$$

This represents a set of linear equations, the first of which is

$$n\,A_0 + A_1\,\Sigma \underline{x}_1 \dots + A_m\,\Sigma \underline{x}_m = \Sigma \underline{y}$$

or

$$A_0 + A_1 \frac{\Sigma \underline{x}_1}{n} \dots + A_m \frac{\Sigma \underline{x}_m}{n} = \frac{\Sigma \underline{y}}{n}, \tag{5-23}$$

so that

$$A_0 + \sum_{i=1}^{m} A_i \bar{\underline{x}}_i = \bar{\underline{y}} \tag{5-24}$$

*) Here $\Sigma \underline{x}_1$ stands for $\sum_{j=1}^{n} x_{1j}$ etc.

But from eq (5-1) it follows, because of its linear character, that

$$A_0 + \sum_{i=1}^{m} A_i \bar{x}_i = \hat{y} \qquad (5\text{-}25)$$

Comparison of eq (5-24) with eq (5-25) shows that

$$\hat{y} = \bar{y} \qquad (5\text{-}26)$$

and, using eq (5-2), then

$$\bar{\Delta}\hat{y} = \bar{y} - \hat{y} = 0 \qquad (5\text{-}27)$$

5.3. The variance of the errors of estimate

The variance of $\Delta\hat{y}$, i.e. the variance of the errors of estimate, in the following called the variance of estimate, can be derived as follows.

Assuming independence between the errors of measurement Δy and those of estimatey $\Delta\hat{y}$ it follows from eq (5-19) that

$$\text{Var}\ \Delta y = \text{Var}\ \Delta y + \text{Var}\ \Delta\hat{y} \qquad (5\text{-}28)$$

Since:

$$\text{Var}\ \Delta y = \varepsilon_y^2$$

it follows that:

$$\text{Var}\ \Delta y = \varepsilon_y^2 + \text{Var}\ \Delta\hat{y} \qquad (5\text{-}29)$$

and:

$$\text{Var}\ \Delta\hat{y} = \text{Var}\ \Delta y - \varepsilon_y^2 \qquad (5\text{-}30)$$

Thus, if Var Δy and ε_y are known, Var $\Delta\hat{y}$ follows easily.

The sample variance of the errors Δy is

$$\text{Var } \Delta y = \frac{\sum_{j=1}^{n} (\Delta y_j - \bar{\Delta}\bar{y})^2}{n}, \tag{5-31}$$

or, because of eq (5-27) ,

$$\text{Var } \Delta y = \frac{\sum_{j=1}^{n} (\Delta y_j)^2}{n}. \tag{5-32}$$

The numerator of eq (5-32) noting eqs (5-9) and (5-11) may be expressed as

$$\begin{aligned} \sum_{j=1}^{n} (\Delta y_j)^2 &= \Delta Y^T \Delta Y \\ &= \underline{Y}^T\underline{Y} - 2A^T\underline{X}^T\underline{Y} + A^T\underline{X}^T\underline{X}A \end{aligned} \tag{5-33}$$

For A the expression of eq (5-12) has been derived. Substitution of this into eq (5-33) leads to

$$\begin{aligned} \sum_{j=1}^{n} (\Delta y_j)^2 &= \underline{Y}^T\underline{Y} - 2\,\{(\underline{X}^T\underline{X})^{-1}\,\underline{X}^T\underline{Y}\}\,{}^T\underline{X}^T\underline{Y} + \{(\underline{X}^T\underline{X})^{-1}\underline{X}^T\underline{Y}\}^T\underline{X}^T\underline{X}\,(\underline{X}^T\underline{X})^{-1}\,\underline{X}^T\underline{Y} \\ &= \underline{Y}^T\underline{Y} - 2\,\{(\underline{X}^T\underline{X})^{-1}\,\underline{X}^T\underline{Y}\}\,{}^T\underline{X}^T\underline{Y} + \{(\underline{X}^T\underline{X})^{-1}\,\underline{X}^T\underline{Y}\}\,{}^T\underline{X}^T\underline{Y} \\ &= \underline{Y}^T\underline{Y} - \{(\underline{X}^T\underline{X})^{-1}\,\underline{X}^T\underline{Y}\}\,{}^T\underline{X}^T\underline{Y} \\ &= \underline{Y}^T\underline{Y} - (\underline{X}^T\underline{Y})^T\,\{(\underline{X}^T\underline{X})^{-1}\}\,{}^T\underline{X}^T\underline{Y} \end{aligned} \tag{5-34}$$

In Annex III it is explained that an expression for the variance of Δy can be further developed, resulting in:

$$\text{Var } \Delta y = \frac{|R\,(\underline{XY})|}{|R\,(\underline{XX})|}\,\text{Var } \underline{y} \tag{5-35}$$

where $|R\,(\underline{XY})|$ and $|R\,(\underline{XX})|$ denote the determinants of the correlation matrices as defined in eq (III- 18). Eq (5-35) expresses the unexplained variance of $\underline{y}$.

From this the standard error is

$$\sigma\Delta y = \sqrt{\frac{|R(\underline{X}\,\underline{Y})|}{|R(\underline{X}\,\underline{X})|}}\;\sigma\underline{y} \qquad (5\text{-}36)$$

From eqs (5-35) and (5-30) it easily follows that

$$\text{Var}\;\Delta\hat{y} = \frac{|R(\underline{X}\,\underline{Y})|}{|R(\underline{X}\,\underline{X})|}\;\text{Var}\;\underline{y} - \varepsilon_y^2 \qquad (5\text{-}37)$$

and:

$$\sigma\Delta\hat{y} = \sqrt{\frac{|R(\underline{X}\underline{Y})|}{|R(\underline{X}\,\underline{X})|}\;\text{Var}\;\underline{y} - \varepsilon_y^2}. \qquad (5\text{-}38)$$

5.4 Effect of measurement error on the estimate ŷ and on the variance of estimate

The regression matrix A, calculated according to eq (5-12) is based on measured values, occurring in the matrices $\underline{X}$ and $\underline{Y}$. The value $\hat{y}$, estimated later on according to eq (5-13) is influenced by measurement errors introduced by the values in $\vec{\underline{x}}$, given by eq (5-14).

5.4.1. The expected value of a measured level and its variance of estimate.

First consider the values of the independent variables, i.e. those produced by the network stations. For a single variable x, the 'true' value x_t and the measured value $\underline{x}$ are related by (see also eq (2-4))

$$\underline{x} = x_t + \Delta\underline{x}. \qquad (5\text{-}39)$$

If the true value x_t is assumed to be independent of the measurement error $\Delta\underline{x}$, then the variance of $\underline{x}$ is

$$\text{Var}\;\underline{x} = \text{Var}\;x_t + \text{Var}\Delta\underline{x} \qquad (5\text{-}40)$$

or

$$\text{Var}\;\underline{x} = \text{Var}\;x_t + \varepsilon_x^2, \qquad (5\text{-}41)$$

where

ε_x = the standard error of measurement of x.

The covariance between $\underline{x}$ and x_t can be calculated using

$$\text{Cov}\ \underline{x}\ x_t = \frac{\Sigma\ (\underline{x} - \bar{\underline{x}})(x_t - \bar{x}_t)_j}{n} {}^{*)} \tag{5-42}$$

where $\bar{\underline{x}}$ and $\bar{x}_t$ denote mean values.

Further elaboration yields

$$\begin{aligned}
\text{Cov}\ \underline{x}\ x_t &= \frac{\Sigma\underline{x}\ x_t - n\ \bar{\underline{x}}\bar{x}_t}{n} \\
&= \frac{\Sigma\bar{\underline{x}}\ x_t}{n}\ \bar{\underline{x}}\ \bar{x}_t \\
&= \frac{\Sigma\ (x_t + \Delta\underline{x})\ x_t}{n} - (\bar{x}_t + \bar{\Delta}\bar{x})\ \bar{x}_t \\
&= \frac{\Sigma x_t^2 + \Sigma\Delta\underline{x}x_t}{n} - \bar{x}_t^2 - \bar{\Delta}\bar{\underline{x}}\ \bar{x}_t \\
&= \frac{\Sigma x_t^2}{n} - \bar{x}_t^2 + \frac{\Sigma\Delta\underline{x}x_t}{n} - \bar{\Delta}\bar{\underline{x}}\ \bar{x}_t \\
&= \text{Var}\ x_t + \text{Cov}\ \Delta\underline{x}\ x_t
\end{aligned} \tag{5-43}$$

or, since it was assumed that x_t and $\Delta\underline{x}$ were independent the second right hand term of eq (5-43) is 0, so

$$\text{Cov}\ \underline{x}\ x_t = \text{Var}\ x_t \tag{5-44}$$

The following relation between x_t and $\underline{x}$ can be used to obtain the expected value of x_t:

$$E\ (x_t - \bar{x}_t) = \frac{\text{Cov}\ \underline{x}\ x_t}{\text{Var}\ \underline{x}}\ (\underline{x} - \bar{\underline{x}}) \tag{5-45}$$

*) The sign Σ stands for $\sum_{j=1}^{n}$

Substitution of eqs (5-41) and (5-44) into eq (5-45) yields

$$E\,(x_t\text{-}\bar{x}_t) \quad = \frac{\text{Var } x_t}{\text{Var } x_t + \varepsilon_x^{\,2}}\,(\underline{x} - \bar{\underline{x}}) \tag{5-46}$$

If there are no systematic measurement errors then

$$\bar{x}_t = \bar{\underline{x}}. \tag{5-47}$$

The variance of $E\,(x_t\text{-}\bar{x}_t)$ is

$$\text{Var } E\,(x_t\text{-}\bar{x}_t) = \text{Var } E\,(x_t), \tag{5-48}$$

and also:

$$\text{Var }(\underline{x} - \bar{\underline{x}}) = \text{Var } \underline{x} \tag{5-49}$$

It follows, using eq (5-46) that

$$\text{Var } E\,(x_t) \quad = \left(\frac{\text{Var } x_t}{\text{Var } x_t + \varepsilon_x^{\,2}}\right)^2 \text{Var } \underline{x} \tag{5-50}$$

This reduction of Var $\underline{x}$ to Var E $(\underline{x}_t)$ is stronger than that of Var $\underline{x}$ to Var x_t, which, obviously, is

$$\text{Var } x_t \quad = \frac{\text{Var } x_t}{\text{Var } x_t + \varepsilon_x^{\,2}}\,\text{Var } \underline{x} \tag{5-51}$$

The reason for this difference in reduction is that eq (5-50) is based on a unique relation, according to eq (5-46), but eq (5-51) is based on regression considerations which include stochastic elements.

This means that the variance of the expected values is somewhat smaller than the variance of the true values themselves; further elaboration of eq (5-50) yields

$$\begin{aligned} \text{Var } E\,(x_t) \quad &= \left(\frac{\text{Var } x_t}{\text{Var } x_t + \varepsilon_x^{\,2}}\right)^2 (\text{Var } x_t + \varepsilon_x^{\,2}) \\ &= \frac{\text{Var } x_t}{\text{Var } x_t + \varepsilon_x^{\,2}}\,\text{Var } x_t. \end{aligned} \tag{5-52}$$

Note that the reduction has the same ratio as the reductions, found in eqs (5-46) and (5-51).

The correlation coefficient between $\underline{x}$ and x_t can be found according to:

$$\varrho(\underline{x}\ x_t) = \frac{\mathrm{Cov}\ \underline{x}\ x_t}{\sqrt{\mathrm{Var}\ \underline{x}\ \mathrm{Var}\ x_t}} \tag{5-53}$$

Substitution of eqs (5-41) and (5-44) into eq (5-53) yields

$$\varrho(\underline{x}\ x_t) = \frac{\mathrm{Var}\ x_t}{\sqrt{(\mathrm{Var}\ x_t + \varepsilon_x^2)\mathrm{Var}\ x_t}}$$

$$= \sqrt{\frac{\mathrm{Var}\ x_t}{\mathrm{Var}\ x_t + \varepsilon_x^2}}, \tag{5-54}$$

or, expressed in terms of $\underline{x}$:

$$\varrho(\underline{x}\ x_t) = \sqrt{1 - \frac{\varepsilon_x^2}{\mathrm{Var}\ \underline{x}}} \tag{5-55}$$

From this the residual (i.e. unexplained) variance, i.e. the variance of $E(x_t|\underline{x})$ can be calculated*):

$$\mathrm{Var}\ E\ (x_t|\underline{x}) = \{1 - \varrho^2(\underline{x}x_t)\}\ \mathrm{Var}\ x_t$$

$$= \frac{\varepsilon_x^2}{\mathrm{Var}\ \underline{x}}\ \mathrm{Var}\ x_t$$

$$= \frac{\mathrm{Var}\ x_t}{\mathrm{Var}\ x_t + \varepsilon_x^2}\ \varepsilon_x^2. \tag{5-56}$$

Note that the same reduction factor as in eqs (5-46) and (5-52) appears again. In fact $\mathrm{Var}\ E\ (x_t|\underline{x})$ is the variance of estimate of the level, for which measurements are also made.

*) $E\ (x_t|\underline{x})$ denotes the expected value of x_t if for this value a measurement $\underline{x}$ is found.

5.4.2. The expected value of a calculated level (in the two-dimensional case).

So far the influence of measurement error to an independent variable has been considered. Now the effect of measurement errors of a dependent variable will be discussed.

Consider a two dimensional relation

$$E\,(y_t-\bar{y}_t) = A_1\,(\underline{x}-\bar{\underline{x}}) \tag{5-57}$$

where according to eq (5-12), A_1 is given by

$$A_1 = \frac{\text{Cov}\ \underline{x}\ \underline{y}}{\text{Var}\ \underline{x}} \tag{5-58}$$

If the measurement errors of both variables are independent of the true values and if also these measurement errors are mutually independent, then

$$\text{Cov}\ \underline{x}\ \underline{y} = \text{Cov}\ x_t\ \underline{y} = \text{Cov}\ \underline{x}\ y_t = \text{Cov}\ x_t\ y_t \tag{5-59}$$

In the following the covariance will simply be denoted by Cov xy. Thus eq (5-58) may be written as

$$A_1 = \frac{\text{Cov}\ xy}{\text{Var}\ \underline{x}} \tag{5-60}$$

If for both variables the 'true' values are considered, then the regression coefficient becomes

$$A_{1t} = \frac{\text{Cov}\ xy}{\text{Var}\ x_t}, \tag{5-61}$$

so that

$$A_{1t} = \frac{\text{Var}\ \underline{x}}{\text{Var}\ x_t}\ A_1. \tag{5-62}$$

Now consider the covariance between the expected value $E\,(x_t)$ and the true value y_t, i.e. $\text{Cov}\ \{E(x_t),\ y_t\}$. Then

$$\text{Cov}\ \{E\,(x_t),\ y_t\} = \frac{\Sigma E\,(x_t-\bar{x}_t).\ (y_t-\bar{y}_t)}{n} \tag{5-63}$$

or, because of eq (5-46)

$$\text{Cov}\{E(x_t), y_t\} = \frac{\text{Var } x_t}{\text{Var } x_t + \varepsilon_x^2} \cdot \frac{\Sigma(\underline{x}-\bar{\underline{x}})(y_t-\bar{y}_t)}{n}$$

$$= \frac{\text{Var } x_t}{\text{Var } x_t + \varepsilon_x^2} \text{ Cov } xy \qquad (5\text{-}64)$$

The regression coefficient A_{1E} between the expected value $E(x_t)$ and the true value y_t can be calculated using

$$A_{1E} = \frac{\text{Cov}\{E(x_t). y_t\}}{\text{Var } E(x_t)}. \qquad (5\text{-}65)$$

Substitution of eqs (5-50) and (5-64) into eq (5-65) yields

$$A_{1E} = \frac{\text{Var } x_t}{\text{Var } x_t + \varepsilon_x^2} \left(\frac{\text{Var } x_t + \varepsilon_x^2}{\text{Var } x_t} \right)^2 \frac{\text{Cov } xy}{\text{Var } \underline{x}}$$

$$= \frac{\text{Cov } xy}{\text{Var } x_t} \qquad (5\text{-}66)$$

This is equal to the 'true' regression coefficient A_{1t}, given by eq (5-61).

For the calculation of $E(y_t-\bar{y}_t)$ one might prefer to use this 'true' regression coefficient A_{1t} instead of the 'raw' regression coefficient A_1 of eq (5-60). But in that case use should be made of the true value $x_t-\bar{x}_t$ in the application of eq (5-57). However, since this true value is unknown, its expected value $E(x_t-\bar{x}_t)$ must be used. This makes no difference to the subsequent calculations since A_{1t} and A_{1E} are equal, according to eqs (5-61) and (5-66).

Eq (5-57) is elaborated as follows:

$$E(y_t-\bar{y}_t) = A_{1t}E(x_t-\bar{x}_t) \qquad (5\text{-}67)$$

Using eqs (5-46) and (5-62) then

$$E(y_t-\bar{y}_t) = \frac{\text{Var } \underline{x}}{\text{Var } x_t} A_1 \frac{\text{Var } x_t}{\text{Var } x_t + \varepsilon_x^2} (\underline{x}-\bar{\underline{x}})$$

$$= A_1(\underline{x}-\bar{\underline{x}}) \qquad (5\text{-}68)$$

This is the same result as was found when using directly the measured value $\underline{x}$ (see eq (5-57)). It therefore makes no sense to transform the measured value into the expectation of the true value.

The foregoing considerations were based on the assumption that that measurement errors $\Delta \underline{x}$ in the data series for which A_1 was calculated (the period of analysis), had the same variance as the measurement errors in the period, when the model is used operationally to calculate E $(y_t - \bar{y}_t)$. If those errors do not have the same variance, then the final result is affected, as can be seen from

$$E\,(y_t-\bar{y}_t) = \frac{\mathrm{Var}\;x_t + \varepsilon_{xa}^2}{\mathrm{Var}\;x_t + \varepsilon_{xo}^2}\; A_1\,(x-\bar{x}), \qquad (5\text{-}69)$$

where ε_{xa} = the standard error of measurement in the period of analysis,
ε_{xo} = the standard error of measurement in the period of operation.

Now consider the systematic errors of measurement, Δy_s. If there are no such errors then

$$E\,(\bar{y}_t) = \bar{\underline{y}}. \qquad (5\text{-}70)$$

If such errors do occur, then

$$E\,(\bar{y}_t) = \bar{\underline{y}} - \Delta y_s. \qquad (5\text{-}71)$$

Now eq (5-68) can be written as

$$E\,(y_t) = A_1\underline{x} - A_1\bar{\underline{x}} + \bar{y} - \Delta y_s. \qquad (5\text{-}72)$$

The last three terms are in fact the constant term A_0 of eq (5-1). If Δy_s is unknown one can only calculate the value

$$E\,(y_t + \Delta y_s) = \bar{A}_0 + A_1\underline{x}, \qquad (5\text{-}73)$$

where, in this case,

$$\bar{A}_0 = -\,A_1\,\bar{\underline{x}} + \bar{\underline{y}}. \qquad (5\text{-}74)$$

If Δy_s is known, this can be incorporated into A_o so that

$$A_0 = \bar{A}_0 - \Delta y_s, \qquad (5\text{-}75)$$

and therefore

$$A_0 = - A_1 \bar{\underline{x}} + \bar{\underline{y}} - \Delta y_s. \tag{5-76}$$

Since systematic errors of $\underline{y}$ directly influence the result of $E(y_t)$ it is of importance to reduce them as much as possible. If these errors are known (or equal to 0) then using eq (5-76)

$$E(y_t) = A_0 + A_1\underline{x} \tag{5-77}$$

Since the expected value $E(y_t)$ is obtained as the calculated value $\hat{y}$, eq (5-77) is the two dimensional case of eq (5-1):

$$\hat{y} = A_0 + A_1\underline{x} \tag{5-78}$$

5.4.3. The variance of estimate of a calculated level (in the two-dimensional case).

This concerns the variance of $\Delta\hat{y}$ when $\hat{y}$ is calculated using the expected values $E(x_t)$ instead of the measured values $\underline{x}$. Recall the definition of eq (2-5), i.e.

$$\Delta\hat{y} = \hat{y} - y_t$$

which, for a specific value of y, becomes

$$\Delta\hat{y} = (\hat{y} - y_t) \mid \hat{y}$$

or

$$\Delta\hat{y} = \hat{y}|\hat{y} - y_t|\hat{y}. \tag{5-79}$$

Since the first right hand term is fixed, then

$$\text{Var } \Delta\hat{y} = \text{Var } y_t|\hat{y} \tag{5-80}$$

Since $\hat{y}$ is derived from $\underline{x}$ through the fixed relation of eq (5-78), then

$$\text{Var } y_t|\hat{y} = \text{Var } y_t|\underline{x}. \tag{5-81}$$

This can be calculated using the relation

$$\text{Var } y_t|\underline{x} = \{1 - \varrho^2(\underline{x}\ y_t)\} \text{ Var } y_t{}^{*)} \tag{5-82}$$

*) Var $y_t|\underline{x}$ is the unexplained variance of y_t when correlated with the independent variable $\underline{x}$.

where $\varrho(\underline{x}\ y_t)$ is the correlation coefficient between $\underline{x}$ and y_t. This is

$$\varrho(\underline{x}\ y_t) = \frac{\text{Cov } \underline{x}\ y_t}{\sqrt{\text{Var } \underline{x} \text{ Var } y_t}}$$

or, because of eq (5-59)

$$\varrho(\underline{x}\ y_t) = \frac{\text{Cov } xy}{\sqrt{\text{Var } \underline{x} \text{ Var } y_t}} \tag{5-83}$$

When working with the expected values of x_t, i.e. $E(x_t)$, instead of with the measured values $\underline{x}$, then the correlation coefficient between y_t and $E(x_t)$, should be used; this is

$$\varrho\{E(x_t),\ y_t\} = \frac{\text{Cov } E(x_t),\ y_t}{\sqrt{\text{Var } E(x_t), \text{ Var } y_t}}. \tag{5-84}$$

The covariance was derived, according to eq (5-64), as

$$\text{Cov } E(x_t),\ y_t = \frac{\text{var } x_t}{\text{Var } x_t + \varepsilon_x^2} \text{ Cov } xy.$$

Further, referring to eq (5-52),

$$\text{Var } E(x_t) = \frac{\text{Var } x_t}{\text{Var } x_t + \varepsilon_x^2} \text{ Var } x_t$$

Substitution of eqs (5-52) and (5-64) into eq (5-84) yields

$$\varrho\{E(x_t),\ y_t\} = \frac{\dfrac{\text{Var } x_t}{\text{Var } x_t + \varepsilon_x^2} \text{ Cov } xy}{\sqrt{\dfrac{\text{Var } x_t}{\text{Var } x_t + \varepsilon_x^2} \text{ Var } x_t \text{ Var } y_t}} = \frac{\text{Cov } xy}{\sqrt{\text{Var } \underline{x} \text{ Var } y_t}} \tag{5-85}$$

or, with regard to eq (5-83):

$$\varrho\{E(x_t),\ y_t\} = \varrho(\underline{x}\ y_t). \tag{5-86}$$

Therefore it does not matter what correlation coefficient is used in eq (5-82) for calculating the variance of estimate Var $\Delta\hat{y}$. Eqs (5-80), (5-81), (5-82) and (5-86) show that

$$\text{Var}\ \Delta\hat{y} = \text{Var}\ y_t|\underline{x} = \text{Var}\ y_t|E(x_t). \qquad (5\text{-}87)$$

It makes no difference to the calculation of Var $\Delta\hat{y}$ whether the measured or the expected values of x are used.

5.4.4. The expected value of a calculated level and its variance of estimate (in the generalized case).

For the generalized case with more independent variables x_i (i = 1...m) similar results as in the case of only one variable x can be found.

For the estimation of y the following relation is used (with reference to eq (5-13))

$$E\ (y_t) = \hat{y} = \underline{\vec{x}}\ A, \qquad (5\text{-}88)$$

or

$$E\ (y_t) = A_0 + \sum_{i=1}^{m} A_i\ x_i. \qquad (5\text{-}89)$$

As in eq (5-17) A_0 includes possible systematic errors of $\underline{y}$.

According to eq (5-12) the matrix A was determined by:

$$A = (\underline{X}^T\underline{X})^{-1}\ (\underline{X}^T\underline{Y})$$

For the expected values of x_i and the true values of y this matrix transforms into

$$A_E = \{E\ (X_t)^T\ E(X_t)\}^{-1}\ \{E(X_t)^T Y_t\}, \qquad (5\text{-}90)$$

where $E(X_t)$ consists of elements

$$e_{ij} = x_{ijt} \qquad i = 1...m,\ \text{(columns)} \quad j = 1...n\ \text{(rows)} \qquad (5\text{-}91)$$

together with values 1 in the first column, as is the case in eq (5-5). Recall that m concerns the number of stations and n the number of measurements at a station.

Now let D be a matrix which transforms the transposed matrix of the measured values into that of the expected values according to

$$E\,(X_t)^T = D\,\underline{X}^T \qquad (5\text{-}92)$$

Substituting this into eq (5-90): gives

$$\begin{aligned} A_E &= \{D\,\underline{X}^T\,(D\,\underline{X}^T)^T\}^{-1}\,(D\,\underline{X}^T Y_t) \\ &= \{D\,(\underline{X}^T\underline{X})\,D^T\}^{-1}\,(D\,\underline{X}^T Y_t) \\ &= (D^T)^{-1}\,(\underline{X}^T\underline{X})^{-1}\,D^{-1}\,D\,(\underline{X}^T Y_t) \\ &= (D^T)^{-1}\,(\underline{X}^T\underline{X})^{-1}\,(\underline{X}^T Y_t) \end{aligned} \qquad (5\text{-}93)$$

Also

$$\begin{aligned} \hat{y} &= E\,(\vec{x}_t)\,A_E \\ &= (D\,\underline{\vec{x}}^T)^T\,A_E \\ &= \underline{\vec{x}}\,D^T.\,(D^T)^{-1}\,(\underline{X}^T\underline{X})^{-1}\,(\underline{X}^T Y_t) \qquad (5\text{-}94) \\ &= \underline{\vec{x}}\,(\underline{X}^T\underline{X})^{-1}\,(\underline{X}^T Y_t). \end{aligned}$$

Since the covariances do not change when replacing the true values by the measured values, neither do the sums of products $\Sigma x_i y_t$ for the corresponding matrix result holds:

$$\underline{X}^T Y_t = \underline{X}^T \underline{Y} \qquad (5\text{-}95)$$

Thus eq (5-94) can be replaced by

$$\hat{y} = \underline{\vec{x}}\,(\underline{X}^T\underline{X})^{-1}\,(\underline{X}^T\underline{Y}) \qquad (5\text{-}96)$$

which is the same result as given by eq (5-15). So, in this generalized case too, it makes no difference to $\hat{y}$ whether measured values $\underline{x}$ or expected values $E(x_t)$ are used.

For the sum of squares of the deviations Δy_j ($j = 1 \ldots n$), eq (5-34) was obtained

$$\sum_{j=1}^{n} \Delta y_j^2 = \underline{Y}^T\underline{Y} - (\underline{X}^T\underline{Y})^T\,(\underline{X}^T\underline{X})^{-1}\,\underline{X}^T\underline{Y}.$$

If expected values $E(X_t)$ were used in this computation instead of measured values $\underline{X}$, then one could calculate:

$$\underline{Y}^T\underline{Y} - \{E(\underline{X}_t)^T\underline{Y}\}^T \{E(X_t)^T E(X_t)^{-1}\} \{E(X_t)^T\underline{Y}\}$$

and, using eq (5-92), this is equal to

$$= \underline{Y}^T\underline{Y} - \{(D\underline{X}^T)\underline{Y}\}^T \{(D\underline{X}^T)(D\underline{X}^T)^T\}^{-1} \{(D\underline{X}^T)\underline{Y}\}$$

$$= \underline{Y}^T\underline{Y} - \{(D\underline{X}^T\underline{Y})^T\} \{(D\underline{X}^T\underline{X}D^T)^{-1}\} D\underline{X}^T\underline{Y}$$

$$= \underline{Y}^T\underline{Y} - (\underline{X}^T\underline{Y})^T D^T (D^T)^{-1} (\underline{X}^T\underline{X})^{-1} D^{-1} D\underline{X}^T\underline{Y}$$

$$= \underline{Y}^T\underline{Y} - (\underline{X}^T\underline{Y})^T (\underline{X}^T\underline{X})^{-1} \underline{X}^T\underline{Y},$$

which is equal to eq (5-34) and thus gives $\Sigma\Delta y_j^2$. Consequently the sum of squares of Δy_j does not change when $\underline{X}$ is replaced by $E(X_t)$. Thus Var Δy is also not influenced by this replacement, since according to eq (5-32)

$$\text{Var } \Delta y = \frac{\sum_{j=1}^{n} \Delta y_j^2}{n} .$$

Further, because of eq (5-23),

$$\text{Var } \Delta\hat{y} = \text{Var } \Delta y - \varepsilon_y^2,$$

so the variance of estimate also remains uninfluenced when the measured values $\underline{X}$ are replaced by the expected values $E(X_t)$.

5.5. Estimation of values of y and their confidence interval

After establishing the model parameters A_i ($i = 0...m$) the model will be applied later on to estimate $\hat{y}$, making use of a set of measured independent variables $\underline{x}_i$, ($i = 1...m$). In matrix form, expressed by eq (5-13):

$$\hat{y} = \underline{\vec{x}}\, A$$

where, as expressed by eqs (5-14) and (5-5),

$\underline{\vec{x}} = [1\ \underline{x}_1 \ldots \underline{x}_m]$,

$$A = \begin{bmatrix} A_0 \\ A_1 \\ . \\ . \\ . \\ A_m \end{bmatrix}$$

For each actual case, $\underline{\vec{x}}$ consists of given, fixed values. Matrix A, besides being based on a set of measurements of the dependent and the independent variables, is in fact the result of a sample drawn from a great number of combinations $(\underline{x}_1 \ldots \underline{x}_m, \underline{y})$. Therefore A includes standard errors, which in turn affect $\hat{y}$, for given measured values $\underline{\vec{x}}$.

When keeping $\underline{\vec{x}}$ fixed, the corresponding $\hat{y}$ value is

$$\hat{y} = E\ (y_t | \underline{\vec{x}}), \tag{5-97}$$

and therefore

$$E\ (y_t | \underline{\vec{x}}\ ) = \underline{\vec{x}} A. \tag{5-98}$$

The variance may be calculated as

$$\mathrm{Var}\ E\ (y_t | \underline{\vec{x}}\ ) = \underline{\vec{x}}\ V_A\ \underline{\vec{x}}^T \tag{5-99}$$

where

V_A = covariance matrix of A.

Matrix A was derived by eq (5-12) as

$$A = (\underline{X}^T \underline{X})^{-1}\ (\underline{X}^T \underline{Y})$$

or, defining

$$F = (\underline{X}^T \underline{X})^{-1}\ \underline{X}^T, \tag{5-100}$$

as

$$A = F\ \underline{Y}\ . \tag{5-101}$$

The covariance matrix is

$$V_A = F V_Y|_X F^T, \quad (5\text{-}102)$$

where $V_Y|_X$ = covariance matrix of $\underline{Y}$. The vector $\underline{Y}$ has the form (see also eq (5-5)):

$$\underline{Y} = \begin{bmatrix} \underline{y}_1 \\ . \\ . \\ . \\ \underline{y}_n \end{bmatrix}$$

Now $V_Y|_X$ can be written as:

$$V_Y|_X = \begin{bmatrix} \text{Var } \underline{y}_1|\underline{X} & \text{Cov } \underline{y}_1\underline{y}_2|\underline{X} & \ldots & \text{Cov } \underline{y}_1\underline{y}_n|\underline{X} \\ \text{Cov } \underline{y}_1\underline{y}_2|\underline{X} & \text{Var } \underline{y}_2|\underline{X} & & . \\ . & & & . \\ . & & & . \\ . & & & \\ \text{Cov } \underline{y}_1\,\underline{y}_n|X & \ldots & & \text{Var } \underline{y}_n|\underline{X} \end{bmatrix} \quad (5\text{-}103)$$

Var $\underline{y}_j|\underline{X}$ (j = 1...n) is the variance of $\underline{y}_j$ when the corresponding set of $\underline{x}_{ij}$ values (i = 1...m) is fixed, or because of the relation used, when $\hat{y}_j$ is fixed. Thus it is the variance of y_j-$\hat{y}_j$, i.e. of Δy_j, so that

$$\text{Var } \underline{y}_j|\underline{X} = \text{Var } \Delta y_j \quad (5\text{-}104)$$

Assuming also an equal value Var Δy for all Var Δy_j (j = 1...n) eq (5-103) can be converted into:

$$V_Y|_X = \begin{bmatrix} 1 & \varrho\Delta y_1\,\Delta y_2 & \ldots & \varrho\Delta y_1\Delta y_n \\ \varrho\Delta y_1\Delta y_2 & 1 & & . \\ . & & & . \\ . & & & . \\ . & & & \\ \varrho\Delta y_1\Delta y_n & & \ldots & 1 \end{bmatrix} \text{Var}\Delta y$$

(5-105)

or

$$V_Y|_X = R_{\Delta Y} \text{Var } \Delta y, \quad (5\text{-}106)$$

where $R_{\Delta Y}$ denotes the correlation matrix of the Δy-values. Since it concerns a series of subsequently measured y-values this is in fact the autocorrelation matrix.

Substituting eq (5-106) into eq (5-102) gives

$$V_A = F\, R_{\Delta Y}\, F^T \,\text{Var}\,\Delta y \qquad (5\text{-}107)$$

and substituting this into eq (5-99) yields

$$\text{Var E}\,(y_t|\underline{\vec{x}}) = \underline{\vec{x}}\, F\, R_{\Delta y} F^T \underline{\vec{x}}^T.\ \text{Var}\,\Delta y. \qquad (5\text{-}108)$$

This is the variance of the expected value of y_t. For an individual case, this variance should be increased by an amount (see eq (5-16))

$$\text{Var}\,(\hat{y}\text{-}y_t) = \text{Var}\,\Delta\hat{y}. \qquad (5\text{-}109)$$

Thus, for an individual estimation $\hat{y}_k|\underline{\vec{x}}$

$$\text{Var}\,\hat{y}_k|\underline{\vec{x}} = \underline{\vec{x}}\, F\, R_{\Delta Y}\, F^T \underline{\vec{x}}^T\, \text{Var}\,\Delta y + \text{Var}\,\Delta\hat{y} \qquad (5\text{-}110)$$

or, because of eq (5-30),

$$\text{Var}\,\hat{y}_k|\underline{\vec{x}} = (\underline{\vec{x}}\, F\, R_{\Delta Y}\, F^T \underline{\vec{x}}^T + 1)\,\text{Var}\,\Delta y - \varepsilon_y^2. \qquad (5\text{-}111)$$

A special case occurs if no autocorrelation exists between the Δy-values. Then

$$R_{\Delta Y} = I \qquad (5\text{-}112)$$

where: I = the n × n unit matrix,

which converts eq (5-111) into

$$\text{Var}\,\hat{y}_k|\underline{\vec{x}} = \underline{\vec{x}}\, FF^T\, \underline{\vec{x}}^T + 1)\,\text{Var}\,\Delta y - \varepsilon_y^2, \qquad (5\text{-}113)$$

or, substituting eq (5-100),

$$\text{Var}\,\hat{y}_k|\underline{\vec{x}} = [\underline{\vec{x}}\,\{(\underline{X}^T\underline{X})^{-1}\,\underline{X}^T\}\,\{(\underline{X}^T\underline{X})^{-1}\underline{X}^T\}^{\,T}\underline{\vec{x}}_T + 1]\,\text{Var}\,\Delta y - \varepsilon_y^2. \qquad (5\text{-}114)$$

Now

$$\begin{aligned}&\{(\underline{X}^T\underline{X})^{-1}\,\underline{X}^T\}\,\{(\underline{X}^T\underline{X})^{-1}\,\underline{X}^T\}^{\,T}\\ &= (\underline{X}^T\underline{X})^{-1}\,(\underline{X}^T\underline{X})\,\{(\underline{X}^T\underline{X})^{-1}\}^{\,T}\\ &= \{(\underline{X}^T\underline{X})^{-1}\}^{\,T} = (\underline{X}^T\underline{X})^{-1}\end{aligned} \qquad (5\text{-}115)$$

which, substituted into eq (5-114), yields

$$\text{Var } \hat{y}_k | \underline{\vec{x}} = [\vec{x} \{(\underline{X}^T\underline{X})^{-1}\} \underline{\vec{x}}^T + 1] \text{ Var } \Delta y - \varepsilon_y^2 \qquad (5\text{-}116)$$

If the model consists of two terms only, so

$$\hat{y} = A_0 + A_1\underline{x}, \qquad (5\text{-}117)$$

then

$$\underline{\vec{x}} = [1\ \underline{x}] \text{ and } A = \begin{bmatrix} A_0 \\ . \\ A_1 \end{bmatrix} \qquad (5\text{-}118)$$

The matrix $\underline{X}^T\underline{X}$ is now*):

$$\underline{X}^T\underline{X} = \begin{bmatrix} n & \Sigma\underline{x} \\ \Sigma\underline{x} & \Sigma\underline{x}^2 \end{bmatrix} \qquad (5\text{-}119)$$

If the coordinate system is chosen in such a way that

$$\Sigma\underline{x} = 0, \qquad (5\text{-}120)$$

then

$$\underline{X}^T\underline{X} = \begin{bmatrix} n & 0 \\ 0 & \Sigma\underline{x}^2 \end{bmatrix}$$

and

$$(X^TX)^{-1} = \begin{bmatrix} \frac{1}{n} & 0 \\ 0 & \frac{1}{\Sigma\underline{x}^2} \end{bmatrix} \qquad (5\text{-}121)$$

Substitution of eqs (5-119) and (5-121) into eq (5-116) leads to:

*) Σx stands for $\sum_{j=1}^{n} x_j$

$$\text{Var } \hat{y}_k|\underline{\vec{x}} = \left(\begin{bmatrix} 1 & \underline{x} \end{bmatrix} \begin{bmatrix} \frac{1}{n} & 0 \\ 0 & \frac{1}{\Sigma \underline{x}^2} \end{bmatrix} \begin{bmatrix} 1 \\ \underline{x} \end{bmatrix} + 1 \right) \text{Var } \Delta y\text{-}\varepsilon_y^{\,2}$$

$$= \left(\frac{1}{n} + \frac{\underline{x}^2}{\Sigma \underline{x}^2} + 1 \right) \text{ Var } \Delta y - \varepsilon_y^{\,2} \tag{5-122}$$

The expected value of y_t may be obtained along similar lines but from eq (5-108) instead of eq (5-116), giving

$$\text{Var E } (y_t|\underline{\vec{x}}) = \left(\frac{1}{n} + \frac{\underline{x}^2}{\Sigma \underline{x}^2} \right) \text{ Var } \Delta y. \tag{5-123}$$

The eqs (5-122) and (5-123) show, when transformed into standard errors, the well known hyperbolic curves describing the confidence band (see Fig. 5-1).

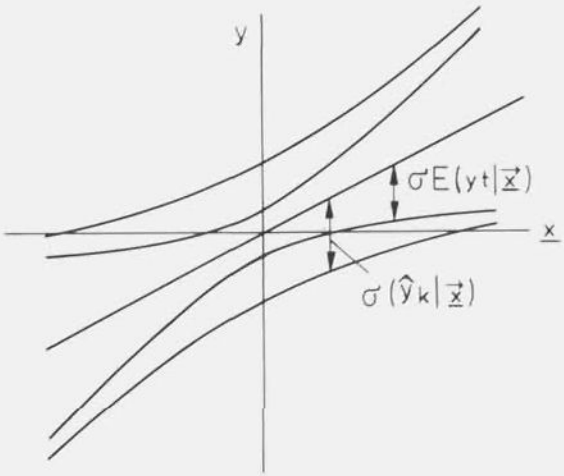


Fig. 5-1 Standard error of expected value $E(y_t|\underline{x})$ and of individual value $\hat{y}_k|\underline{x}$ against $\underline{x}$

Example

A simple example with two input variables, x_2 and x_2, and one variable to be examined, y will be discussed here and illustrated using data from some gauging

Feb 1980	IJsselkop $\underline{x}_1$	Doesburg $\underline{x}_2$	De Steeg					
			$\underline{y}$	$\hat{y}$	$\Delta y = \underline{y} - \hat{y}$		$\sigma E(y_{jt}\|\underline{\bar{x}})$	
	NAP+cm	NAP+cm	NAP+cm	NAP+cm	cm	cm	cm	cm
1	2	3	4	5	6	7¹)	8²)	9³)
1	932	726	800	804	- 4	0,763	1,742	2,195
2	952	736	816	818	- 2	0,723	1,697	2,139
3	1060	810	907	905	2	0,630	1,616	2,037
4	1134	886	975	982	- 7	0,584	1,674	2,110
5	1184	918	1020	1021	- 1	0,621	1,716	2,162
6	1222	946	1052	1053	- 1	0,705	1,797	2,265
7	1257	956	1074	1071	3	0,890	1,926	2,427
8	1293	971	1098	1094	4	1,152	2,162	2,724
9	1331	994	1120	1122	- 2	1,336	2,362	2,977
10	1329	1004	1122	1129	- 7	1,148	2,211	2,787
11	1310	998	1119	1118	1	1,007	2,075	2,615
12	1274	986	1100	1098	2	0,843	1,943	2,449
13	1239	968	1078	1073	5	0,799	1,900	2,394
14	1205	946	1048	1047	1	0,772	1,859	2,343
15	1173	926	1023	1022	1	0,780	1,854	2,337
16	1141	910	1000	1001	- 1	0,910	1,915	2,459
17	1110	894	978	979	- 1	1,062	2,081	2,623
18	1081	872	952	954	- 2	1,053	2,066	2,604
19	1052	846	928	927	1	0,964	1,975	2,489
20	1024	819	900	899	1	0,860	1,870	2,356
21	1002	797	878	876	2	0,782	1,796	2,264
22	985	778	860	858	2	0,719	1,602	2,019
23	967	759	840	839	1	0,701	1,708	2,153
24	950	739	818	819	- 1	0,718	1,703	2,147
25	930	718	798	798	0	0,786	1,744	2,198
26	922	703	784	785	- 1	0,894	1,809	2,280
27	917	695	777	778	- 1	0,968	1,861	2,346
28	912	689	772	772	0	1,014	1,899	2,393
29	897	677	762	759	3	1,060	1,943	2,448
1	886	664	748	746	2	1,164	2,034	2,563
mean squared error (cm^2)						0,808	3,588	5,699
resulting Var Δy (cm^2)						8,075	10,855	12,966
number of degrees of freedom						27	20	17

Table 5-1 Examination of confidence interval (River IJssel stations, February 1980).

1) based on uncorrelated Δy, calculated with 27 degrees of freedom
2) based on autocorrelation of Δy during 1979...1981, also calculated with 27 degrees of freedom.
3) like 2), but calculated with 17 degrees of freedom.

stations along the river IJssel (Fig. 2-1). IJsselkop and Doesburg are the two network stations concerned, and respectively provide measured values x_1 and x_2; De Steeg is the station to be examined. The calculations were carried out for an arbitrary period of n = 30 days, from 1 February 1980 to 1 March 1980. The data comprised water levels measured daily at 0800 h.

The measured water levels are given in Table 5-1.
This yielded the matrices:

$$\underline{X}^T\underline{X} = \begin{bmatrix} 30 & 32671 & 25331 \\ 32671 & 36218437 & 28067120 \\ 25331 & 28067120 & 21761549 \end{bmatrix}$$

$$\text{and } \underline{X}^T\underline{Y} = \begin{bmatrix} 28147 \\ 31197654 \\ 24183847 \end{bmatrix}$$

which gave, according to eq. (5-12)

$$A = \begin{bmatrix} -7{,}13784 \\ 0{,}34074 \\ 0{,}68014 \end{bmatrix}$$

The values of $\hat{y}_j$ (j = 1...30), calculated according to eq (5-13), are also given in Table 5-1, together with the departures according to eq (5-2) namely

$$\Delta y = \underline{y} - \hat{y}.$$

The mean value of y, as might be expected, is

$$\bar{\Delta y} = \frac{\sum_{j=1}^{30} \Delta y_j}{30} = 0 \text{ cm}$$

and the mean squared error is

$$\frac{\sum_{j=1}^{30} \Delta y_j^2}{30} = \frac{218}{30} = 7{,}267 \text{ cm}^2 \qquad (5\text{-}124)$$

This corresponds to an expected variance

$$\text{Var } \Delta y = \frac{218}{n\text{-}3} = \frac{218}{27} = 8{,}074 \text{ cm}^2. \tag{5-125}$$

and to a standard error $\sigma\Delta y = 2.841$ cm.

Since there are two input variables the number of degrees of freedom must be reduced by 2 + 1 = 3, leading to n - 3 = 30 - 3 = 27 degrees of freedom.

Now consider the confidence limits for the $E(y_{jt}|\underline{\vec{x}})$ values at each $\underline{x}_1$, $\underline{x}_2$ combination of the series concerned. At first no autocorrelation in the Δy-series will be taken into account.

For each of the measured values, eq (5-108) can be used. For $R_{\Delta Y} = I$ this converts into

$$\text{Var } E(y_{jt}|\underline{\vec{x}}) = \underline{\vec{x}} \{(\underline{X}^T\underline{X})^{-1}\} \underline{\vec{x}}^T \text{ Var } \Delta y \tag{5-126}$$

The standard errors $\sigma E\,(y_{jt}|\underline{\vec{x}})$, calculated as the square roots of eq (5-126), are given in Table 5-1, Column 7. The mean of the variances of eq (5-126) gives the uncertainty of the regression relation (plane) as a whole, and it amounts to 0,808 cm. This is just the difference between the mean of the squared errors Δy_j (7,267 cm^2; see eq (5-124)) and Var Δy (8,074 cm^2; see eq (5-125)), which is based on n-3=27 degrees of freedom. The uncertainty in the regression plane is the reason for the decrease of the number of degrees of freedom from n to n-3.

Now consider the case where autocorrelation is present between the subsequent differences Δy. In that case measurements at subsequent times provide partly the same information. This means that a series of n subsequent measurements will produce an information content which could have been produced by a number of n' (n' <n) independent measurements as well. This will be demonstrated, although not exactly proved*), by the following. For this purpose the correlogram (i.e. the relation between the autocorrelation coefficient $\varrho\Delta y(\tau)$ and the time lag τ) should be known.

For this example a correlogram was used based on three years measurements (1979...1981) from the same stations considered, yielding 1096 water level readings at each of the three stations.

*) For further examination of this problem a more detailed time series analysis would be required, and possibly the application of generalized least squares (Wonnacott and Wonnacott, 1979).

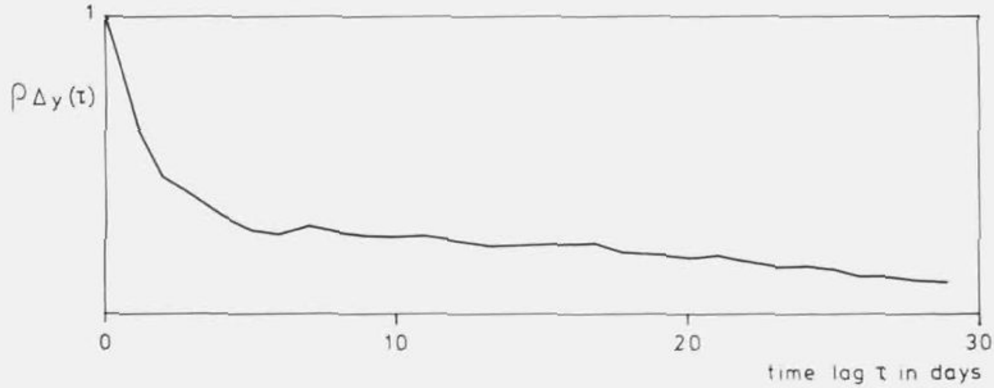


Fig. 5-2 Correlation of Δy-values at De Steeg, calculated from IJsselkop and Doesburg (1979...1981).

The autocorrelation coefficients are given in table 5-2 and graphically presented in Fig. 5-2. The correlogram shows a gradually decreasing shape.

From these correlation coefficients a correlation matrix was constructed as follows:

$$R_{\Delta Y} = \begin{bmatrix} 1 & \varrho_{\Delta y}(1) & \cdots & \varrho_{\Delta y}(n-1) \\ \varrho_{\Delta y}(1) & 1 & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \\ \varrho_{\Delta y}(n-1) & & \cdots & 1 \end{bmatrix} \qquad (5\text{-}127)$$

$$(n=30)$$

Subsequently Var $E(y_{jt}|\underline{\vec{x}})$ was calculated using eq (5-108). If the value of 8,074 cm^2 is used for Var Δy (which is based on 27 degrees of freedom) then one will find errors $\sigma E(y_{jt}|\underline{\vec{x}})$ as given in Column 8 of Table 5-1. The mean of the squares of these errors, shown in the last line but one in this table amounts to 3,588 cm^2. When this is added to the mean of the squares of Δy_j (7,267 cm^2, see eq (5-124)) in the same way as was done in the correlation free case, then one finds a value of Var Δy of 10,855 cm^2. This value would also be found if there were only 20 degrees of freedom instead of 27, namely

$$\frac{\sum_{j=1}^{n} \Delta y_j^2}{20} = \frac{218}{20} = 10{,}9 \text{ cm}^2.$$

Apparantly the autocorrelation has a similar effect on Var Δy as a smaller number of degrees of freedom. But there is a difference between the initially used number (27) and the find number (20). Therefore one has to repeat the calculation with another number of degrees of freedom until the initial and find numbers are the same. This is found for 17 degrees of freedom, the results of which are given in Column 9 of table 5-1. This time the mean of the squared errors amounts to 5,699 cm^2, which leads to a variance Var Δy of (5,699 + 7,267) cm^2 = 12,966 cm^2, which value corresponds approximately to 17 degrees of freedom, namely

$$\frac{\sum_{j=1}^{n} \Delta y_j^2}{17} = \frac{218}{17} = 12{,}825 \text{ cm}^2$$

It is found that the autocorrelation has an increasing effect on the standard error $\sigma\Delta y$, which is expressed in a loss of degrees of freedom. This indicates a loss of information. The remaining number of degrees of freedom might be called the 'effective numer of independent observations' (Schilperoort, 1986).

This example has demonstrated that the correlation structure can be of great importance to Var Δy, and thus also to Var $\Delta\hat{y}$. It is clear that this fact should be taken into account when testing the feasibility of the network.

5.6. Lagrange functions

5.6.1. Application to linear regression interpolation.

The coefficients A_i (i = 1...m) of eq (5-1) are in fact values of Lagrange functions. Such functions were discussed in various sections of Chapter 4, where they concerned several types of mathematical interpolation methods.

By means of the coefficients A_i it is possible to express the levels, y, at a certain site in terms of levels x_i, at a series of m network stations. For the present interpolation by regression these coefficients A_i have the same meaning as those for the interpolation methods of Chapter 4. However this time, only values can be given for those sites where earlier measurements have been carried out in order to determine the interpolation equations (model).

For the network stations the A_i-coefficients are either 1, if it concerns the station itself, or 0, if it concerns another network station. This is the case if the measured data are maintained without modification to the most probable values, taking into account the uncertainties of the measurements. However such modification can easily be made. If this is the case, then one has a type of approximated interpolation, which is comparable with the concept of the same name discussed in Chapter 4.

τ days	$\varrho_{\Delta y}(\tau)$
0	1
1	0,62787
2	0,45848
3	0,39609
4	0,32501
5	0,27538
6	0,25932
7	0,29318
8	0,27810
9	0,25855
10	0,25790
11	0,26133
12	0,24429
13	0,22112
14	0,22612
15	0,22769
16	0,23009
17	0,23375
18	0,20267
19	0,19979
20	0,18841
21	0,19724
22	0,16952
23	0,15597
24	0,15849
25	0,14613
26	0,13081
27	0,12419
28	0,11402
29	0,10347

Table 5-2. Autocorrelation coefficients of Δy values at De Steeg, calculated from IJsselkop and Doesburg (1979...1981) (see also Fig. 5-2).

If there is only one network station, the expected value of a measurement $\underline{x}$ can be determined as described in Subsection 5.4.1, and in particular by eq (5-46):

$$E\,(x_t - \bar{x}_t) = \frac{\mathrm{Var}\ x_t}{\mathrm{Var}\ x_t + \varepsilon_x^2}\,(\underline{x} - \bar{x})$$

so that:

$$A_t = \frac{\text{Var } x_t}{\text{Var } x_t + \varepsilon_x^2}, \qquad (5\text{-}128)$$

and, after some elaboration

$$A_o = -(\bar{\underline{x}} - \bar{x}_t) + \frac{\varepsilon_x^2}{\text{Var } x_t + \varepsilon_x^2} \bar{\underline{x}} \qquad (5\text{-}129)$$

in which the term in brackets denotes the systematic error. If no systematic errors occur, so

$$\bar{\underline{x}} = \bar{x}_t, \qquad (5\text{-}130)$$

then

$$A_0 = \frac{\varepsilon_x^2}{\text{Var } x_t + \varepsilon_x^2} \bar{\underline{x}} \qquad (5\text{-}131)$$

The residual variance is given by eq (5-56) (where also reference is made to eq (5-80))

$$\text{Var } \Delta\hat{x} = \text{Var E } (x_t|\underline{x}) \quad = \frac{\text{Var } x_t}{\text{Var } x_t + \varepsilon_x^2} \varepsilon_x^2. \qquad (5\text{-}132)$$

The smoothing effect is evident since this residual variance is smaller than the variance of measurement i.e. ε_x^2. Also the accuracy of the measured data becomes somewhat improved.

For multidimensional regression, the expression for the water level to be calculated for the network station considered is, according to eq (5-13),

$$\hat{y} = \vec{\underline{x}} \, A \qquad (5\text{-}133)$$

where, according to eq (5-12),

$$A = (\underline{X}^T\underline{X})^{-1} (\underline{X}^T\underline{Y}). \qquad (5\text{-}134)$$

Now let $\hat{y}$ be the water level to be derived for the i^{th} network station, i.e.

$$\hat{y} = E\,(x_{it}).$$

In that case the matrix $\underline{X}^T\underline{Y}$, which generally is composed as follows:

$$\underline{X}^T\underline{Y} = \begin{bmatrix} \Sigma y \\ \Sigma \underline{x}_1 \, \underline{y} \\ . \\ . \\ . \\ \Sigma \underline{x}_i \, \underline{y} \\ . \\ . \\ . \\ \Sigma \underline{x}_m \, \underline{y} \end{bmatrix} \qquad (5\text{-}135)$$

becomes

$$\underline{X}^T\underline{Y} = \begin{bmatrix} \Sigma \underline{x}_i \\ \Sigma \underline{x}_1 \, \underline{x}_i \\ . \\ . \\ . \\ \Sigma \underline{x}_i \underline{x}_i - n\varepsilon_x^2 \\ . \\ . \\ . \\ \Sigma \underline{x}_m \, \underline{x}_i \end{bmatrix} \quad . \qquad (5\text{-}136)$$

The change of the element including $\Sigma \underline{x}_i \underline{x}_i$ in this column matrix can be explained as follows. Consider this element to be originated from an element $\Sigma \underline{x}_i \underline{x}_k$ for k→i. Then

$$\Sigma \underline{x}_i \underline{x}_k = \Sigma (x_{it} + \Delta \underline{x}_{it})\,(x_{kt} + \Delta \underline{x}_{kt})$$

$$= \Sigma x_{it} x_{kt} + \Sigma x_{it}\,\Delta \underline{x}_{kt} + \Sigma x_{kt}\,\Delta \underline{x}_{it} + \Sigma \Delta \underline{x}_{it}\,\Delta \underline{x}_{kt} \qquad (5\text{-}137)$$

The increments in eq (5-137) are composed as follows

$$\left.\begin{array}{l} \Delta \underline{x}_{it} = \alpha\varepsilon_x - \Delta \underline{x}_{is} \\ \\ \Delta \underline{x}_{kt} = \beta\varepsilon_x - \Delta \underline{x}_{ks} \end{array}\right\} \qquad (5\text{-}138)$$

where α and β are standard normally distributed stochastic variables and $\Delta \underline{x}_{is}$ and $\Delta \underline{x}_{ks}$ are systematic errors. The errors, both stochastic as well as systematic, are considered to be independent of the true values. The various terms of eq (5-137) are further expanded as follows:

$$\begin{aligned}\Sigma x_{it}\,\Delta\underline{x}_{kt} &= \Sigma x_{it}\,(\alpha\varepsilon_x + \Delta\underline{x}_{ks}) \\ &= \varepsilon_x\Sigma\alpha x_{it} + \Delta\underline{x}_{ks}\,\Sigma x_{it} \\ &= 0 + \Delta\underline{x}_{ks}\,\Sigma x_{it} \\ &= \Delta\underline{x}_{ks}\,\Sigma x_{it}\end{aligned} \tag{5-139}$$

$$\Sigma x_{kt}\Delta\underline{x}_{it} = \Delta\underline{x}_{is}\,\Sigma x_{kt} \tag{5-140}$$

$$\begin{aligned}\Sigma\Delta\underline{x}_{it}\,\Delta\underline{x}_{kt} &= \Sigma\,(\alpha\varepsilon_x + \Delta\underline{x}_{is})\,(\beta\varepsilon_x + \Delta\underline{x}_{ks}) \\ &= \varepsilon_x^2\Sigma\alpha\beta + \varepsilon_x\,(\Delta\underline{x}_{ks}\Sigma\alpha + \Delta\underline{x}_{is}\Sigma\beta) + n\,\Delta\underline{x}_{is}\,\Delta\underline{x}_{ks} \\ &= \varepsilon_x^2\Sigma\alpha\beta + n\,\Delta\underline{x}_{is}\,\Delta\underline{x}_{ks}\end{aligned} \tag{5-141}$$

The sum $\Sigma\alpha\beta$ is related to the correlation between measurement errors at two stations. For the relevant correlation coefficient the symbol ϱ_Δ was used (see eq (3-53)). Since α and β have standard normal distributions, with mean zero and standard deviation one, it holds that

$$\varrho_\Delta = \frac{\Sigma\alpha\beta/n}{\sigma_\alpha\,\sigma_\beta} = \frac{\Sigma\alpha\beta}{n}$$

and

$$\Sigma\alpha\beta = n\varrho_\Delta \tag{5-142}$$

Thus eq (5-141) becomes, after substitution of eq (5-142)

$$\Sigma\Delta\underline{x}_{it}\,\Delta\underline{x}_{kt} = n\,(\varepsilon_x^2\varrho_\Delta + \Delta\underline{x}_{is}\,\Delta\underline{x}_{ks}) \tag{5-143}$$

Substitution of eqs (5-139), (5-140) and (5-143) into eq (5-137) yields

$$\Sigma\underline{x}_i\underline{x}_k = \Sigma x_{it}x_{kt} + \Delta\underline{x}_{ks}\,\Sigma x_{it} + \Delta\underline{x}_{is}\,\Sigma x_{kt} + n\,(\,\varepsilon_x^2\,\varrho_\Delta + \Delta\underline{x}_{is}\,\Delta\underline{x}_{ks}) \tag{5-144}$$

So if $k \rightarrow i$, then

$$\Sigma\underline{x}_i\underline{x}_i = \Sigma x_{it}^2 + 2\,\Delta\underline{x}_{is}\,\Sigma x_{it} + n\,(\varepsilon_x^2\varrho_\Delta + \Delta\underline{x}^2_{is}) \tag{5-145}$$

If k is fully identical with i, so that it relates to the same station, then $\varrho_\Delta = 1$, so that

$$\Sigma\underline{x}_i^2 = \Sigma x_{it}^2 + 2\,\Delta\underline{x}_{is}\,\Sigma x_{it} + n\,(\varepsilon_x^2 + \Delta x_{is}^2) \tag{5-146}$$

Subtraction of eq (5-146) from eq (5-145), and bringing Σx_i^2 to the right hand side, yields

$$\Sigma \underline{x}_i \underline{x}_i = \Sigma x_i^2 - n\,(1 - \varrho_\Delta)\,\varepsilon_x^2 \tag{5-147}$$

In fact an existing and an imaginary station are compared. The scope of the introduction of an imaginary station was to reduce the influence of measurement errors as much as possible. This can best be arrived at by assessing $\varrho_\Delta = 0$, since then measurement errors at the existing station are not carried over to the values, to be calculated for the imaginary station. Thus

$$\Sigma \underline{x}_i \underline{x}_i = \Sigma x_i^2 - n\varepsilon_x^2 \tag{5-148}$$

which is the expression used in eq (5-136)

It should be noted that, for the two dimensional case (see Section 5.4.1), where an estimate for the level at the same station where measurements are carried out was made, the result was found to be given by eq (5-46). Implicitely the value $\varrho_\Delta = 0$ was applied in this case.

Now, substituting eq (5-136) into eq (5-134) gives

$$A = (\underline{X}^T\underline{X})^{-1} \cdot \begin{bmatrix} \Sigma \underline{x}_i \\ \Sigma \underline{x}_1 x_i \\ \cdot \\ \cdot \\ \cdot \\ \Sigma \underline{x}_i^2 - n\varepsilon_x^2 \\ \cdot \\ \cdot \\ \cdot \\ \Sigma \underline{x}_m x_i \end{bmatrix} \tag{5-149}$$

$$= (\underline{X}^T\underline{X})^{-1} \cdot \left\{ \begin{bmatrix} \Sigma \underline{x}_i \\ \Sigma \underline{x}_1 x_i \\ \cdot \\ \cdot \\ \cdot \\ \Sigma \underline{x}_i^2 \\ \cdot \\ \cdot \\ \cdot \\ \Sigma \underline{x}_m x_i \end{bmatrix} - \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ n\varepsilon_x^2 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \right\}$$

The first column matrix in brackets is in fact the i^{th} column of the matrix $\underline{X}^T\underline{X}$, that is of the inverse of the matrix before the brackets. Multiplication of this column matrix by $\underline{X}^T\underline{X}$ yields another column matrix with elements 0, except for the i^{th} element, which is 1. Thus

$$A = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} - (\underline{X}^T \underline{X})^{-1} \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ n\varepsilon_x^2 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} . \tag{5-150}$$

The second term of eq (5-150) multiplies the column for x_i in $(\underline{X}^T\underline{X})^{-1}$ by $n\,\varepsilon_x^2$. This is subtracted from the first matrix, which contains 1 for A_i and 0 for the other coefficients A. As a rule $n\,\varepsilon_x^2$ will be small. Thus the value A_i will remain close to 1, whereas the others have relatively small values.

When calculating the variance of Δx_i in the multidimensional regression, the determinant expression of eq (5-35) should be used. In the determinant of the numerator, y is to be replaced by $\underline{x}_i$. Thus, according to eq (III-15), it follows that

$$|R(\underline{X}\ \underline{Y})| = \left| \begin{array}{c|cccc} 1 & \varrho(\underline{x}_1\underline{x}_i) & \dots & \varrho(\underline{x}_i\underline{x}_i) \dots & \varrho(\underline{x}_m\underline{x}_i) \\ \hline \varrho(\underline{x}_1\underline{x}_i) & 1 & & \varrho(\underline{x}_1\underline{x}_i) & \varrho(\underline{x}_1\underline{x}_m) \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \varrho(\underline{x}_i\underline{x}_i) & \varrho(\underline{x}_1\underline{x}_i) & & 1 & \varrho(\underline{x}_i\underline{x}_m) \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \varrho(\underline{x}_m\underline{x}_i) & \varrho(\underline{x}_1\underline{x}_m) & \dots & \varrho(\underline{x}_m\underline{x}_i) \dots & 1 \end{array} \right| \tag{5-151}$$

Note, that in the first row and in the first column at the position for the i^{th} element $\varrho(\underline{x}_i\underline{x}_i)$ the substituted value is not 1 since it involves the correlation coefficient between two independent measurements of the same value x_{it}. This relates to the so called internal correlation coefficient, between two independently measured values, given by

$$\varrho(\underline{x}_i\underline{x}_i) \underset{k=i}{=} \varrho(\underline{x}_k\underline{x}_i) = \frac{\text{Cov } \underline{x}_k\underline{x}_i}{\text{Var } \underline{x}_i} . \qquad (5\text{-}152)$$

For Cov $\underline{x}_k\underline{x}_i$ (k = i) it can be derived that

$$\text{Cov } \underline{x}_k\underline{x}_i = \frac{\Sigma\underline{x}_k\underline{x}_i - n\,\bar{\underline{x}}_k\bar{\underline{x}}_i}{n}$$

$$= \frac{\Sigma\underline{x}_k\underline{x}_i}{n} - \bar{\underline{x}}_k\bar{\underline{x}}_i$$

$$= \frac{\Sigma(x_{it} + \Delta\underline{x}_k)(x_{it} + \Delta\underline{x}_i)}{n} - (\bar{x}_{it} + \bar{\Delta}\bar{\underline{x}}_k)(\bar{x}_{it} + \bar{\Delta}\bar{\underline{x}}_i)$$

$$= \frac{\Sigma x_{it}^2 + \Sigma x_{it}\,\Delta\underline{x}_i + \Sigma x_{it}\,\Delta\underline{x}_k + \Sigma\Delta\underline{x}_k\,\Delta\underline{x}_i}{n}$$

$$- (\bar{x}_{it}^2 + \bar{x}_{it}\,\bar{\Delta}\bar{\underline{x}}_k + \bar{x}_{it}\,\bar{\Delta}\bar{\underline{x}}_i + \bar{\Delta}\bar{x}_k\,\bar{\Delta}\bar{x}_i)$$

$$= \frac{\Sigma x_{it}^2}{n} - \bar{x}_{it}^2 + \frac{\Sigma x_{it}\,\Delta\underline{x}_i}{n} - \bar{x}_{it}\,\bar{\Delta}\bar{\underline{x}}_i$$

$$+ \frac{\Sigma x_{it}\,\Sigma\underline{x}}{n} - \bar{x}_{it}\,\bar{\Delta}\bar{\underline{x}}_k + \frac{\Sigma\Delta\underline{x}_i\,\Delta\underline{x}_k}{n} - \bar{\Delta}\bar{x}_i\,\bar{\Delta}\bar{x}_k$$

$$= \text{Var } x_{it} + \text{Cov } x_{it}\,\Delta\underline{x}_i + \text{Cov } x_{it}\,\Delta\underline{x}_k + \text{Cov } \Delta\underline{x}_i\,\Delta\underline{x}_k$$

Since all covariances are 0, because of the imposed independence, then

$$\underset{k=i}{\text{Cov}}\ \underline{x}_k\ \underline{x}_i = \text{Var } x_{it} \qquad (5\text{-}153)$$

Further, because of eq (5-30):

$$\text{Var } \underline{x}_i = \text{Var } x_{it} + \varepsilon_x^2 \qquad (5\text{-}154)$$

so that

$$\varrho(\underline{x}_i\underline{x}_i) = \frac{\text{Var } x_{it}}{\text{Var } x_{it} + \varepsilon_x^2} \tag{5-155}$$

Now, substracting in eq (5-151) the $(i+1)^{th}$ row from the first row and subsequently the $(i+1)^{th}$ column from the first column, so the value of $|R(\underline{X}\underline{Y})|$ is not changed, it follows that

$$|R(\underline{X}\ \underline{Y})| = \left|\begin{array}{c|c} 2\{1 - \varrho(\underline{x}_i\underline{x}_i)\} & 0 \ldots 0 \ \ -\{1 - \varrho(\underline{x}_i\underline{x}_i)\} \ \ 0 \ldots 0 \\ \hline 0 & \\ \vdots & \\ 0 & \\ -\{(1 - \varrho(\underline{x}_i\underline{x}_i)\} & |R(\underline{X}\ \underline{X})| \\ 0 & \\ \vdots & \\ 0 & \end{array}\right| \tag{5-156}$$

The minor indicated by $|R\ (\underline{X}\ \underline{X})|$ is the correlation matrix of the m network stations.

Reduction of $|(R\ (\underline{X}\ \underline{X})|$ *by deleting its* i^{th} *row and its* i^{th} *column yields another minor* $|M|$ of the order (m-1), which is the correlation matrix of the network stations, except with the i^{th}station excluded. Now the value of $|R\ (\underline{X}\ \underline{Y})|$ can be calculated as follows:

$$|R\ (\underline{X}\ \underline{Y})| = 2\ (1 - \varrho(\underline{x}_i\underline{x}_i))\ |R(\underline{X}\ \underline{X})| - \{1 - \varrho(\underline{x}_i\underline{x}_i)\}^2|M| \tag{5-157}$$

Substitution of eq (5-155) into eq (5-157) yields

$$|R\ (\underline{X}\ \underline{Y})| = \frac{2\ \varepsilon_x^2}{\text{Var } x_{it} + \varepsilon_x^2} \cdot |R\ (\underline{X}\ \underline{X})| - \left(\frac{\varepsilon_x^2}{\text{Var } x_{it} + \varepsilon_x^2}\right)^2 |M| \tag{5-158}$$

Making use of this result Var $\Delta\hat{x}_i$ will now be calculated. For this variance it follows from eq (5-30) that

$$\text{Var } \Delta\hat{x}_i = \text{Var } \Delta x_i - \varepsilon_x^2 \tag{5-159}$$

The variance Var Δx_i can be calculated according to eq (5-35) as

$$\text{Var } \Delta x_i = \frac{|R\ (\underline{X}\ \underline{Y})|}{|R\ (\underline{X}\ \underline{X})|} \text{Var } \underline{x}_i. \tag{5-160}$$

Substitution of eq (5-158) into eq (5-160) leads to (since Var $\underline{x}_i$ = Var $x_{it} + \varepsilon_x^2$):

$$\text{Var}\,\Delta x_i = 2\,\varepsilon_x^2 - \frac{\varepsilon_x^4}{\text{Var}\,x_{it} + \varepsilon_x^2}\;\frac{|M|}{|R\,(\underline{X}\,\underline{X})|} \tag{5-161}$$

and, finally, regarding eq (5-159):

$$\text{Var}\,\Delta\hat{x}_i = \varepsilon_x^2 - \frac{\varepsilon_x^4}{\text{Var}\,x_{it} + \varepsilon_x^2}\;\frac{|M|}{|R\,(\underline{X}\,\underline{X})|} \tag{5-162}$$

The result is smaller than the variance of measurement ε_x^2, since the second term consist only of positive elements, viz. the two variances ε_x^2 and Var x_{it} and the determinants of correlation matrices, which are always positive. This means that by recalculating the values at the network stations in the manner described, the variance of estimate Var $\Delta\hat{x}_i$ shows a slight improvement with respect to the variance of measurement ε_x^2, as was already shown for the case with two variables by eq (5-132).

5.6.2. Consideration of a series of examined stations and network stations

So far, only one examined station has been considered against a series of network stations. There is no objection to considering a number of examined stations against the same series of network stations. In that case the whole system of stations along a river reach could be studied at one time (Fig. 5-3).

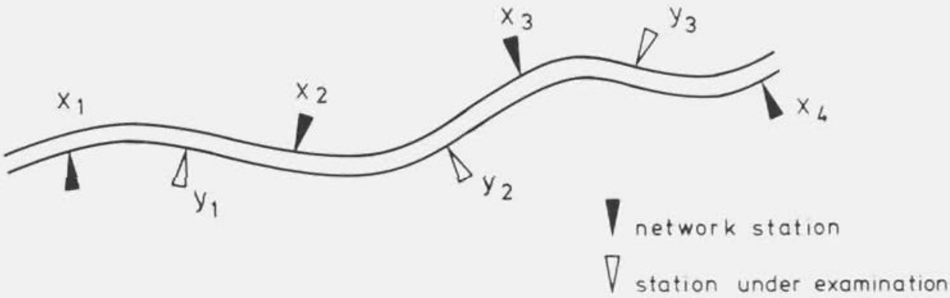


Fig. 5-3. River reach with network stations and stations under examination.

The (n×1) matrix $\underline{Y}$ in eq (5-5) which gives the measured data of one examined station can be replaced now by a (n×m) matrix $\underline{Y}_r$ for the whole set of M examined stations along the river:

$$\underline{Y}_r = \begin{bmatrix} y_{11} \cdots y_{M1} \\ \\ y_{1n} \cdots y_{Mn} \end{bmatrix} \tag{5-163}$$

Now a (m×M) matrix A_r can be calculated, the elements of which are the A-coefficients for all examined stations, and a single column relates to a certain examined station, so

$$A_r = \begin{bmatrix} A_{10} \cdots A_{M0} \\ \\ A_{1m} \cdots A_{Mm} \end{bmatrix} \tag{5-164}$$

Each row of A_r concerns the Lagrange function of a network station, each column the set of Lagrange functions used to derive the level at a station to be examined from the set of network stations.

In accordance with eq (5-12) A_r can be calculated as follows:

$$A_r = (\underline{X}^T\underline{X})^{-1} (\underline{X}^T\underline{Y}_r) \tag{5-165}$$

This method can be further extended by inclusion of the values for the network stations as well. Then a number (n) of columns like eq (5-136) are inserted at the places in the matrix $\underline{X}^T\underline{Y}_r$, relevant to the sequence of stations. In doing so the network stations are treated like examined stations. Now the matrix A_r will include the values of the Lagrange functions at the sites of the network stations too. The order of $\underline{Y}_r$ will be n × (M+m); $\underline{X}^T\underline{Y}_r$ and A_r will both be of order m × (M+m).

Example

The network of the river IJssel will be used as an example, and a situation map of the network is shown in Fig. 2-1. The stations IJsselkop, Doesburg, Zutphen, Olst and Katerveer are designated as network stations; stations to be examined are De Steeg, Dieren, Deventer and Wijhe. The calculations are based on water level data for every third day during the years 1977 and 1978, starting from 3 January 1977. Each station hass 243 data values.

The matrices $\underline{X}^T\underline{X}$ and $\underline{X}^T\underline{Y}_r$ calculated from these data aregiven in Tables 5-3 and 5-4 respectively. In the latter the values in italics are the square sums, Σx_i^2, for the network stations. From these values an amount $n\varepsilon_x^2$ has to be substracted in order to account for the errors of measurement.For a standard error of measurement ε_x = 1,5 cm for all network stations, this yields

	Katerveer	Wijhe	Olst	Deventer	Zutphen	Dieren	Doesburg	De Steeg	IJsselkop
Sum	17212	42537	56425	77587	112928	153889	163534	184700	220555
Katerveer	*1706796* (1706249)	3709571	4779764	6362402	8903088	11832167	12523514	14022396	16561062
Olst	4779764	11028466	*14400517* (14399970)	19462419	27736154	37298442	39551784	44464342	52785185
Zutphen	8903088	21107323	27736154	37749513	*54260576* (54260029)	73361424	77861548	87698661	104359305
Doesburg	12523514	30021232	39551748	53982078	77861548	105506707	*112020676* (112020129)	126273088	150411121
IJsselkop	16561062	39993062	52785185	72180433	104359305	141628002	150411142	169646750	*202221153* (202220606)

Table 5-4 Matrix $\underline{X}^T\underline{Y}r$ calculated from river IJssel water levels measured every third day over the period 1977…1978.

$n\,\varepsilon_x^2 = 243 \times (1{,}5)^2\ cm^2 = 547\ cm^2$.

The reduced values, $\Sigma x_i^2 - n\,\varepsilon_x^2$, are given in Table 5-4, in brackets. Using these data in $\underline{X}^T\underline{Y}_r$, eq (5-165) was used to calculate A_r, and the result is given in Table 5-5. In Fig. 5-4 the values are depicted in an analogous way to the graphs of Lagrange functions presented in Chapter 4. However, now only discrete values of the Lagrange functions can be given.

It should be noted that A_0 has a constant value, whereas $A_1 \ldots A_5$ are the factor values, by which the water level data of the network stations listed below, are to be multiplied:

A_1: Katerveer
A_2: Olst
A_3: Zutphen
A_4: Doesburg
A_5: IJsselkop

	Sum	Katerveer	Olst	Zutphen	Doesburg	IJsselkop
Sum	243	17212	56425	112928	163534	220555
Katerveer	17212	1706796	4779764	8903088	12523514	16561062
Olst	56425	4779746	14400517	27736154	39551784	52785185
Zutphen	112928	8903088	27736154	54260576	77861548	104359305
Doesburg	163534	12523514	39551748	77861548	112020676	150411121
IJsselkop	220555	16561062	52785185	104359305	150411142	202221153

Table 5-3 Matrix $\underline{X}^T\underline{X}$ calculated for the river IJssel network stations using water levels measured every third day over the period 1977...1978 (values in cm^2)

The value of the propagation function of the measurement errors, by analogy with Chapter 4, is given by A^TA; this is presented in Table 5-5 and depicted in Fig. 5-4. This function shows the influence of the propagated measurement errors of the control stations on the variance of Δy, i.e. the difference between the measured and the calculated value at the examined station. If the standard errors of measurement are all equal to ε_x, then the contribution of these errors to Var Δy amounts to:

$$\text{Var } m = A^TA\,\varepsilon_x^2 \qquad (5\text{-}166)$$

From the results, given in Table 5-5 and Fig. 5-4 it appears that only for one case does the value of A^TA exceed the value 1 by a small amount, viz for the station Wijhe. For the other cases A^TA is smaller than 1, which means that the contribution of the propagated measurement errors is smaller than the variance of measurement itself.

	Katerveer	Wijhe	Olst	Deventer	Zutphen	Dieren	Doesburg	De Steeg	IJsselkop
A_0 (cm)	1,78036	12,80174	8,33237	- 8,47972	2,37815	18,49224	3,96997	- 15,01153	0,34969
A_1 - Katerveer	0,93515	0,11256	0,08621	- 0,13976	- 0,02479	- 0,04450	- 0,03901	0,00383	0,01846
A_2 - Olst	0,08621	0,96529	0,82060	0,78915	0,11875	0,03784	0,02314	- 0,05866	- 0,02085
A_3 - Zutphen	- 0,02479	- 0,21002	0,11875	0,30428	0,70344	- 0,24681	0,13865	0,09602	- 0,03390
A_4 - Doesburg	- 0,03334	0,19269	0,01133	0,11624	0,14647	0,77614	0,79098	0,52317	0,07537
A_5 - IJsselkop	0,01846	- 0,11232	- 0,02085	- 0,07184	- 0,03390	0,03070	0,07674	0,43160	0,96498
A^tA	0,88399	1,03831	0,69547	0,75354	0,66687	0,66767	0,65282	0,47266	0,93880

Table 5-5 Matrix $A_r = (\underline{X}^T\underline{X})^{-1}\,\underline{X}^T\underline{Y}r$ defining the regression coefficients for the river IJssel stations

correlation coefficients

	Kampen	Katerveer	Wijhe	Olst	Deventer	Zutphen	Dieren	Doesburg	De Steeg	IJsselkop
Kampen		0,902604	0,848124	0,842832	0,832904	0,827921	0,818122	0,815635	0,811760	0,802126
Katerveer	15		0,986463	0,984105	0,976336	0,970518	0,962680	0,960305	0,954007	0,941783
Wijhe	30	15		0,999080	0,996653	0,992652	0,984492	0,983708	0,975144	0,959401
Olst	39	24	9		0,998490	0,995840	0,990707	0,988277	0,980760	0,966296
Deventer	51	36	21	12		0,998014	0,993980	0,991657	0,984140	0,969619
Zutphen	67	52	31	28	16		0,997669	0,996097	0,990317	0,977585
Dieren	84	69	54	45	33	17		0,999395	0,996244	0,986843
Doesburg	89	74	59	50	38	22	5		0,997903	0,990330
De Steeg	97	82	67	58	46	30	13	8		0,996424
IJsselkop	112	97	82	73	61	34	28	23	15	

distances (km)

Table 5-6 Correlation coefficients and station distances (km) for the river IJssel stations

To find the variance of estimate Var $\Delta\hat{y}$, the above variance must be augmented by the variance of the model errors, Var r (see also eq (2-18)). This cannot be calculated separately. However, Var Δy can be calculated as a whole using the determinants quotient of eq (5-35). Substracting the variance of measurement, ε_y^2, and the contribution of the propagated errors, $A^TA\ \varepsilon_x^2$, from Var Δy yields the variance of the model error:

$$\text{Var r} = \text{Var } \Delta y - A^TA\ \varepsilon_x^2 - \varepsilon_y^2 \qquad (5\text{-}167)$$

To make use of eq (5-35) the correlation coefficients between each pair of stations (both network and examined stations) are required. These correlation coefficients, calculated for the same data series as the A-values, are given in Table 5-6.

The internal correlation coefficients of the network stations are also required. These are derived from eq (5-155), and further elaborated as follows:

$$\varrho(x_i x_i) = \frac{\text{Var } x_{it}}{\text{Var } x_{it} + \varepsilon_x^2} = 1 - \frac{\varepsilon_x^2}{\text{Var } \underline{x}_i} . \qquad (5\text{-}168)$$

In this way the internal correlation coefficients given in Table 5-7 were calculated, using the overall standard deviations of the water levels given in Table 5-8, whereas ε_x = 1,5 cm. The results are shown in Columns 2 and 3 of Table 5 - 9.

	Var $\underline{x}_i$ (cm²)	$\varrho(x_i x_i)$
Katerveer	$(44,9)^2$	0,998884
Olst	$(73,3)^2$	0,999581
Zutphen	$(85,8)^2$	0,999694
Doesburg	$(90,1)^2$	0,999723
IJsselkop	$(91,8)^2$	0,999733

Table 5-7 Internal correlation coefficient of river IJssel network stations.

Station	Mean water level NAP + cm	Overall standard deviation cm
Kampen	1,7	21,9
Katerveer	70,8	44,9
Wijhe	175,0	65,0
Olst	232,2	73,3
Deventer	319,3	81,7
Zutphen	464,7	85,8
Dieren	633,3	89,1
Doesburg	673,0	90,1
De Steeg	760,1	90,7
IJsselkop	907,6	91,8

Table 5-8 Mean water levels and standard deviations for the river IJssel stations

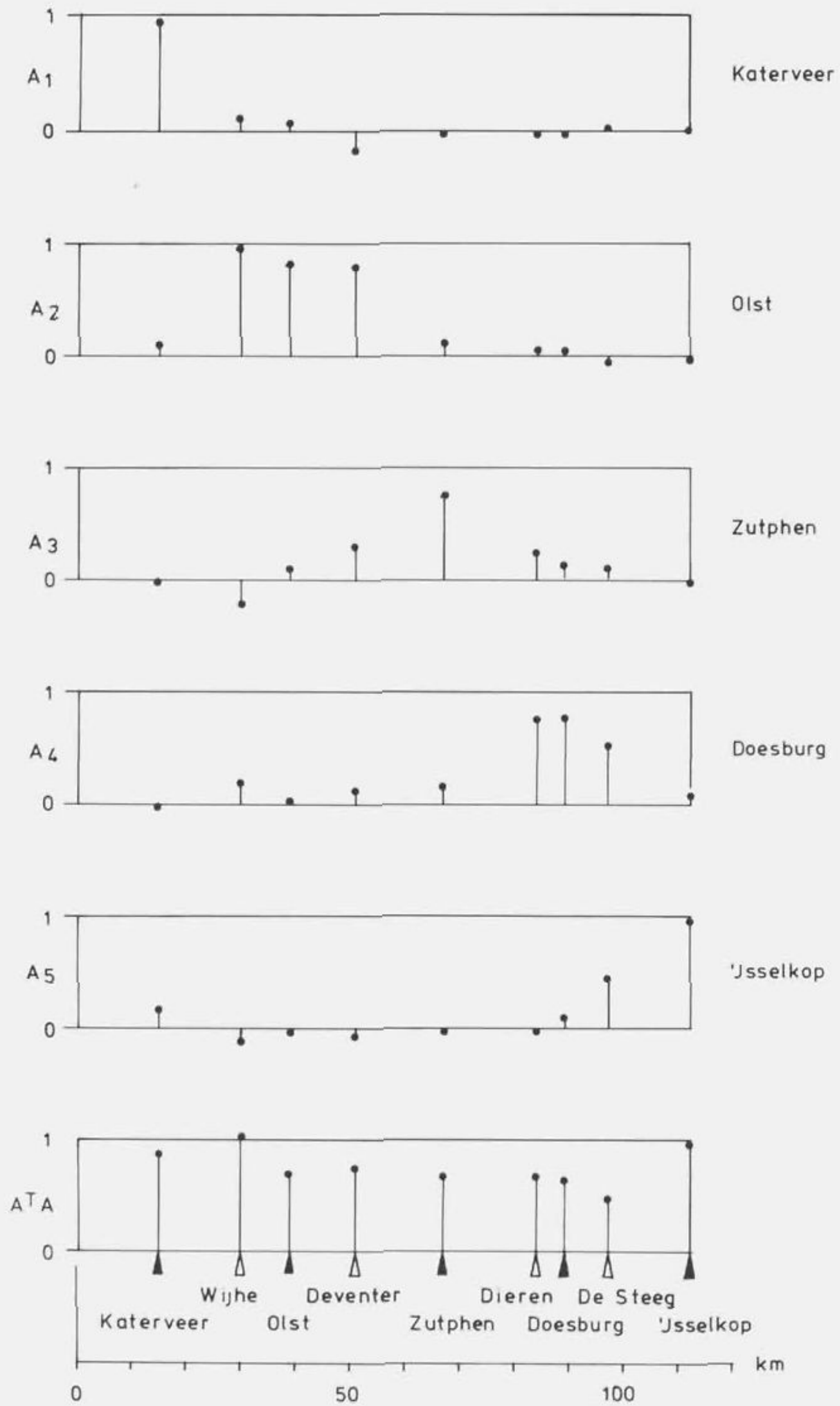


Fig. 5-4 Lagrange functions A_i and propagation function A^TA for linear regression interpolation (river IJssel)

Station	Var Δy	$\sigma\Delta y$	Var $\Delta\hat{y}$	$\sigma\Delta\hat{y}$	$A^TA\ \varepsilon_x^2$
	cm^2	cm	cm^2	cm	cm
1	2	3	4	5	6
Katerveer	4,356	2,087	2,106	1,451	1,989
Wijhe	4,178	2,044	1,928	1,389	2,336
Olst	4,098	2,024	1,848	1,359	1,565
Deventer	7,497	2,738	5,247	2,291	1,695
Zutphen	4,076	2,019	1,826	1,351	1,500
Dieren	4,558	2,135	2,308	1,519	1,502
Doesburg	4,036	2,009	1,786	1,336	1,469
De Steeg	5,813	2,411	3,563	1,888	1,063
IJsselkop	4,423	2,103	2,173	1,474	2,112

Table 5-9 Elements of Var Δy for river IJssel stations.

If the standard error of measurement ε_y *is also 1,5 cm, then the variance of estimate* can be calculated using eq (5-30) as

$$\text{Var}\ \Delta\hat{y} = \text{Var}\ \Delta y - \varepsilon_y^2$$

These variances are given in Column 4 of Table 5-9 and the corresponding standard errors in Column 5. For comparison the variances of the propagated measurement errors are also given (Column 6). Since these values form only part of the variance of Column 4, the values of Column 6 must be smaller than those of Column 4. This is true for the network stations (in italics) and for most examined stations, with the exception of Wijhe. Note that the local standard error of measurement of this station is smaller than $\varepsilon_y = 1{,}5$ cm. If $\varepsilon_y = 1$ cm, then one would find that

$$\text{Var}\ \Delta\hat{y} = 4{,}178 - 1 = 3{,}178\ \text{cm},$$

which is greater that $A^TA\ \varepsilon_x^2 = 2{,}336\ cm^2$.

For the station Deventer there is a big difference between the Columns 4 and 6. This could be due to model errors, but could also be attributable to a large local standard error of measurement. A standard error of say $\varepsilon_y = 2$ cm would lead to

$$\text{Var}\ \Delta\hat{y} = 7{,}497 - 4 = 3{,}497\ \text{cm}^2$$

5.7 Extension of the method to include measurements at different points in time

In the preceding sections use was made of simultaneous measurements only. The method described can, without principal problems, be extended to include measurements at the network stations made at other times to the time at which water levels at the site under examination are calculated. In this case the levels, denoted by $y(t_o)$, are considered as a function of a number of levels $x_i\ (t_o+h\ \Delta t)$ at each of the network stations. Here Δt denotes the measurement or sampling interval and h an integer number: for instance, h=-3...0 or h=-3...+3. What values of h are taken depends on the time that the water level at a network station can be considered to influence the level at the station under examination. For a network station, upstream of the station under examination, earlier water levels will influence the relation between both stations: these concern negative values of h. On the other hand, for a downstream network station, later levels will be more important. This implies positive h values.

A relation between water levels at the station under examination and, for instance, two upstream and two downstream network stations, might be written as

$$\begin{aligned} \hat{y}(t) = A_o \quad & + A_1x_1(t) && + A_2x_2(t) && + A_3x_3(t) + A_4x_4(t) \\ & + B_1x_1(t-2\ \Delta t) && + B_2x_2(t-2\ \Delta t) && \\ & + C_1x_1(t-\Delta t) && + C_2x_2(t-\Delta t) && \\ & + D_3x_3(t+\Delta t) && + D_4x_4(t+\Delta t) && \\ & + E_3x_3(t+2\Delta t) && + E_4x_4(t+2\Delta t) && \end{aligned} \qquad (5\text{-}169)$$

Such a relation can be expressed in a time-distance diagram as shown in fig. 5-5.

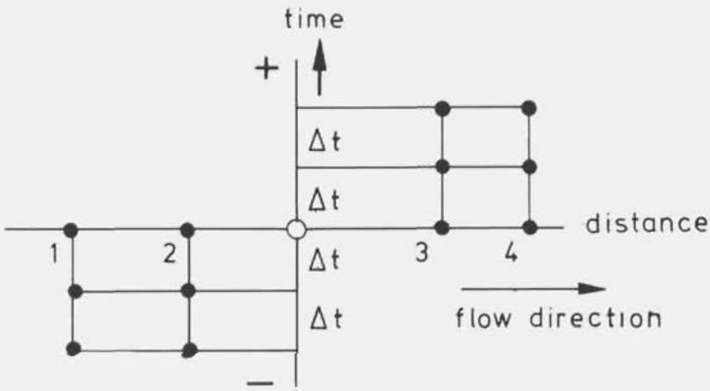


Fig. 5-5 Time-distance diagram for relation of eq (5-169).

If a series of stations to be examined is considered against a fixed series of network stations it is, for reasons of uniformity, often expedient to use a fixed time-distance diagram for all sites under examination. In this case weak relations will also be included in the formulae for $\hat{y}(t)$. This might be a drawback if the time series are short, since the number of degrees of freedom will be reduced considerably. However, in practice, relatively long time series are used for analysing networks. For instance one year of daily measurements comprises 365 elements, and one month of hourly measurements comprises 720 elements. As a rule much longer series will be used, so the problem of losing degrees of freedom is only of minor importance.

A time-distance diagram for a series of sites under examination within a fixed network may be similar to that shown in Fig. 5-6. An advantage of a symmetrical scheme like this is that the incoming waves, as well as the reflected waves, are taken into account.

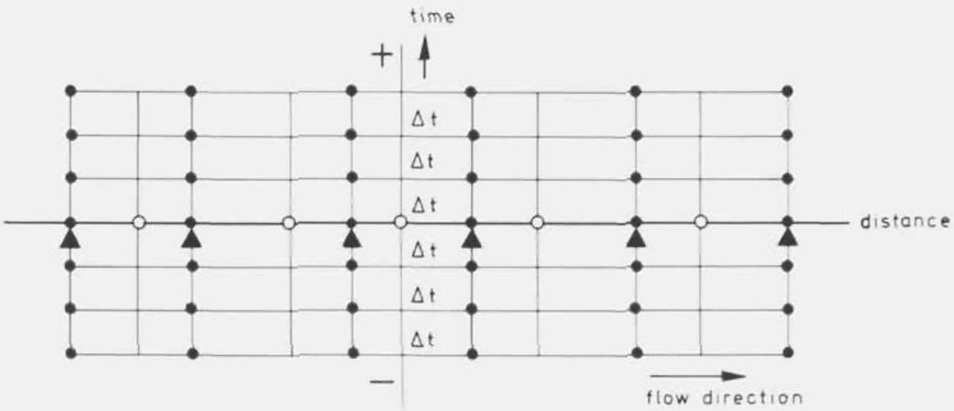


Fig. 5-6 Time-distance diagram for a series of stations to be examined and a fixed network.
- ● combinations, used in the calculation
- ○ stations to be examined
- ▲ network stations

Compared with the case, in which only simultaneous measurements were used, there are no major differences. The measurements made at the network stations for time points other than t=0 are considered as ordinary input variables, as are the measurements made at t=0. Each input variable has its own regression coefficient in the relation to be used.

In the correlation matrix there will be:
- spatial correlation coefficients between simultaneous measurements at different stations,
- autocorrelation coefficients between time lagged measurements at the same station,
- cross correlation coefficients between time lagged measurements at different stations.

The calculation of these correlation coefficients is straightforward.

Now, as was done for simultaneous measurements the standard error $\sigma\Delta y$ of the differences between measured and calculated values can be derived, using eq (5-36):

$$\sigma\Delta y = \sqrt{\frac{|R(\underline{X}\ \underline{Y})|}{|R(\underline{X}\ \underline{X})|}}\ \sigma\underline{y}.$$

The determinant of the denominater $|R(\underline{XX})|$ only contains the correlation coefficients of the measurements at the network stations, this time for all t-values considered. The determinant of the numerator includes, moreover, the correlation coefficients between the water levels at the stations under examination and the water levels at each of the network stations, at each time t considered.

Examples

1. The river Rhine in Germany

Consider the series of gauging stations indicated on the map of Fig. 3-13. Partial sets of this series were examined as stations of the network, whereas for the remaining stations the water levels were thought to be derived from the levels at the network stations. In this way six partial sets were examined as alternatives, as given in Tables 5-10, 5-11 and 5-12. In the calculations use was made of the daily (5 a.m.) water levels during the years 1976, 1977 and 1978. The following cases were distinguished:

- simultaneous measurements only: t=0 (Table 5-10);
- measurements at the network stations at t=-1d, t=0 and t=+1d (Table 5-11);
- measurements at the network stations at t=-2d, t=-1d, t=0, t=+1d, t=+2d (Table 5-12).

The tables show the values $\sigma\Delta y$ for each station to be examined, together with their overall (marginal) standard deviations $\sigma\underline{y}$. The tables show how accurate the water levels at the stations under examination can be calculated by multiple linear regression. If a limit value for $\sigma\Delta y$ is assessed, then one can judge whether a certain network can produce acceptable values at intermediate sites. If, for instance, the following requirement is to be satisfied:

$\sigma\Delta y < 3{,}5$ cm,

then, in alternative 1, for simultaneous measurements (Table 5-10), the stations Mannheim, Mainz and Rees would fulfil this requirement. For alternative 2 this requirement would hold for the stations Mannheim, Bingen and Rees, and for alternative 3 would hold for the stations Worms, Bingen, Köln and Rees.

Station	Distance to Basel (km)	Station distance (km)	Alternative 1	2	3	4	5	6	σy (cm)
Plittersdorf	340		s	s	s	s	s	s	77,1
		22							
Maxau	362		s	4,7	s	5,2	6,2	6,9	83,7
		39							
Speyer	401		s	s	6,5	7,7	12,4	16,3	96,8
		24							
Mannheim	425		2,8	2,8	s	s	10,2	15,6	96,3
		18							
Worms	443		s	s	2,8	3,5	9,2	15,3	93,5
		38							
Nierstein-Opp.	481		s	3,8	4,7	4,9	6,5	15,7	91,8
		17							
Mainz	498		3,1	s	s	s	s	12,0	77,7
		30							
Bingen	528		s	2,3	2,0	2,7	3,0	10,1	69,4
		18							
Kaub	546		5,8	s	s	7,7	8,1	14,2	83,7
		10							
St. Goar	556		4,3	4,2	4,2	6,1	6,2	14,8	92,7
		15							
Boppard	571		s	6,9	s	7,9	8,0	16,0	95,9
		21							
Koblenz	592		s	s	s	s	s	s	95,1
		22							
Andernach	614		4,6	5,2	s	5,7	9,3	15,8	106,0
		16							
Linz	630		s	3,9	4,3	4,3	8,3	16,7	113,8
		25							
Bonn	655		s	s	3,8	2,6	6,2	14,7	102,6
		33							
Köln	688		s	s	3,4	s	5,2	14,8	110,9
		56							
Düsseldorf	744		s	s	s	3,6	s	11,5	108,4
		37							
Ruhrort	781		s	s	4,1	s	4,1	10,3	121,0
		33							
Wesel	814		s	3,7	4,7	3,8	4,9	6,4	128,3
		23							
Rees	837		2,9	2,7	2,7	3,2	3,4	3,4	132,2
		15							
Emmerich	852		s	s	s	s	s	s	122,2
Number of network stations			15	11	10	7	5	3	

Table 5-10 Standard errors $\sigma\Delta y$ in cm for various series of network stations according to multiple linear regression (simultaneous measurements).

s = network station

Station	Distance to Basel (km)	Station distance (km)	Alternative 1	2	3	4	5	6	σy (cm)
Plittersdorf	340		s	s	s	s	s	s	77,1
		22							
Maxau	362		s	4,2	s	4,7	4,6	5,1	83,7
		39							
Speyer	401		s	s	5,5	6,9	7,1	7,9	96,8
		24							
Mannheim	425		2,5	2,6	s	s	6,8	7,7	96,3
		18							
Worms	443		s	s	2,6	3,4	7,0	8,0	93,5
		38							
Nierstein-Opp.	481		s	2,2	3,0	3,2	4,2	7,5	91,8
		17							
Mainz	498		2,2	s	s	s	s	7,5	77,7
		30							
Bingen	528		s	2,0	1,7	2,2	2,5	6,1	69,4
		18							
Kaub	546		5,0	s	s	6,9	7,4	9,6	83,7
		10							
St. Goar	556		3,6	3,5	3,3	4,7	5,0	9,0	92,7
		15							
Boppard	571		s	5,5	s	6,7	6,7	10,9	95,9
		21							
Koblenz	592		s	s	s	s	s	s	95,1
		22							
Andernach	614		4,1	4,6	s	4,9	6,1	9,9	106,0
		16							
Linz	630		s	3,5	3,6	3,8	5,0	9,9	113,8
		25							
Bonn	655		s	s	2,9	2,3	3,6	8,0	102,6
		33							
Köln	688		s	s	2,6	s	3,4	7,8	110,9
		56							
Düsseldorf	744		s	s	s	2,7	s	6,5	108,4
		37							
Ruhrort	781		s	s	2,9	s	3,0	4,8	121,0
		33							
Wesel	814		s	2,9	3,1	3,1	3,3	3,4	128,3
		23							
Rees	837		2,6	2,5	2,5	3,1	3,2	3,3	132,2
		15							
Emmerich	852		s	s	s	s	s	s	122,2
Number of network stations			15	11	10	7	5	3	

Table 5-11 Standard errors $\sigma\Delta y$ in cm for various series of network stations according to multiple linear regression (measurements at 3 days).

s = network station

Station	Distance to Basel (km)	Station distance (km)	Alternative 1	2	3	4	5	6	σy (cm)
Plittersdorf	340		s	s	s	s	s	s	77,1
		22							
Maxau	362		s	4,0	s	4,6	5,0	5,2	83,7
		39							
Speyer	401		s	s	5,3	6,4	7,8	8,4	96,8
		24							
Mannheim	425		2,4	2,5	s	s	7,3	7,4	96,3
		18							
Worms	443		s	s	2,5	3,3	7,0	7,4	93,5
		38							
Nierstein-Opp.	481		s	2,1	2,9	3,1	3,8	5,6	91,8
		17							
Mainz	498		2,0	s	s	s	s	6,4	77,7
		30							
Bingen	528		s	1,9	1,7	2,2	2,4	4,7	69,4
		18							
Kaub	546		4,6	s	s	6,6	7,1	8,0	83,7
		10							
St. Goar	556		3,5	3,4	3,2	4,6	4,8	6,9	92,7
		15							
Boppard	571		s	5,1	s	6,7	6,8	9,2	95,9
		21							
Koblenz	592		s	s	s	s	s	s	95,1
		22							
Andernach	614		4,1	4,4	s	4,8	6,1	9,4	106,0
		16							
Linz	630		s	3,4	3,5	3,7	4,9	9,3	113,8
		25							
Bonn	655		s	s	2,8	2,3	3,4	7,6	102,6
		33							
Köln	688		s	s	2,4	s	3,2	7,5	110,9
		56							
Düsseldorf	744		s	s	s	2,7	s	6,4	108,4
		37							
Ruhrort	781		s	s	2,7	s	3,0	4,7	121,0
		33							
Wesel	814		s	2,9	3,2	3,1	3,3	3,4	128,3
		23							
Rees	837		2,5	2,4	2,4	3,0	3,1	3,2	132,2
		15							
Emmerich	852		s	s	s	s	s	s	122,2
Number of network stations			15	11	10	7	5	3	

Table 5-12 Standard errors σΔy in cm for various series of network stations according to multiple linear regression (measurements at 5 days).

s = network station

The series where more points of time are used show smaller values of $\sigma\Delta y$. This can be seen by comparison of Tables 5-11 and 5-12. In the case of 5 points of time for alternative 3 (Table 5-12), the levels at nearly all stations under examination can be reconstituted within the assessed requirement, apart from the station Speyer.

It also appears that the number of network stations is not the only factor affecting the quality of the network. From Table 5-12, for instance, it is seen that the alternatives 1 and 2, with 15 and 11 network stations respectively, produce greater $\sigma\Delta y$ values than alternative 3 with 10 network stations. Apparantly the locations of the network stations are of great importance as well. Clearly, this fact surely should be taken into account when designing a network.

2. The Rio Magdalena in Colombia

Similar alternatives to those considered in example 1 have been examined for the Rio Magdalena, for the series of gauging stations shown on the map of Fig. 3-10. The resulting $\sigma\Delta y$ values for 10 alternative networks, once again for the case with simultaneous measurements only, and for the cases with additional measurements at 1 and 2 days before, as well as after, the days considered, are given in Tables 5-13A, 5-13B and 5-13C. The calculations are based on daily water level measurements made during the year 1974. The $\sigma\Delta y$ values are considerably greater than the values found for the river Rhine in example 1, a fact which is partly due to the longer distances between the stations. Local conditions of the regime, and unknown to the author, also probably play a role.

If a criterion of 3,5 cm were chosen as a design value, none of the alternatives would constitute an adequate network. Whether the full series of stations satisfies this requirement can only be judged if measurements at intermediate sites could be made available. An extensive examination of the local conditions also would be required, in particular focussing on those places where discontinuities in the flow conditions may be expected, such as at points where tributaries join the main river.

3. Tidal waters in the Netherlands

The methods previously described and illustrated in the Examples 1 and 2 can also be applied to tidal waters. Since the water level variations occur over much shorter time spans in tidal waters than in tideless rivers or lakes, use is made of water level data, measured at more frequent intervals. In the following examples hourly water level data are used. The data are from network stations measured at 7 points of time (i.e. from 3 hours before until 3 hours after the hour considered).

station	t = 0			Alternatives					σy
	1	2	3	4	5	6	7	8	
Casc Simon Bolivar	s	s	s	s	s	s	s	s	45,2
Salado Blanco	16,08	16,08	16,15	s	s	s	s	16,03	43,7
Puente Balseadero	s	s	s	9,53	9,40	9,33	9,39	s	35,2
Puente Momico	16,65	16,65	16,55	s	s	s	s	15,77	54,1
Betanía	27,97	28,05	s	s	28,16	s	28,05	s	62,6
Puente Santander	s	s	s	16,07	s	s	s	s	43,4
Purificación	13,89	13,80	13,85	15,13	13,28	13,28	13,07	s	40,2
Girardot-2	s	s	s	s	s	s	s	s	143,2
Arranchaplumas	17,81	s	s	s	s	s	s	s	129,5
Puerto Salgar	s	9,68	9,64	9,73	9,73	9,64	9,57	9,43	59,4
Puerto Immarco	17,00	17,15	17,06	17,16	17,09	17,02	s	s	78,2
Puerto Berria	s	s	s	s	s	s	s	s	53,4
Number of stations	6	6	7	7	7	8	8	9	

Table 5-13A simultaneous measurements

station	t = -1d; 0; +1d			Alternatives					σy
	1	2	3	4	5	6	7	8	
Casc Simon Bolivar	s	s	s	s	s	s	s	s	45,2
Salado Blanco	15,58	15,63	15,52	s	s	s	s	15,38	43,7
Puente Balseadero	s	s	s	8,83	8,13	8,10	8,10	s	35,2
Puente Momico	15,95	16,04	15,86	s	s	s	s	15,03	54,1
Betanía	27,41	27,48	s	s	26,92	s	26,86	s	62,6
Puente Santander	s	s	s	13,53	s	s	s	s	43,4
Purificación	11,05	11,12	11,09	12,82	10,93	10,91	10,01	s	40,2
Girardot-2	s	s	s	s	s	s	s	s	143,2
Arranchaplumas	16,64	s	s	s	s	s	s	s	129,5
Puerto Salgar	s	8,63	8,49	8,51	8,60	8,43	8,30	7,88	59,4
Puerto Immarco	13,70	13,45	13,43	13,63	13,39	13,38	s	s	78,2
Puerto Berria	s	s	s	s	s	s	s	s	53,4
Number of stations	6	6	7	7	7	8	8	9	

Table 5-13B measurements at 3 days

In tidal waters the high and low tide levels, indicated here by HW and LW respectively, are also of importance. These can be calculated in a similar way to simultaneously occuring levels, although the HW and LW levels do not occur simultaneously, but are subject to a travel time between the sites concerned.

station	t = -2d; -1d; 0; +1d; +2d			alternatives					σy
	1	2	3	4	5	6	7	8	
Casc Simon Bolivar	s	s	s	s	s	s	s	s	45,2
Salado Blanco	15,13	15,06	15,02	s	s	s	s	14,72	43,7
Puente Balseadero	s	s	s	8,36	7,66	7,64	7,63	s	35,2
Puente Momico	15,43	15,56	15,35	s	s	s	s	14,45	54,1
Betanía	26,49	26,46	s	s	25,95	s	25,81	s	62,6
Puente Santander	s	s	s	12,77	s	s	s	s	43,4
Purificación	10,10	10,47	10,36	12,30	10,32	10,21	9,20	s	40,2
Girardot-2	s	s	s	s	s	s	s	s	143,2
Arranchaplumas	15,75	s	s	s	s	s	s	s	129,5
Puerto Salgar	s	8,39	8,18	8,26	8,34	8,12	8,05	7,56	59,4
Puerto Immarco	13,66	13,07	12,90	13,01	12,79	12,76	s	s	78,2
Puerto Berria	s	s	s	s	s	s	s	s	53,4
Number of stations	6	6	7	7	7	8	8	9	

Table 5-13C measurements at 5 days

Table 5-13 Values of σΔy (cm) for various network alternatives along the Rio Magdalena (Colombia).

s = network station

The following examples are considered:
- the western part of the Wadden Sea;
- the Western Scheldt estuary.

THE WESTERN PART OF THE WADDEN SEA

The geographical situation of the area is shown in Fig. 5-7. The tidal gauging network includes the following stations:
- Den Helder
- Oude Schild
- Vlieland-haven
- West Terschelling
- Harlingen
- Kornwerderzand
- Den Oever

In Table 5-14 the resulting σΔy values are given for three network alternatives, both for the calculated hourly water levels (h), as well as for the HW and LW levels. The hourly values are based on data for the year 1981. A separate calculation of the σΔy values for HW and LW was made using the measured HW and LW values in the period 1976...1980.

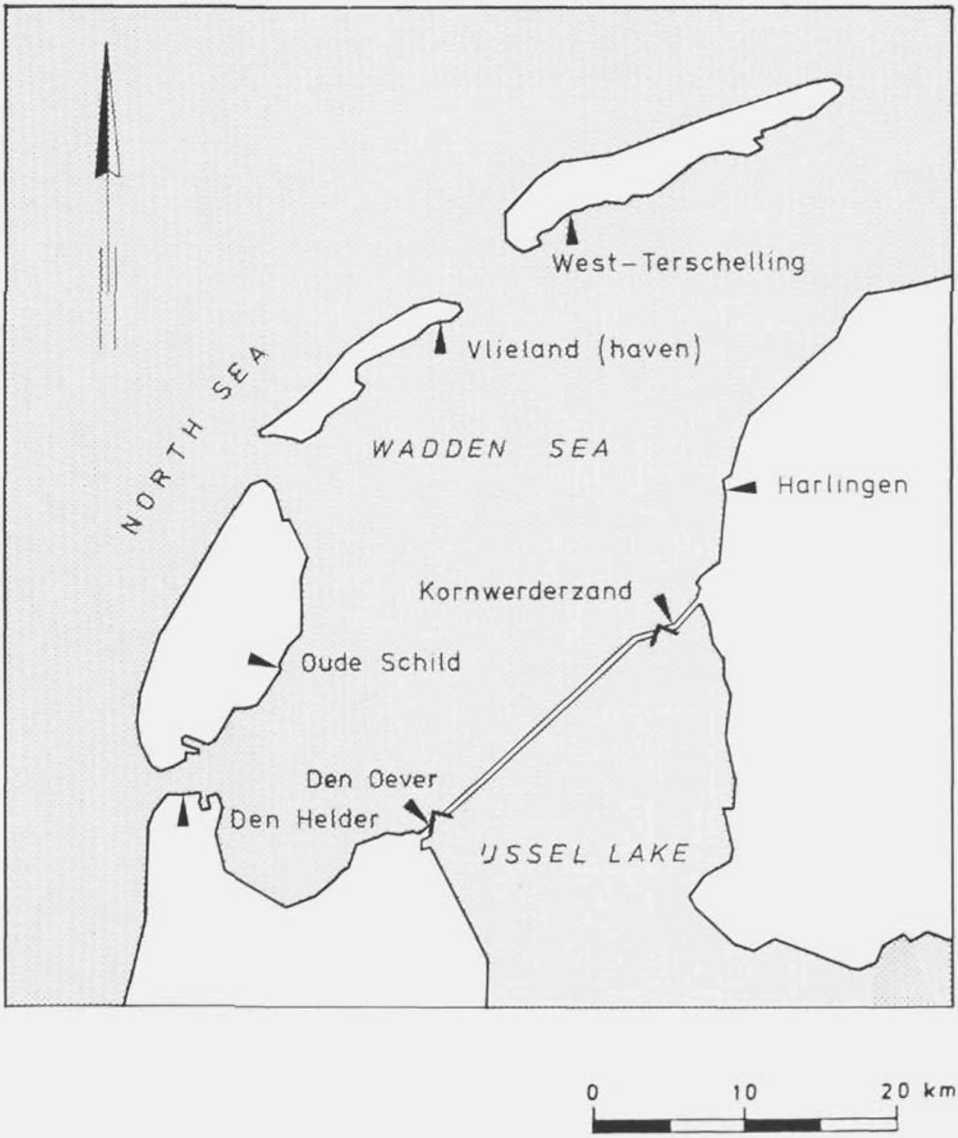


Fig. 5-7 Water level gauging stations in the western Wadden Sea

Station	Network alternatives								
	1			2			3		
	h	HW	LW	h	HW	LW	h	HW	LW
Den Helder	s	s	s	s	s	s	s	s	s
Oude Schild	3,32	2,96	2,81	2,75	2,57	2,72	3,02	2,80	2,70
Vlieland-haven	3,87	3,89	3,80	3,65	3,80	3,52	3,68	3,84	3,78
West Terschelling	s	s	s	s	s	s	s	s	s
Harlingen	s	s	s	s	s	s	s	s	s
Kornwerderzand	4,34	3,17	5,63	3,19	2,86	4,84	s	s	s
Den Oever	5,71	4,54	6,50	s	s	s	4,00	4,10	5,59
Number of stations	3			4			4		

s = network station

Table 5-14 Values of $\sigma\Delta y$ (cm) for three network alternatives in the western part of the Wadden Sea (Netherlands)

The best overall results are obtained from alternative 2.

	1975				1976				1977			
Alternative	1	2	3	4	1	2	3	4	1	2	3	4
Bath	s	s	s	s	s	s	s	s	s	s	s	s
Hansweert	6,2	6,5	s	6,5	5,9	6,4	s	6,2	5,5	6,0	s	5,7
Terneuzen	s	5,1	4,9	s	s	4,8	4,5	s	s	4,2	3,9	s
Vlissingen	s	s	s	4,8	s	s	s	4,9	s	s	s	3,8
Breskens	3,9	3,9	3,8	4,3	3,4	3,5	3,4	3,8	2,6	2,7	2,6	3,2
Cadzand	s	s	s	s	s	s	s	s	s	s	s	s
	1978				1979				1980			
Bath	s	s	s	s	s	s	s	s	s	s	s	s
Hansweert	6,5	6,9	s	6,7	6,9	7,4	s	7,0	5,1	5,4	s	5,4
Terneuzen	s	4,7	4,4	s	s	3,2	3,0	s	s	3,8	3,7	s
Vlissingen	s	s	s	4,7	s	s	s	4,1	s	s	s	4,1
Breskens	3,4	3,6	3,4	3,9	2,9	3,1	3,0	3,1	3,6	3,6	3,6	4,0
Cadzand	s	s	s	s	s	s	s	s	s	s	s	s
	1981				1982				1983			
Bath	s	s	s	s	s	s	s	s	s	s	s	s
Hansweert	3,8	4,5	s	4,1	3,6	5,0	s	4,0	4,3	4,6	s	4,5
Terneuzen	s	3,9	3,4	s	s	3,8	2,9	s	s	3,1	2,9	s
Vlissingen	s	s	s	4,2	s	s	s	3,0	s	s	s	3,0
Breskens	2,8	2,9	2,8	3,6	5,3	5,4	5,3	5,6	2,2	2,5	2,4	2,9
Cadzand	s	s	s	s	s	s	s	s	s	s	s	s

s = network station

Table 5-15 Values of $\sigma\Delta y$ (cm) for four network alternatives for the years 1975...1983 in the Western Scheldt estuary (hourly data).

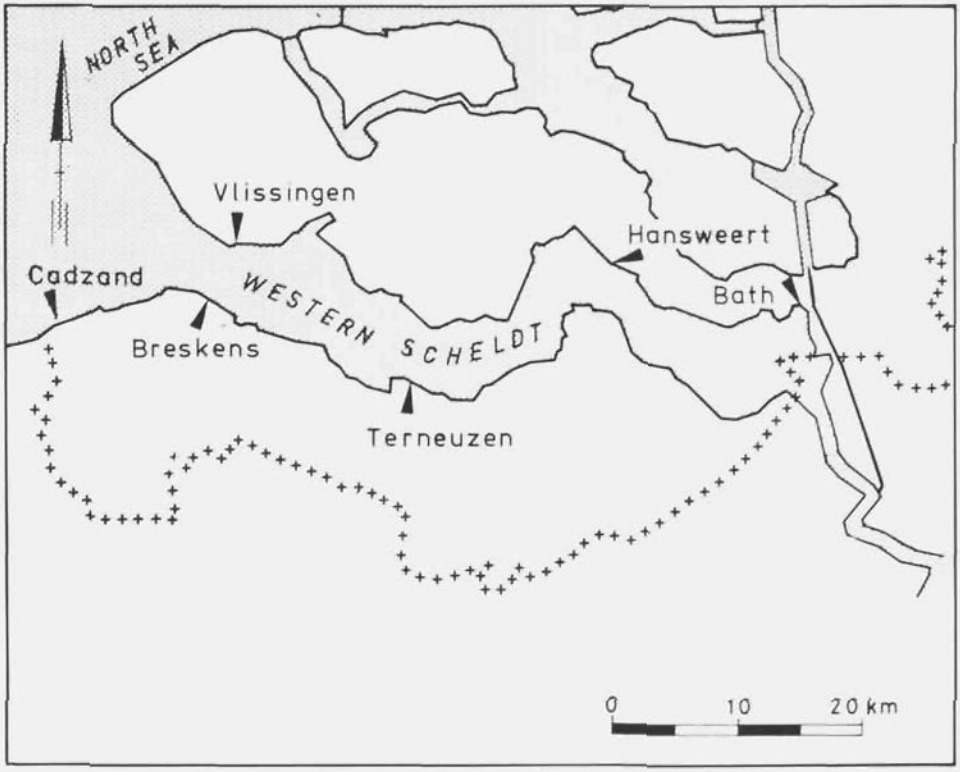


Fig. 5-8 Water level gauging stations in the Western Scheldt estuary

THE WESTERN SCHELDT ESTUARY

The local situation is shown in Fig. 5-8, and the gauging stations considered are the following:

- Bath
- Hansweert
- Terneuzen
- Vlissingen
- Breskens
- Cadzand

In this case the values of $\sigma\Delta y$ for the hourly data are examined for four network alternatives for each year in the period 1975...1983. The results are shown in Table 5-15 and in Fig. 5-9. Values of $\sigma\Delta y$ show some variability over the years, a with a tendency to decrease. The latter is due to the fact that over these years the existing spring clocks have been replaced by more accurate quartz clocks, and processing methods have improved. This influence is particularly clear in Hansweert and Vlissingen. In 1982 the observations at Breskens were strongly disturbed because of technical reasons, but these problems were resolved in the following year.

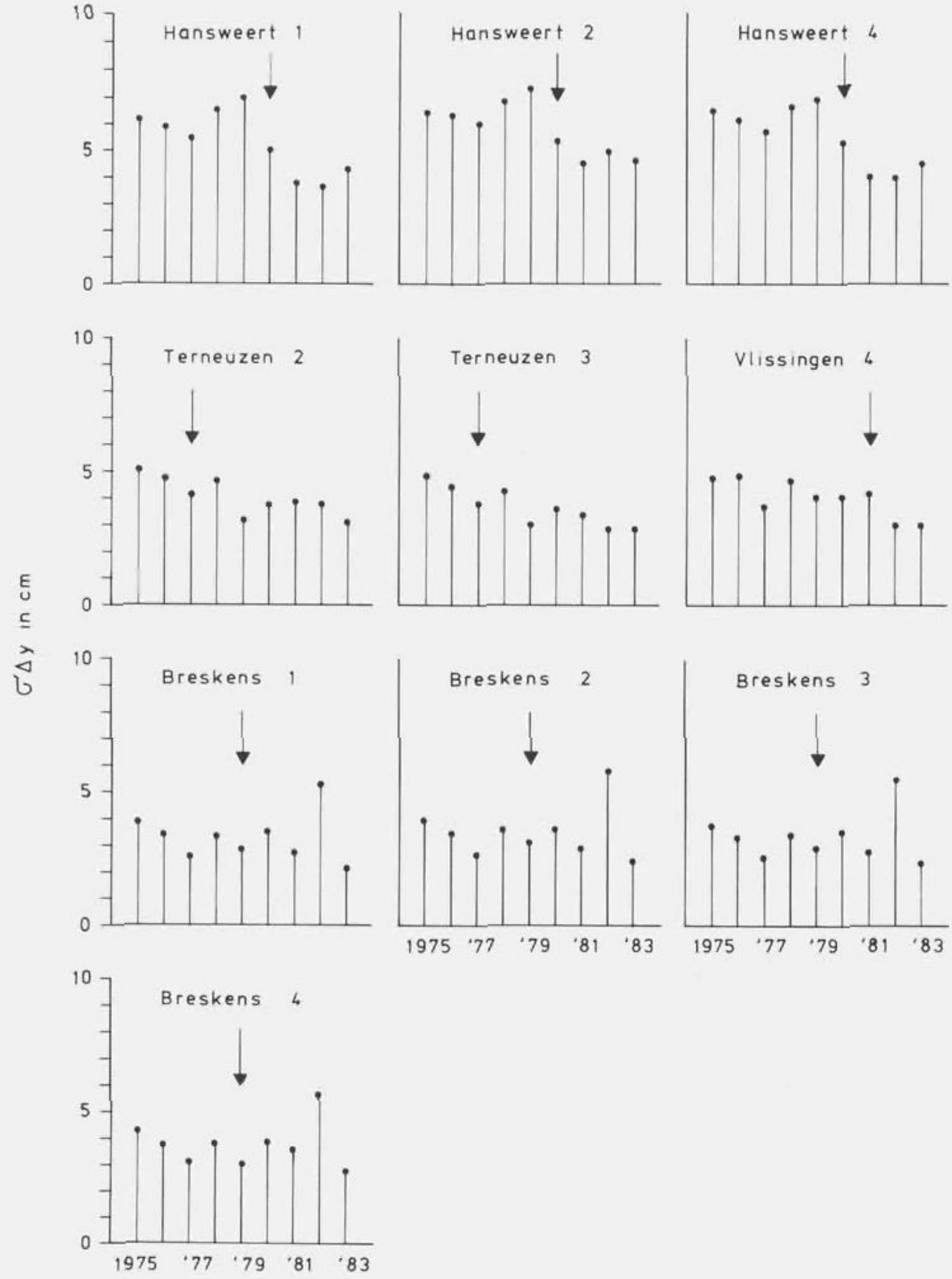


Fig. 5-9 Values of σΔy (cm) for four network alternatives for the years 1975...1983 in the Western Scheldt estuary (hourly data)

↓ Indicates year of introduction of quartz clock

In summary, it can be said that the quality of observation and processing techniques strongly influences the resulting values of $\sigma\Delta y$, in which the errors of estimate of both the network stations and the stations under examination are included. This is very important, since the resulting values may be decisive in resolving the question as to whether a certain station should continue to be maintained or closed down.

6 Regression interpolation along the intermediate reaches between network stations

6.1 General

For sites where no measurements have been or will be carried out (i.e. the sites classified under category c in Section 5.1), a relation, as given by eq (5-1), cannot be derived from local data. If such a relation is required, its coefficients have to be obtained in a different way.

An approach to calculate water level data at such intermediate sites is as follows. The statistical characteristics of the m measured sites (i.e. the network stations), are used to calculate the corresponding characteristics for any site for which data are required. This is done by means of an interpolation method, whether or not account is to be taken of physical considerations. The characteristics of the network stations are:

- the mean values $\bar{\underline{x}}$ (i = 1...m)
- the standard deviations $\sigma\underline{x}_i$ (i = 1...m)
- the correlation coefficients between each pair of network stations ϱ ($\underline{x}_i$, $\underline{x}_k$) (i,k = 1...m).

Interpolations will lead to the characteristics for each intermediate site y considered; that is

- the mean value $\bar{y}$
- the overall standard deviation σy
- the correlation coefficients between y and each of the $\underline{x}_i$ levels ϱ ($\underline{x}_i$,y) (i = 1...m).

The interpolations of the correlation coefficients are depicted in Fig. 6-1. Since as a rule ϱ is close to 1, it is difficult to show the behaviour of the correlation coefficient by a curve expressed in terms of ϱ itself. It is better to draw a curve of the value 1-ϱ ($\underline{x}_i$,y) and use a greater vertical scale.

For interpolation one of the methods described in Chapter 4 can be used e.g. spline functions. In Section 6.4.5 an example will be discussed in which some physical elements are included i.c. tidal components.

The approach is related to techniques like Kriging (Delhomme, 1976) and Optimal Interpolation (Gandin, 1965). These methods are in particular related to areal phenomena like groundwater levels or precipitation, and they have a statical charac-

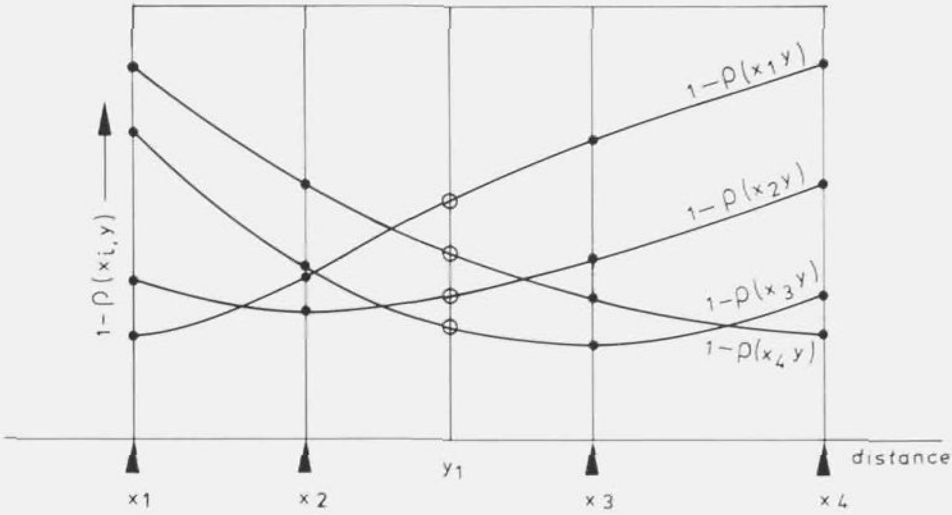


Fig. 6-1 Interpolation of the values 1-ρ (x_i,y)

ter. The technique described here is in particular focussed on surface water levels (in one- as well in two-dimensional applications) and their correlation structure, also with regard to its dynamical character.

6.2 Calculation of the standard error

The standard error, σΔy, can be calculated using eq (5-36), so

$$\sigma\Delta y = \sqrt{\frac{|R(\underline{X}\ \underline{Y})|}{|R(\underline{X}\ \underline{X})|}}\ \sigma\underline{y} \qquad (6\text{-}1)$$

The value of the lower determinant can be calculated from the correlation coefficients between each pair of network stations. In the upper determinant, the values of the correlation coefficients between y and each $\underline{x}_i$ (i = 1...m), which followed from the interpolation, are included. Consequently the value of that determinant can be derived. The value of $\sigma\underline{y}$ also follows from such an interpolation.

Extending this procedure for a series of sites, located at about equal distances along the river, a curve showing the course of the σΔy-values along te river can be derived.

The optimum conditions occur if the network stations are located so that all $\sigma\Delta y$-values are a little below, but not higher than, the design value of eq (2-2). If the values are higher, the network does not fullfil its requirements; if they are far below, the network is not efficient. Examples presented later on will demonstrate the above statements.

The standard error of estimate, $\sigma\Delta\hat{y}$, can be determined using eq (5-30) which corresponds to

$$\sigma\Delta\hat{y} = \sqrt{\sigma^2\Delta y - \varepsilon_y^2} \tag{6-2}$$

This is possible only if the value of the standard error of measurement, ε_y, is known or at least assumed.

6.3. Calculation of the regression coefficients

The regression coefficients are the coefficients A_i ($i = 0,1...m$) of eq (5-1):

$$\hat{y} = A_0 + \sum_{i=1}^{m} A_i x_i \tag{6-3}$$

For the sites with measurements (category b of Section 5.1) these coefficients could be determined according to eq (5-12), as

$$A = (\underline{X}^T\underline{X})^{-1} (\underline{X}^T\underline{Y}). \tag{6-4}$$

It is also possible to set up this equation for the non-measured sites (category c), using the values obtained by interpolation. In this case eq (6-4) has to be re-expressed in terms of means, standard deviations and correlation coefficients.

A number of matrices and vectors needs to be introduced for this development. First

$$R^+(\underline{X}\,\underline{X}) = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & R(\underline{X}\,\underline{X}) \end{array}\right], \tag{6-5}$$

which includes $R(\underline{XX})$ (i.e. the correlation matrix of the measured $\underline{x}_i$-values).

The second matrix

$$S^{+}(\underline{X}) = \left[\begin{array}{c|ccc} 0 & & 0 & \\ \hline & \sigma \underline{x}_1 & & 0 \\ 0 & & \ddots & \\ & 0 & & \sigma \underline{x}_m \end{array}\right]$$

has as its diagonal elements the standard deviations of the measured water levels at the network stations, except for a zero in the first element.

The (m+1) row vector

$$\bar{\underline{X}} = [1 \bar{\underline{x}}_1 \ldots \bar{\underline{x}}_m] \tag{6-7}$$

contains the value 1 and the mean values of the measured levels at the network stations.

The (n+1) column vector

$$R^{+}(\underline{X}Y) = \begin{bmatrix} 0 \\ \varrho(\underline{x}_1 y) \\ \cdot \\ \cdot \\ \cdot \\ \varrho(\underline{x}_m y) \end{bmatrix} \tag{6-8}$$

contains apart from a zero, the correlation coefficients between y and each of the $\underline{x}_i$ values, as found by interpolation.

The (m+1)×(m+1) diagonal matrix

$$S^{+}(Y) = \left[\begin{array}{c|ccc} 0 & & 0 & \\ \hline & \sigma y & & 0 \\ 0 & & \ddots & \\ & 0 & & \sigma y \end{array}\right] \tag{6-9}$$

has the standard deviation of y, found by interpolation, in the diagonal positions, apart form a zero in the first element.

The matrices of eq (6-4) can now be re-expressed as:

$$\underline{X}^T\underline{X} = n\{S^+(\underline{X}) . R^+ (\underline{X}\,\underline{X}) . S^+(\underline{X})^T + \bar{\underline{X}}^T\bar{\underline{X}}\} \tag{6-10}$$

$$\underline{X}^T\underline{Y} = n\,\{S^+(\underline{X}) . S^+(Y) . R^+ (\underline{X}\,Y) + \bar{y}\,\bar{\underline{X}}^T\} \tag{6-11}$$

Here $\bar{y}$ is the mean value of y, derived by interpolation. The value n represents the number of measurements that is supposed to be carried out. In this context it is a non existent value, since the y-values are not measured at all, but are found by interpolation. However n is of no importance, since it disappears when substituting eqs (6-10) and (6-11) into eq (6-4).

Eqs (6-10) and (6-11) can be proven by substituting the matrices of eqs (6-5)...(6-9) into eqs (6-10) and (6-11). This substitution will not be presented here in extenso, but the derivation of eqs (6-10) and (6-11) will become clear if one considers the meaning of the correlation coefficients. For the case of two variables, $\underline{x}_1$ and $\underline{x}_2$, then

$$\varrho(\underline{x}_1, \underline{x}_2) = \frac{\text{Cov}\ \underline{x}_1, \underline{x}_2}{\sqrt{\text{Var}\ \underline{x}_1 . \text{Var}\ \underline{x}_2}}$$

$$= \frac{1/n\ (\Sigma\underline{x}_1\,\underline{x}_n - n\,\bar{\underline{x}}_1\,\bar{\underline{x}}_2)}{\sigma\underline{x}_1 . \sigma\underline{x}_2} \tag{6-12}$$

From this it follows that

$$\Sigma\underline{x}_1\,\underline{x}_2 = n\,\{\sigma\underline{x}_1 . \sigma\underline{x}_2 . \varrho(\underline{x}_1, \underline{x}_2) + \bar{\underline{x}}_1 . \bar{\underline{x}}_2\} \tag{6-13}$$

This equation occurs repeatedly in matrix calculations when deriving eqs (6-10) and (6-11). In fact eq (6-13) is the two variable equivalent of eqs (6-10) and (6-11).

6.4. Some hypothetical cases of networks

6.4.1. Interpolation between two stations at variable distance

Consider a reach of variable length d between two gauging stations. The correlation coefficient between two sites at distance z is assumed to be a gaussian function of this distance, so that

$$\varrho(z) = \varrho(0) . \exp\left\{-\left(\frac{z}{D}\right)^2\right\} \tag{6-14}$$

Here D is a fixed length, called the 'correlation scale', which determines the correlation structure. The correlation coefficient $\varrho(0)$ at distance zero (i.e. the correlation coefficient of a station 'with itself') is taken the correlation coefficient $\varrho(\underline{x}_i\underline{x}_i)$ as given by eq (5-155). It is the correlation coefficient between two independently measured values $\underline{x}$ at the same station. Consequently it may be expressed as

$$\varrho(0) = \varrho(x_i x_i) = \frac{\mathrm{Var}\ x_{it}}{\mathrm{Var}\ x_{it} + \varepsilon_x^2} \tag{6-15}$$

For this case it is assumed that

$$\mathrm{Var}\ x_{it} = \mathrm{Var}\ x_t \ (i = 1,2) \tag{6-16}$$

When expressed in terms of the variance of the measured values $\underline{x}$

$$\mathrm{Var}\ \underline{x} = \mathrm{Var}\ x_t + \varepsilon_x^2 \tag{6-17}$$

the correlation coefficient is

$$\varrho(0) = 1 - \frac{\varepsilon_x^2}{\mathrm{Var}\ \underline{x}} . \tag{6-18}$$

This value is substituted into eq (6-14) for $\varrho(0)$, so

$$\varrho(z) = \left(1 - \frac{\varepsilon_x^2}{\mathrm{Var}\ \underline{x}}\right) . \exp\left\{-\left(\frac{z}{D}\right)^2\right\} \tag{6-19}$$

In this example the following values are assumed:

$$\begin{aligned} \mathrm{Var}\ \underline{x} &= 1 \quad \mathrm{m}^2 \\ \bar{\underline{x}} &= 0 \quad \mathrm{m} \\ \varepsilon_x &= 0{,}025 \quad \mathrm{m} \\ D &= 100 \quad \mathrm{km} \end{aligned}$$

These values are of the same order as found in practice. Now consider subsequently the following distances between the two stations:

d = 10 km; 20 km; 30 km; 40 km; 50 km.

A site at a distance z from the left hand station will be examined (see Fig. 6-2).

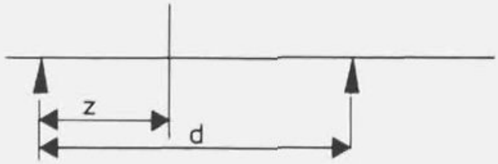


Fig. 6-2. Situation of a site under examination between two gauging stations at a distance d.

For this site the regression coefficients of matrix A of eq (6-4) were calculated, using eqs (6-10) and (6-11). The variance and the mean value of the water level y at the site under examination are assumed to be equal to the corresponding values at the gauging stations. This assumption, together with the numerical values of variance and mean employed, gives according to eqs (6-10) and (6-11)

$$\underline{X}^T\underline{X} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \varrho(d) \\ 0 & \varrho(d) & 1 \end{bmatrix} \tag{6-20}$$

and:

$$\underline{X}^T\underline{Y} = \begin{bmatrix} 0 \\ \varrho(z) \\ \varrho(d-z) \end{bmatrix} \tag{6-21}$$

Now, from eq (6-4)

$$A = \begin{bmatrix} A_0 \\ A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \varrho(d) \\ 0 & \varrho(d) & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \varrho(z) \\ \varrho(d-z) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{1-\varrho^2(d)} & -\dfrac{\varrho(d)}{1-\varrho^2(d)} \\ 0 & -\dfrac{\varrho(d)}{1-\varrho^2(d)} & \dfrac{1}{1-\varrho^2(d)} \end{bmatrix} \begin{bmatrix} 0 \\ \varrho(z) \\ \varrho(d-z) \end{bmatrix} \tag{6-22}$$

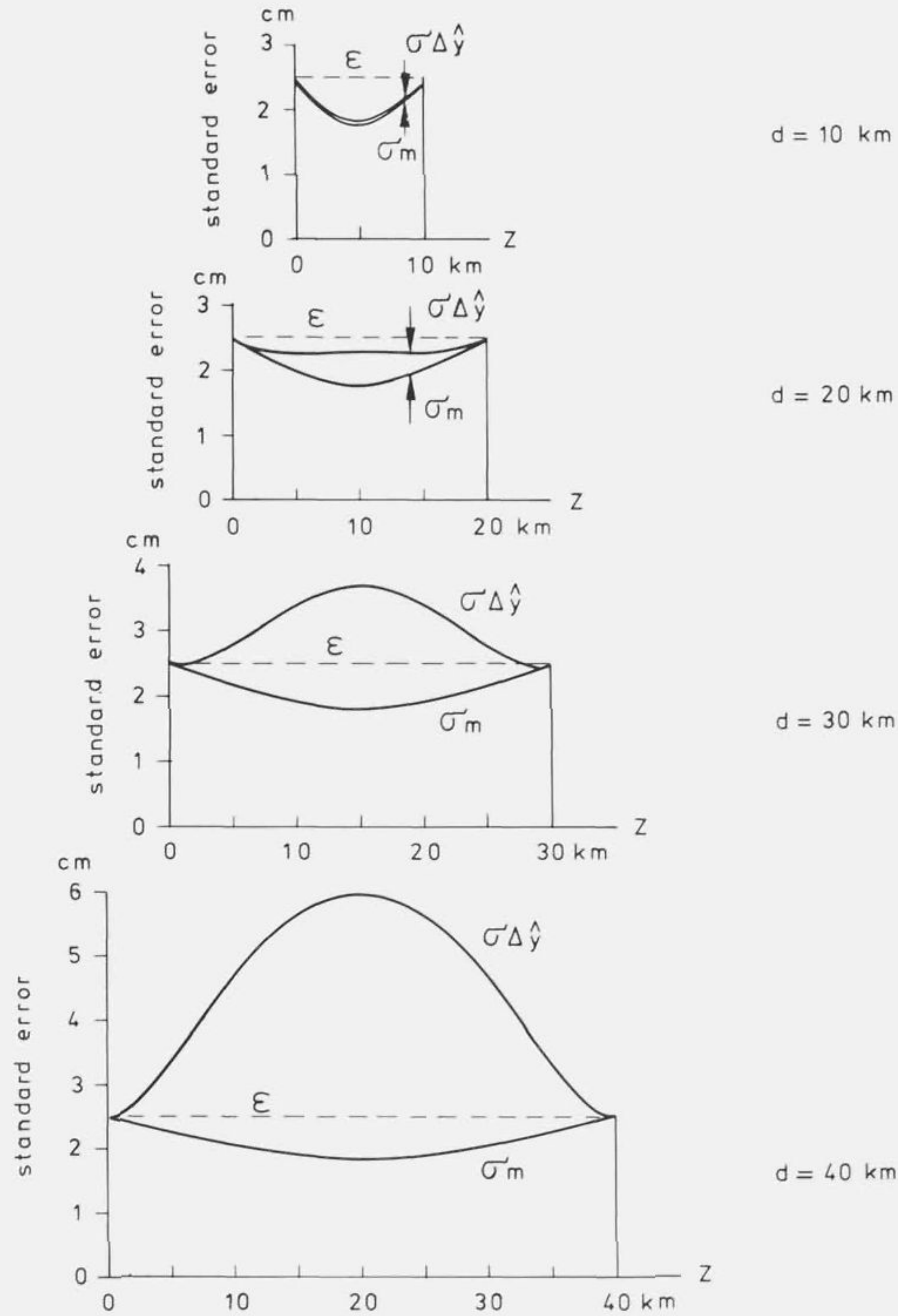


Fig. 6-3 Curves of the standard error of estimate $\sigma\Delta y$ and of the standard error of propagated errors of measurement σ_m for various station distance d

which corresponds to

$$\left.\begin{aligned} A_0 &= 0 \\ A_1 &= \frac{\varrho(z) - \varrho(d).\varrho(d-z)}{1 - \varrho^2(d)} \\ A_2 &= \frac{\varrho(d-z) - \varrho(d).\ \varrho(z)}{1 - \varrho^2(d)}\ . \end{aligned}\right\} \tag{6-23}$$

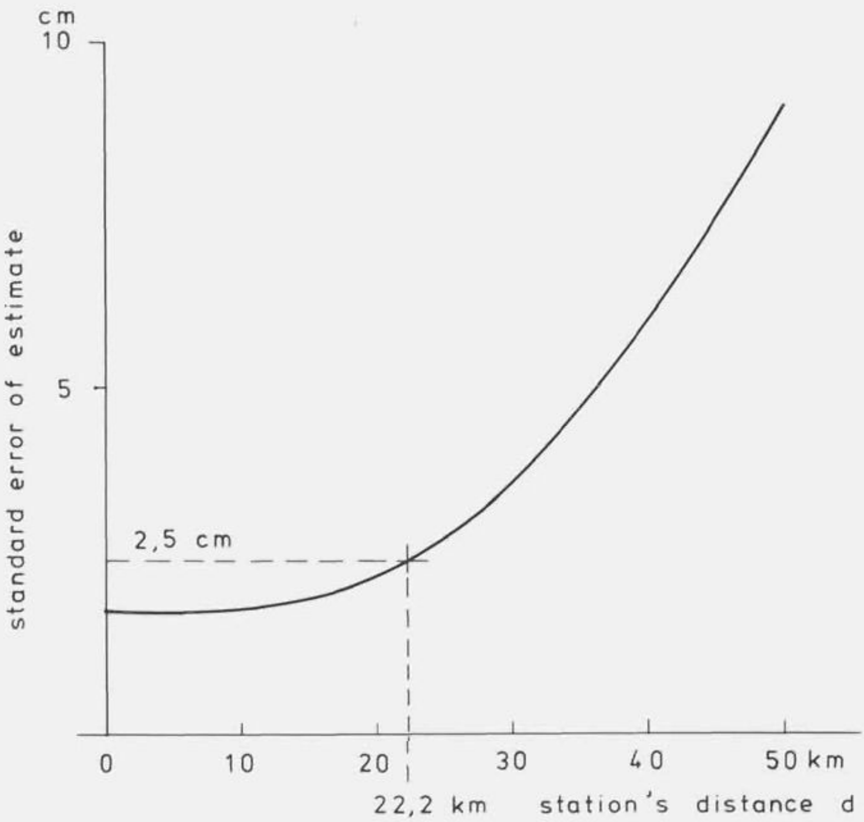


Fig. 6-4 Halfway standard error of estimate vs. station distance

These values were used to calculate the measurement error propagation function, $A^TA = A_1^2 + A_2^2$, according to eq (4-11) as well as its contribution to the standard error of estimate

$$\sigma_m = \varepsilon_x \sqrt{A_1^2 + A_2^2} \tag{6-24}$$

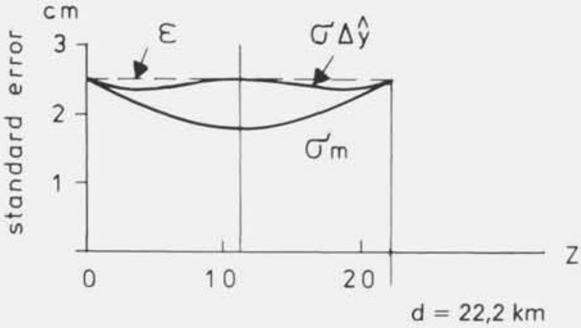


Fig. 6-5 Curves of $\sigma\Delta\hat{y}$ and σ_m for the case $\sigma\Delta\hat{y}$ (1/2 d) = ε

The standard errors of estimatey $\sigma\Delta\hat{y}$ were also calculated using eq (5-38), so

$$\sigma\Delta\hat{y}= \sqrt{\frac{\begin{vmatrix} 1 & \varrho(z) & \varrho(d\text{-}z) \\ \varrho(z) & 1 & \varrho(d) \\ \varrho(\delta\text{-}z) & \varrho\delta(d) & 1 \end{vmatrix}}{1-\varrho^2(d)}\,\sigma^2(y) - \varepsilon_y^2}. \qquad (6\text{-}25)$$

The resulting curves for $\sigma\Delta\hat{y}$ and σ_m, (the standard errors of estimate and the contributions to it from propagated measurement errors at the gauging stations) are plotted in Fig. 6-3 for the station distances considered. At the station sites at the ends of the reach considered all values are approximately equal to the standard error of measurement*). Along the reaches the propagated errors of measurement, expressed in σ_m are reduced until a minimum value is reached halfway along the reach. This serves as a reminder of the behaviour of this error in linear interpolation (Fig. 4-1).

At short station distances (e.g. 10 km), the model error enlarges the standard errors of estimate by a relatively small amount, thus leaving the intermediate values under the values at the ends of the reach. This is still the case for a distance of 20 km, but now the difference with the values at the ends is much smaller. For larger distances, like 30 km and 40 km, the values at the ends are largely exceeded.

In network design the best solution is found when the values of the standard error of estimate are almost equal along the whole reach, or, at the utmost somewhat smaller. In that case the standard error of estimate along the reach is at most equal to

*) At the station site the standard errors are somewhat smaller than the standard errors of measurement, owing to the smoothing effect, expressed by eq (5-162).

that of the measured values. On the other hand, the station distance should be such that a smaller standard error of estimate (thus of higher quality) than that required, is avoided, and with it unnecessary costs and efforts.

In this example the situation with the best equality is obtained for a station distance of 22,2 km. This is shown in Fig. 6-4, where a relation is given between the halfway standard error of estimate and the station distance. Curves for $\sigma\Delta\hat{y}$ and σ_m along the reach for this station distance are given in Fig. 6-5. For this case a distance of 22,2 km leads to an optimum network with about equal standard errors along the whole reach.

6.4.2 Interpolation between four stations

A. SYMMETRICAL NETWORK CONFIGURATION; MIDDLE REACH OF CONSTANT LENGTH.

In this example the middle reach is fixed at 40 km whereas the two outer reaches vary from 50 km to 10 km (see Fig. 6-6). The correlation structure and the standard error of measurement are the same as in Section 6.4.1.

The result is presented in Fig. 6-6 where curves are shown for the standard error of estimate $\sigma\Delta\hat{y}$ and for the propagated errors of measurement

$$\sigma_m = \varepsilon_x \sqrt{A^T A} = \varepsilon_x \sqrt{\sum_{i=1}^{4} A_i^2}. \tag{6-26}$$

Going from the large outer reaches of 50 km to the small outer reaches of 10 km it appears that the standard error of estimate in the outer reaches gradually decreases whereas that of the inner reach at first decreases too, but later on increases. This increase at short outer reaches is due to the increase of the propagated errors of measurement. This behaviour of the standard error becomes clear when the extreme situations are considered. If the outer reach is very long (e.g. infinitely) the outer stations have no influence on the calculated results in the middle reach: then the situation is the same as the two station case of example 1 (middle reach = 40 km; see Fig. 6-3). If the outer reaches are decreased gradually the standard error of estimate $\sigma\Delta\hat{y}$ in the middle reach decreases, owing to the influence of the outer stations. At the same time the propagated measurement error increases, since the outer stations have negative regression coefficients, thus enlarging the regression coefficients of the two middle stations.

The behaviour of the effect in the centre of the middle reach is shown in Fig. 6-7. Below a certain length of the outer reaches the effect of the propagated errors is so

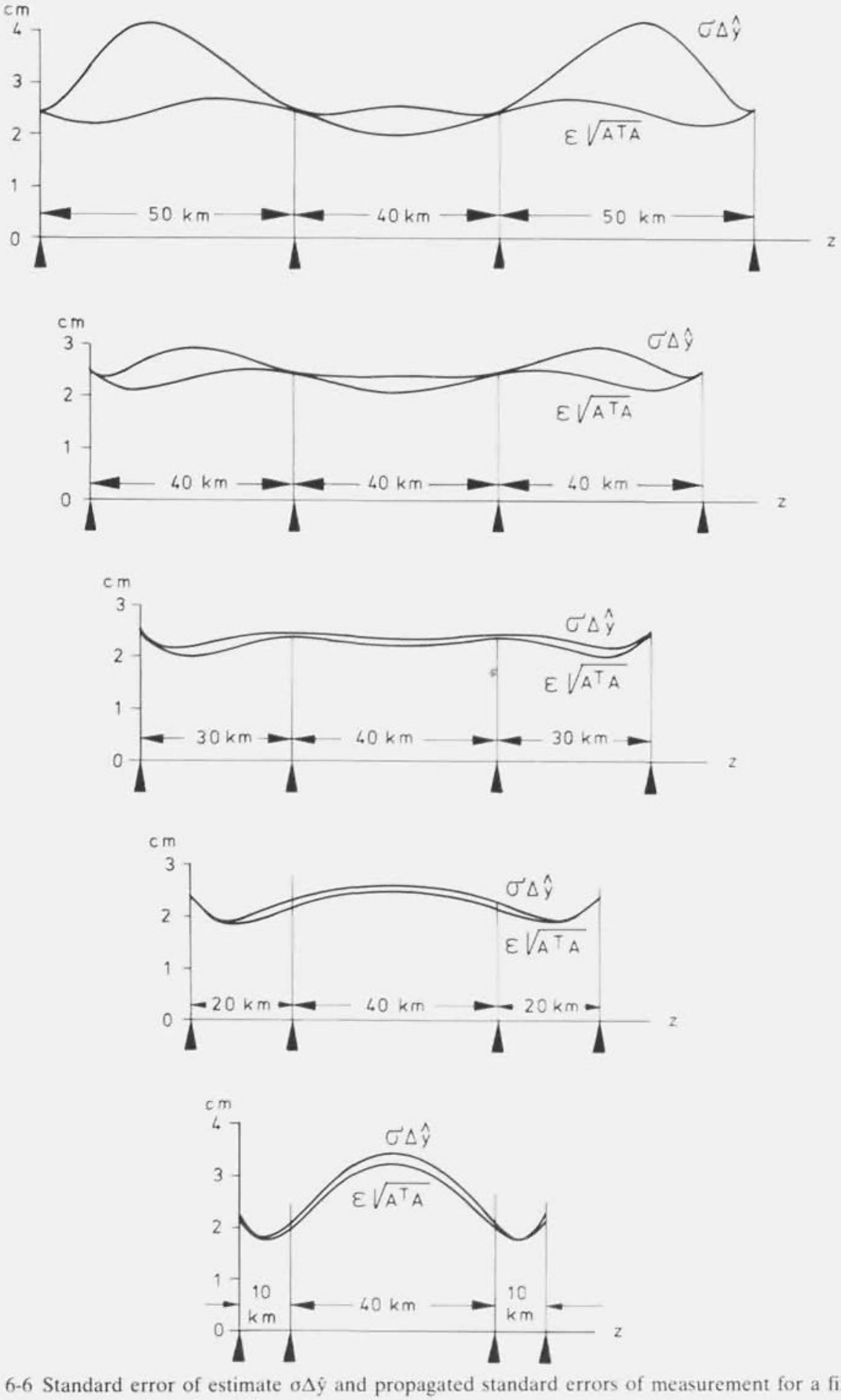


Fig. 6-6 Standard error of estimate $\sigma\Delta\hat{y}$ and propagated standard errors of measurement for a fixed central reach and variable outer reaches

strong, that it causes the standard error of estimate to rise. In the case considered this is below about 30 km. If the outer reach length decreases further to zero the outer and middle stations become subject to averaging which decreases the propagated errors of measurement in the middle reach. However, the standard error of estimate continues to increase,owing to the fact that the outer stations no longer give an essential contribution to the information.

If the outer reach is zero, this is equivalent to a single reach, bounded at both ends by two stations which are in fact averaged. The propagated error of measurement in the centre is now

$$\varepsilon_x \sqrt{\Sigma A^2} = \varepsilon_x \sqrt{4 \times (1/4)^2} = 1/2\ \varepsilon_x. \quad (6\text{-}27)$$

The standard error of estimate $\sigma\Delta\hat{y}$ in the centre has returned to the two station case of example 1.

From Fig. 6-7 it follows that the model errors are very small over a certain range, going in this example from outer reaches between 10 km - 30 km. The optimum network is found at outer reaches of around 30 km as can be seen from Fig. 6-6. In that case the standard error of estimate is of about equal value along the three reaches.

The conclusion for this example 'A' is, that, if gauging stations are located very close together, this will lead to an unfavourable result for the standard error of estimate, due to the fact that the propagated measurement errors are strongly amplified.

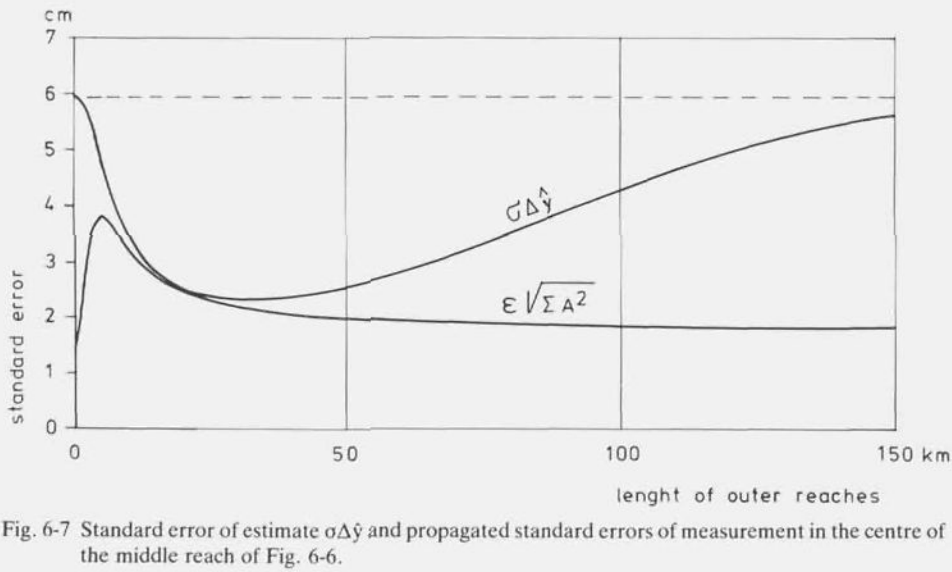


Fig. 6-7 Standard error of estimate $\sigma\Delta\hat{y}$ and propagated standard errors of measurement in the centre of the middle reach of Fig. 6-6.

B. SYMMETRICAL NETWORK CONFIGURATION; TOTAL CONSTANT LENGTH OF THE THREE REACHES TOGETHER.

This is a more practical situation than example 'A', since it may give an approach to the question of where network stations should be located within a given river reach. Three cases are considered, all with a total length of 100 km but with different lengths for the sub reaches. These cases are shown in Fig. 6-8. The result is more or less similar to that of example 'A'. The best solution, just as in example 'A', is obtained with outer reaches of 30 km each and a middle reach of 40 km.

6.4.3. Interpolation between five stations at equal distances.

The stations are located at distances of 50 km each. Two correlation structures are considered, defined by lengths, D, (in eq (6-14)) of 100 km and 150 km respectively. The results are given in Fig. 6-9 for the 100 km case and in Fig. 6-10 for the 150 km case.

In these figures the curves are shown for the different regression (Lagrange) coefficients A_1 ... A_5, for the propagation function A^TA and, in the lower figure, for the standard error of estimate $\sigma\Delta\hat{y}$ and the propagated standard error of measurement $\varepsilon_x \sqrt{A^TA}$.

For the 100 km case (Fig. 6-9) the spatial correlation is relatively small. The results recall to mind the examples in Chapter 4 where the interpolation curve is forced through the measured points (See Fig. 4-16). The propagated error function has an undulating shape, going through the ε_x-values at the station locations. The standard error of estimate is relatively large, in particular in the outer reaches.
In the 150 km case (Fig. 6-10) the spatial correlation is stronger. This leads to a lower weight for the measurements themselves and, subsequently to a certain smoothing. Lower values are obtained for the propagated measurement errors and also for the standard errors of estimate.

6.5. The river IJssel network

In contrast to the above examples, this example is an existing case. This case has already been considered in Section 5.6.2, but then only for the sites of the four additional stations (De Steeg, Dieren, Deventer and Wijhe), for which the water levels could be derived from simultaneous measurements at the five network stations (IJsselkop, Doesburg, Zutphen, Olst and Katerveer). The Lagrange functions A_i and the error propagation function A^TA were shown in Fig. 5-5.

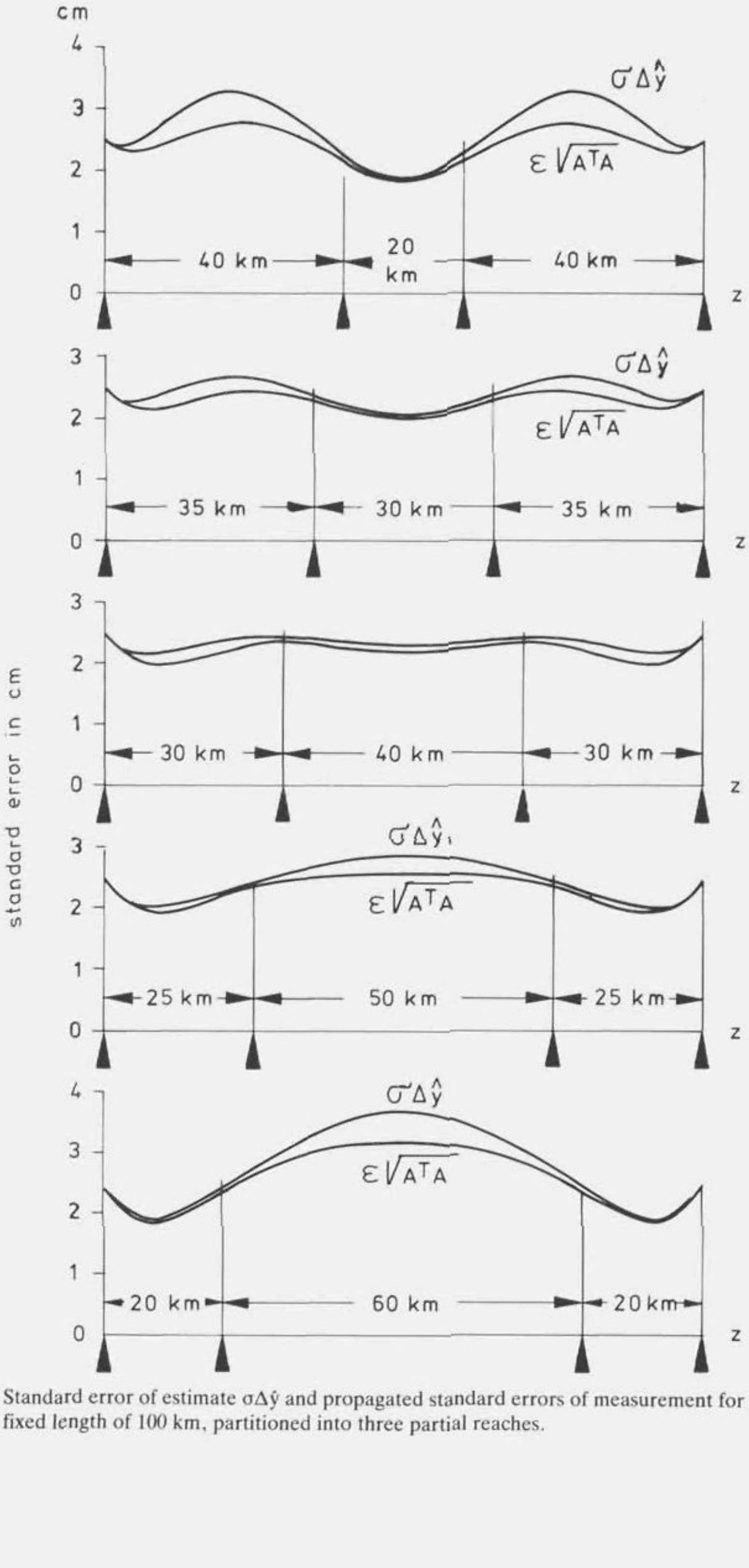
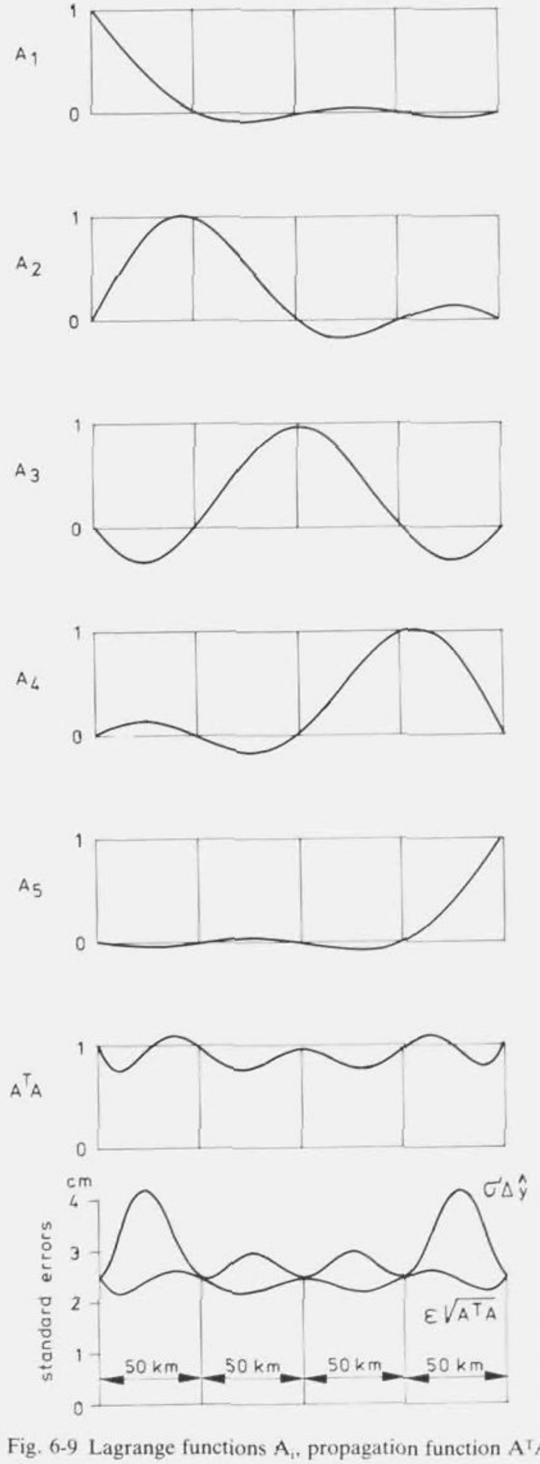


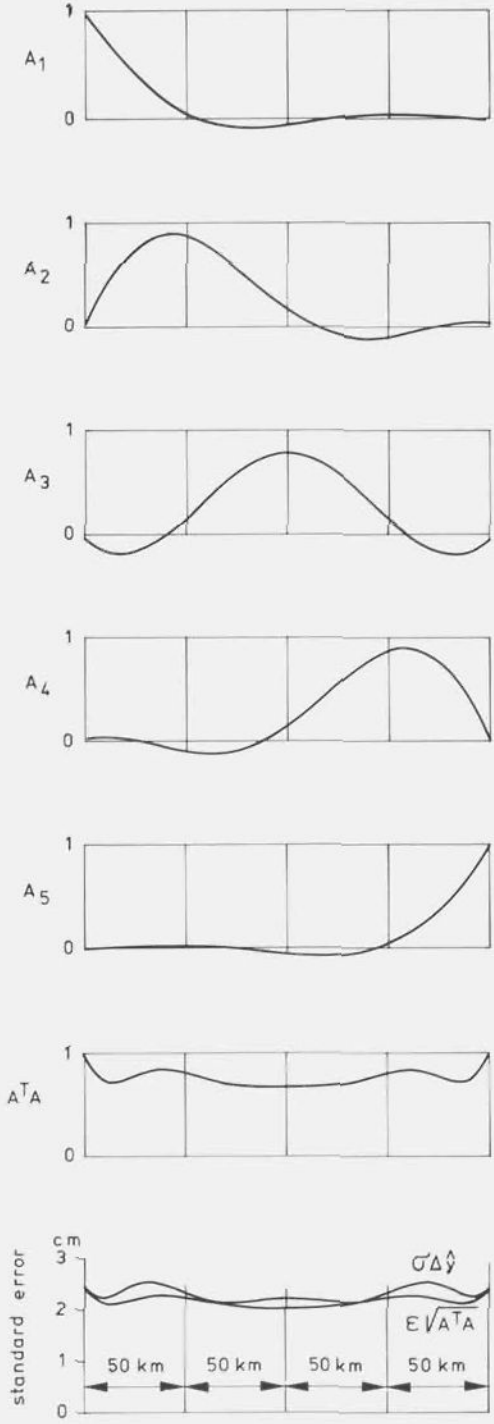
Fig. 6-8 Standard error of estimate σΔŷ and propagated standard errors of measurement for a reach of fixed length of 100 km, partitioned into three partial reaches.



Parameters employed:

-Correlation scale (eq (6-14)): D = 100 km
-standard error of measurement: ε = 2,5 cm

Fig. 6-9 Lagrange functions A_i, propagation function A^TA, and standard errors $\sigma\Delta\hat{y}$ and $\varepsilon\sqrt{A^TA}$



Parameters employed:

-Correlation scale (eq (6-14)): D = 150 km
-standard error of measurement: $\varepsilon = 2{,}5$ cm

Fig. 6-10 Lagrange functions A_i propagation function A^TA, and standard errors $\sigma\Delta\hat{y}$ and $\varepsilon\sqrt{A^TA}$

This time these functions, as well as the standard error of estimate, will be examined for the entire river reach between the stations IJsselkop and Katerveer. For this examination, continuous curves along the reach length are required for the correlation coefficients with regard to the five network stations, as well as those for the standard deviation σ_y and for the mean value $\bar{y}$. These curves were found by spline functions interpolation, as described in Chapter 4. In order to arrive at an acceptable result the following points need to be taken into account:

1. The curves have to fit the plots of the correlation coefficients very closely, in order not to get impossible or improbable combinations of the correlation coefficients at the sites of the network stations and the additional stations. For this reason the weight factor P, used in the spline function interpolation was given a value as high as P = 10.
2. At the site of the station, to which the relevant correlation coefficient applies, the curve should show a maximum value. This implies a horizontal tangent at this site. Care should be taken, however, not to cause a minimum or an inflexion point at that site.
3. Values of the correlation coefficient should not exceed 1. An incorrect interpolation curve might include points with values in excess of 1.
4. No curves with strongly alternating slopes and improbable shapes are acceptable, unless there are special reasons for them.

These four constraints can be taken into account in this phase of the study i.e. when the interpolation curves of the correlation coefficients, of the standard deviations and of the mean values are being constructed. There are some more conditions, but these will be discussed later on.

Because the correlation coefficients ϱ as a rule are very close to 1 (they may have values of the order of 0,999) it is preferable to consider the values $1-\varrho$, and to plot, for instance, the values $1000 \times (1-\varrho)$ in the graphs. In doing so the phenomena can be demonstrated more clearly.

Since the curves have to fit the calculated correlation coefficients closely (condition 1), only high P-values can be used. But a curve closely forced through the plots may, at other places, lead to correlation coefficients, or combinations of correlation coefficients, not fulfilling conditions 3 and 4. The only way to influence the shapes of the curves, in order to obtain an acceptable result is in the selection of dummy points, as described in chapter 4.

Fig. 6-11 shows the curves for the values $1000 \times (1-\varrho)$, concerning the correlation coefficients with respect to the five network stations. The + signs indicate the calculated correlation coefficients at the sites of all network and additional stations. They correspond with the values given in Table 5-6, whereas those at the network

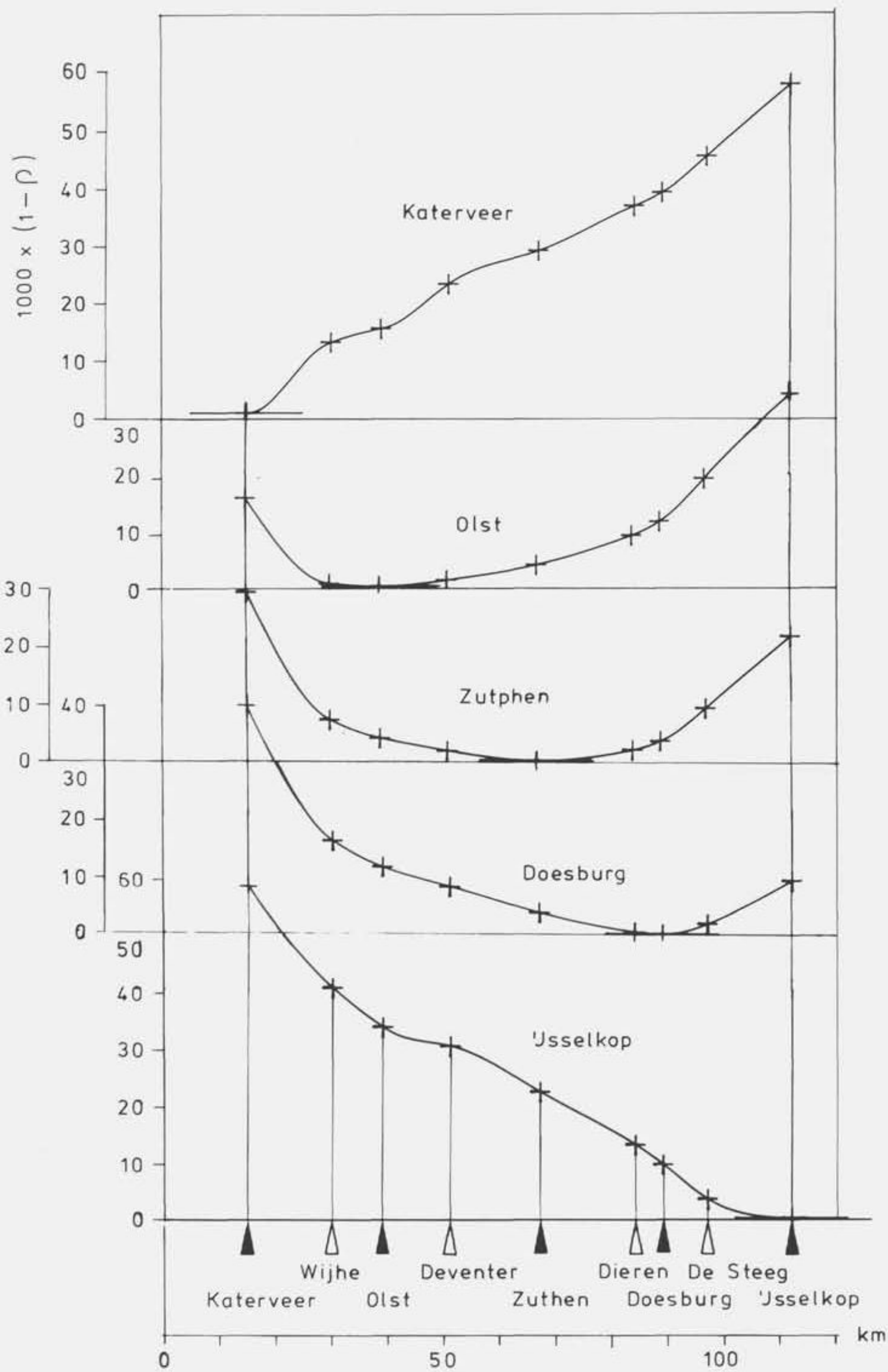


Fig. 6-11 Curves of the value 1000 × (1 – ϱ) along the river IJssel ϱ = correlation coefficient

station sites themselves correspond with values, based on a standard error of measurement of 1,5 cm, as described in Section 5.6.2. The sites along the river where the partial curves (2nd-degree parabolas) change, are indicated in Table 6-1. As a rule the intermediate reaches between the stations were divided into equal parts of 4 to 5 km. However there are two exceptions.

distance to Kampen (km)		indication, transition point
15		Katerveer (network station)
	20	dummy point
	25	dummy point
30		Wijhe (additional station)
39		Olst (network station)
	43	dummy point
	47	dummy point
51		Deventer (additional station)
	55	dummy point
	59	dummy point
	63	dummy point
67		Zutphen (network station)
	71,25	dummy point
	75,50	dummy point
	79,75	dummy point
84		Dieren (additional station)
89		Doesburg (network station)
	93	dummy point
97		De Steeg (additional station)
	101,7	dummy point
	107,3	dummy point
112		IJsselkop (network station)

Table 6-1 Transition points of the partial curves in Fig. 6-11 and Fig. 6-12.

1. The reach Wijhe-Olst (9 km) is not divided at all. A division into two or more parts appeared not to lead to a maximum of the correlation coefficient of Olst at that site, but to an inflexion point, yielding higher correlation coefficients (or lower values of $1000\times(1-\varrho)$) when moving downstream from Olst (see condition 2). This is because of the strong rise of the curve when approaching Katerveer, owing to the strong reduction of the correlation coefficient in that direction. Not applying a division forces the curve to rise from Olst to Wijhe everywhere along that reach.
2. The reach De Steeg-IJsselkop (15 km) is not divided into three equal parts of 5 km each but into three unequal parts. The reason lies in one of the conditions to be discussed later on.

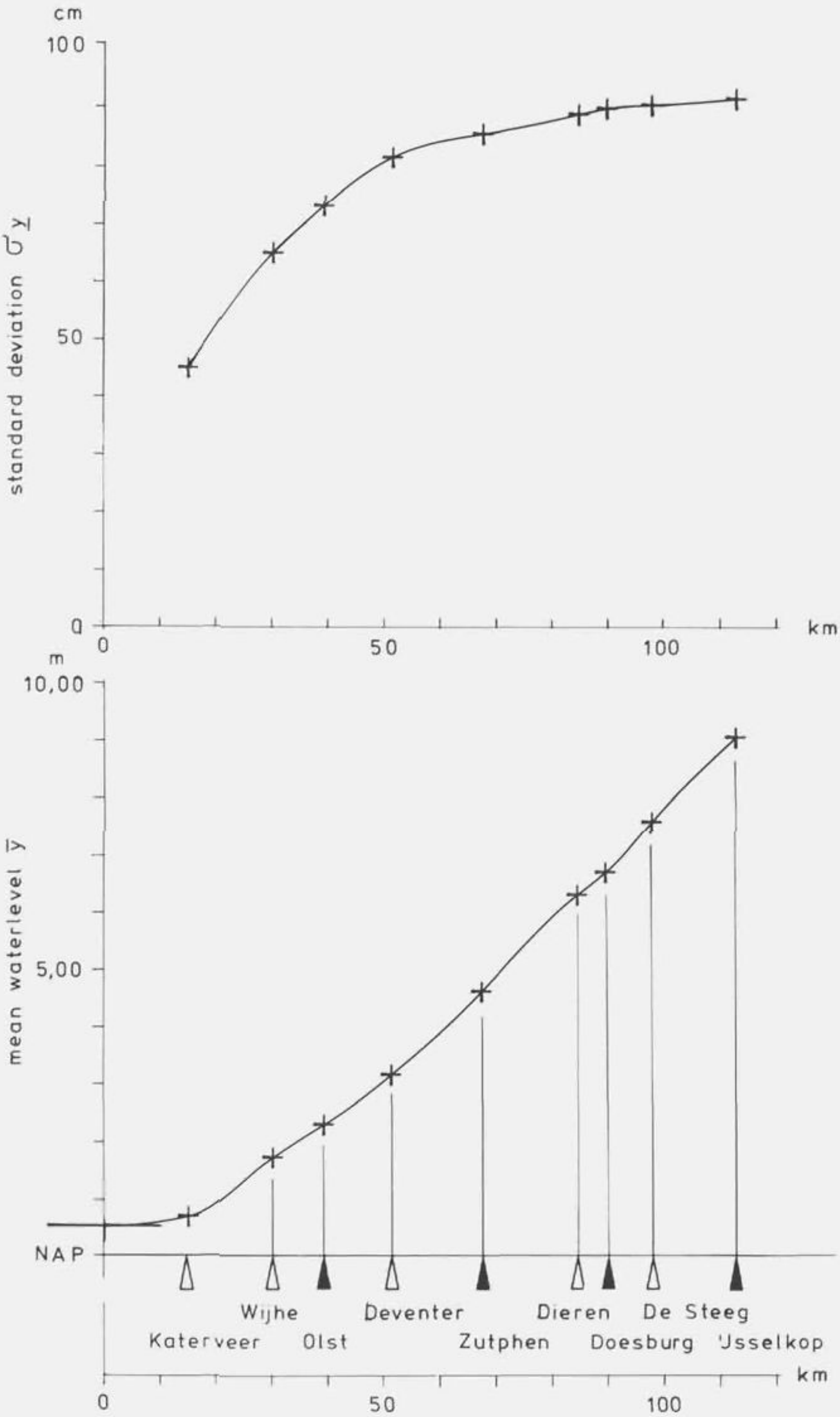


Fig. 6-12 Curves of the standard deviation $\sigma_{\underline{y}}$ and of the mean water level $\underline{y}$ along the river IJssel

Fig. 6-12 shows the curves of the standard deviation σy and of the mean values ȳ. Here, the same sub-division of the river reach, as used for the correlation coefficients was applied.

On the basis of the data represented by the curves of Fig. 6-11 and Fig. 6-12 for each site along the river, the standard error σΔy can be derived by application of eq (6-1). The result for the river IJssel is shown in Fig. 6-13. The curve shows minima for the sites of the network stations and maxima in the intermediate reaches. As a whole the shape of the curve appears acceptable.

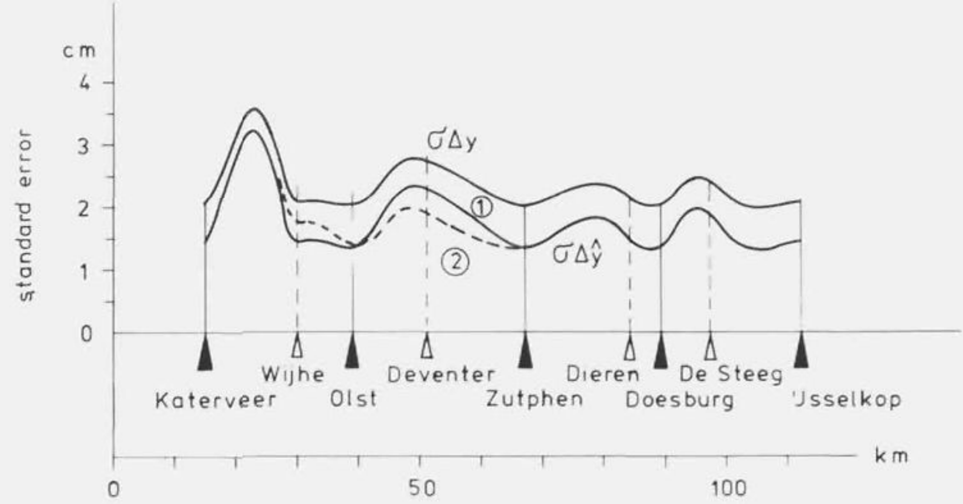


Fig. 6-13 Curves of the standard error σΔy and of the standard error of estimate σΔŷ along the river IJssel
1: σΔŷ, if ε = 1,5 cm for all stations.
2: σΔŷ, if ε = 1 cm for the Wijhe station,
ε = 2 cm for the Deventer station,
ε = 1,5 cm for the other stations.

In order to find a curve for the standard error of estimate σΔŷ the values of the curve of σΔy have to be corrected according to eq (6-2):

$$\sigma\Delta\hat{y} = \sqrt{(\sigma\Delta y)^2 - \varepsilon_y^2}.$$

If a standard error of measurement of $\varepsilon_y = 1{,}5$ cm is applied for all stations, and imagine that this would hold for all sites, the values of σΔŷ are those indicated by the corresponding curve of Fig. 6-13. For most of the stations and the reaches this curve is acceptable. However, as was described in Section 5.6.2, exceptions should be made for the Wijhe and Deventer stations. For the Wijhe station the standard error of measurement can be better assessed as $\varepsilon_y = 1$ cm, whereas for the Deventer station a value of $\varepsilon_y = 2$ cm might be better. The curve of $\sigma\Delta\hat{y}$ should therefore be corrected as indicated in Fig. 6-13. These corrections really improve the shape of the curve of $\sigma\Delta\hat{y}$. The local minimum of the curve of $\sigma\Delta y$ at the Wijhe station was apparently caused by the low standard error of measurement.

The next step was the calculation of the regression coefficients, which act in this case as the Lagrange functions for the interpolations. These were calculated according to eqs (6-4) through (6-11). The result is shown in Fig. 6-14, which presents the coefficients by which the stations data have to be multiplied. The constant term is not shown, but this follows easily by applying the coefficients for the stations to their mean values. This constant term plays no role in error propagation.

In Fig. 6-14 the error propagation function A^TA is also presented. From this the propagated measurement errors can be found by (compare eq (6-25))

$$\sigma_m = \varepsilon_x \sqrt{A^TA} \qquad (6\text{-}28)$$

Applying a value $\varepsilon_x = 1{,}5$ cm at the network stations leads to the corresponding curve in the lower figure of Fig. 6-14. In this figure the corrected curve of Fig. 6-13 is also drawn. Fig. 6-14 as a whole can be compared with Fig. 6-9 and Fig. 6-10 for the hypothetical cases. A general resemblance can be recognized, but the special features and irregularities of the practical case clearly show their influence.

Now consider the division of the De Steeg-IJsselkop reach into three partial reaches. Fig. 6-15 shows the curves of $\sigma\Delta\hat{y}$ if the following subdivisions are applied:

1. km 97 - km 101 - km 108 - km 112.
2. km 97 - km 102 - km 107 - km 112.

Curve 1 shows an unexpected maximum for which no physical reason can be given, whereas curve 2 shows an unrealistic minimum. For the second subdivision, the curve of $\varepsilon_x\sqrt{A^TA}$ showing the propagated measurement errors is also given. These values appear locally to exceed the standard error of estimate which is impossible. The unrealistic shape of the curves is caused by wrong or even impossible values of the correlation coefficients derived from the spline function interpolation.

A more acceptable shape for the $\sigma\Delta\hat{y}$ curve was obtained by varying the subdivision of the De Steeg-IJsselkop reach, leading to the subdivision which was finally chosen. This experience gave rise to the following additional constraints to be used in the selection of the subdivision of the interstation reaches by dummy points:

1. The subdivision should be such that the resulting curve of $\sigma\Delta\hat{y}$ does not show irregularities which cannot be explained.

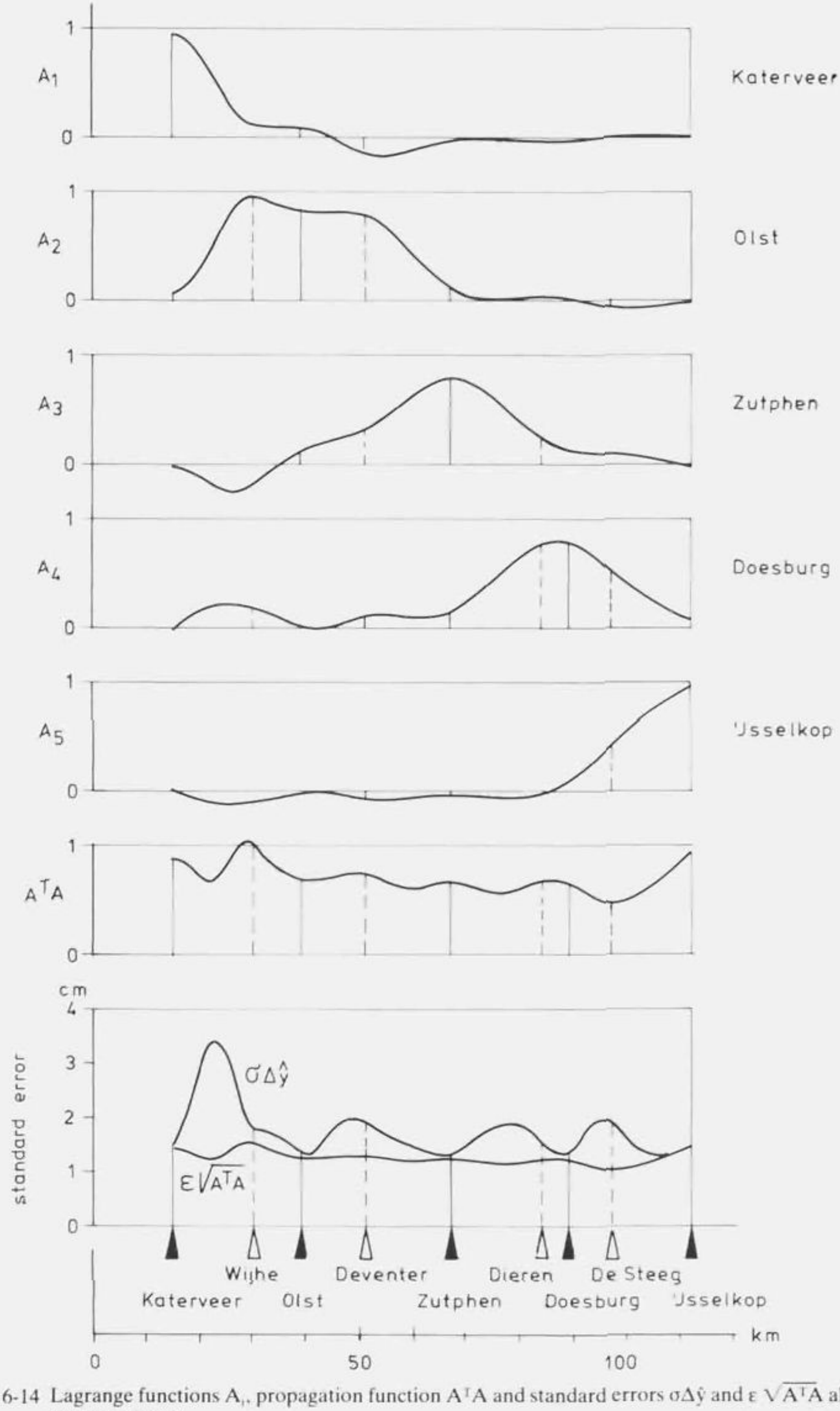


Fig. 6-14 Lagrange functions A_i, propagation function A^TA and standard errors $\sigma\Delta\hat{y}$ and $\varepsilon\sqrt{A^TA}$ along the river IJssel

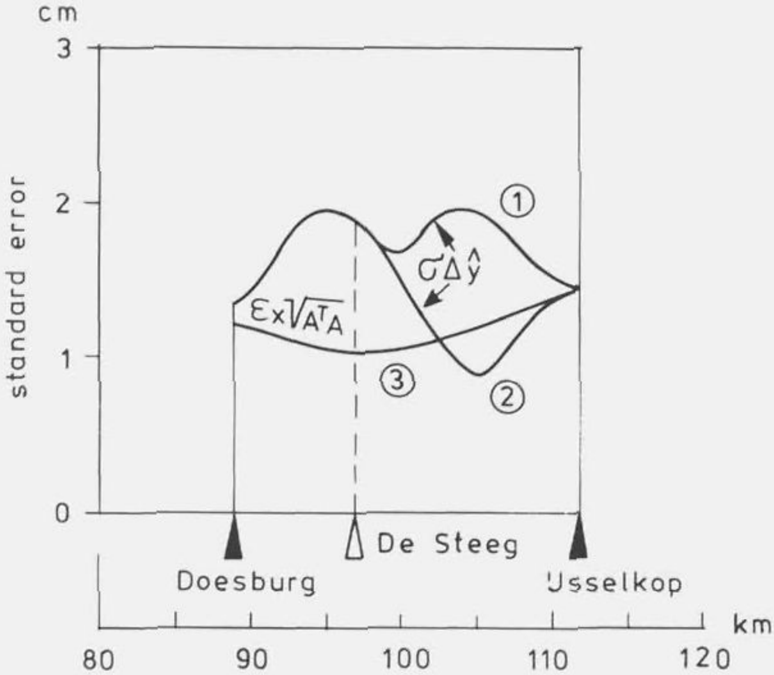


Fig. 6-15 Effect of the distribution of dummmy points on the calculated standard errors along the Doesburg-IJsselkop reach

1: $\sigma\Delta\hat{y}$, for subdivision km 97-101-108-112
2: $\sigma\Delta\hat{y}$, for subdivision km 97-102-107-112
3: $\varepsilon_x\sqrt{A^TA}$, for subdivision km 97-102-107-112

2. The propagated standard errors of measurement, $\varepsilon_x\sqrt{A^TA}$, should nowhere exceed the standard error of estimate $\sigma\Delta\hat{y}$.

It appears that in applying the methods described no straightforward procedure can be given, and a certain amount of trial and error will be needed. At any stage of the examination the results should be considered carefully and adaptions made where appropriate.

The next question is whether the network being examined is acceptable. If a standard error of measurement of $\varepsilon_x = 1{,}5$ cm is assessed as the design criterion, the standard error of estimate appears to be smaller than or equal to that criterion only in the vicinity of the network stations. In such a case an increase in network density would be desirable. If the design criterion is assessed at a higher value, for example at 2,5 cm, the whole network between Wijhe and IJsselkop is acceptable. However additional information would be required downstream of Wijhe. This could possibly be obtained by shifting the Wijhe station downstream to a site near the maximum of the $\sigma\Delta\hat{y}$ curve.

In order to test an alternative network, an estimation is required based on the curves of the correlation coefficients related to the stations of such a network. These curves can be found by a second interpolation between the values found where the sites of the new network stations intersect with the curves of the correlation coefficients of the first network stations and of the additional stations. For the case considered, this procedure was not carried out.

6.6. The Western Scheldt estuary network.

The situation is shown in Fig. 5-9 where a reach between two gauging stations is considered. The water levels along this reach are derived from four measured levels:

- simultaneous measurements at the two stations;
- one measurement at the seaward station, taken a time interval Δt earlier;
- one measurement at the landward station, taken a time interval Δt later.

The time-distance diagram of the type previously shown in Fig. 5-6 now looks as shown in Fig. 6-16.

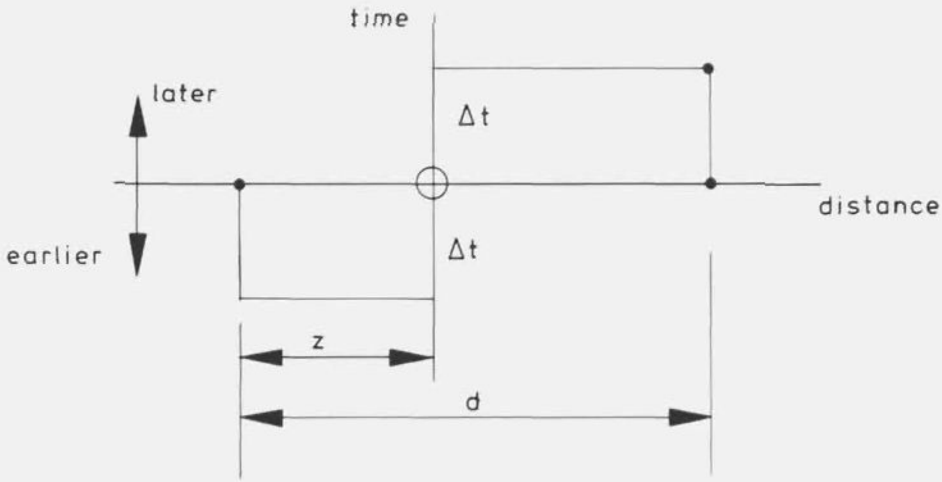


Fig. 6-16 Time distance diagram applied to a tidal reach.

In this picture z denotes the distance between the seaward station and the site under examination, whereas d represents the distance between the two gauging stations.

The correlation structure within the reach is based on eq (II-10) given in Annex II. It is written as follows for this example:

$$\varrho(\underline{y}; z, \Delta t) =$$

$$\frac{1}{\text{Var}\underline{y}}\left[(\text{Var}\underline{y} - \varepsilon^2 - \frac{1}{2}\sum_{i=1}^{N} H_i^2)\exp\left\{-\left(\frac{t-z/c}{To}\right)^2 - \left(\frac{z}{D}\right)^2\right\}\right.$$

$$\left. + \frac{1}{2}\sum_{i=1}^{N} H_i^2 \cos\frac{2\pi(\Delta t - z/c)}{T_i}\right] \qquad (6\text{-}29)$$

where:

$\varrho(\underline{y}; z, \Delta t)$	=	the correlation coefficient between measured water levels at two sites at distance z and with a time interval Δt.
Var $\underline{y}$	=	the variance of water levels at the site under examination, if measurements were carried out there. This value is to be derived by interpolation between the gauging stations.
ε	=	the standard error of measurement.
H_i	=	the amplitude of the i^{th} tidal component.
c	=	the celerity of the tidal wave.
T_o	=	an empirical base time, governing the relation between random noise and time interval, based on a gaussian function.
D	=	an empirical base distance, governing the relation between random noise and distance, based on a gaussian function.
T_i	=	the period of the i^{th} tidal component.
N	=	number of tidal components employed.

Comparing eq (6-29) with eq (II-10) the following remarks can be made:

1. Var $\underline{y}$ of eq (6-29) includes the variance of measurements ε^2 and the variance of all tidal components, $1/2\ \Sigma H_i^2$.
2. The factor $4\pi^2$ in the exponent in eq (II-10) has been deleted; this is incorporated in the empirical values T_o and D.

The correlation structure is thus assumed to be such that the variance of the difference between two measurements at different places and times is governed by:
- the harmonic components of the tidal movement;
- the remaining variance, which exists besides the variance of these components.

The noise correlation, that is the part of the correlation which is influenced by stochastic noise, but not by the tidal components (the first term in square brackets of eq (6-29)), decreases with increasing distance and/or increasing time. This is assumed to obey a gaussian function,

- in time, following the propagation of the wave, based on a fixed time parameter T_o;
- in distance, at a fixed time, based on a fixed distance parameter D.

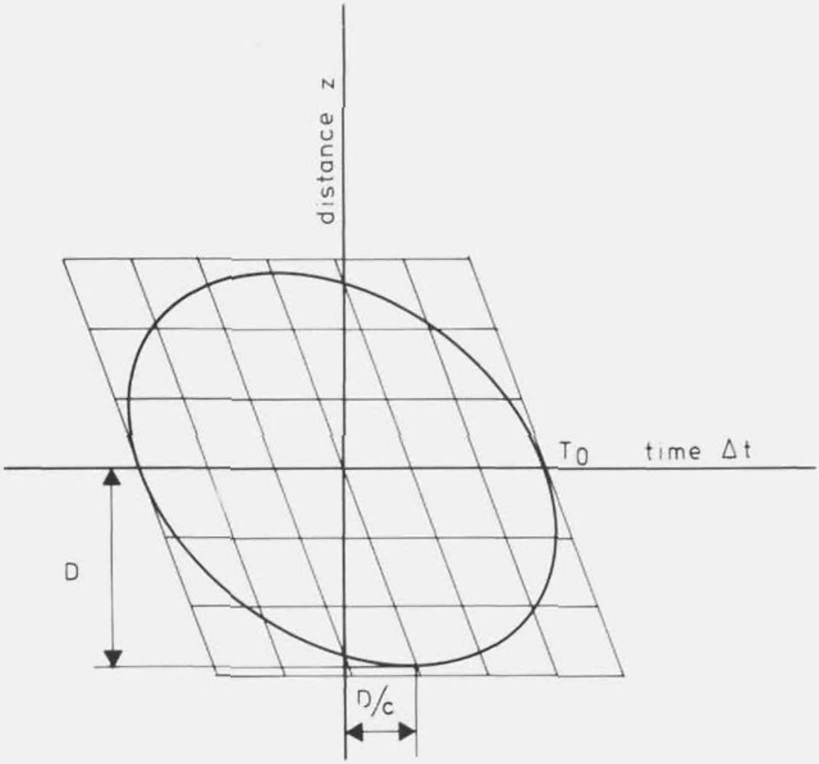


Fig. 6-17 Noise correlation isoline in the time-distance plane.

The noise correlation follows elliptic isolines in the time-distance plane. Fig. 6-17 shows the isoline for the value e^{-1} for the exponential function in eq (6-29).

Of the terms appearing in eq (6-29) the values of T_o, D and c have to be determined empirically on the basis of measurements.

Now consider the following actual case. The water levels at the Terneuzen station, located halfway the Western Scheldt estuary, are derived from measurements at the pairs of stations Vlissingen-Hansweert and Cadzand-Bath.

The time intervals used were $\Delta t = 1$ h, 2 h and 3 h and also only simultaneous measurements were considered. Data for the year 1983 were used to calculate the standard error $\sigma\Delta y$ presented in Table 6-2.

	distance	standard error $\sigma\Delta y$ (cm)			
reach	(km)	simultaneous	Δt= 1h	Δt = 2h	Δt = 3h
Cadzand-Bath*)	78,4	15,95	5,24	15,93	14,49
Vlissingen-Hansweert**)	42,0	5,70	5,60	4,53	5,45

Table 6-2 Standard errors $\sigma\Delta y$ for the Terneuzen station.

*) distance Cadzand-Terneuzen = 37,1 km.
**) distance Vlissingen-Terneuzen = 20,5 km.

In the calculation of eq (6-27) the amplitudes of 16 harmonic components were used. For reasons of simplicity, these were grouped into four classes corresponding with frequencies of about 2×, 4×, 6× and 8× per day. The amplitudes of each of these groups were calculated according to

$$H_i^2 = \sum_{j=1}^{N^*} H_{ij}^2, \tag{6-30}$$

where a number of N^* components was used in the group of frequency i× per day. In this way the following amplitudes were found:

H_2 = 198,98 cm
H_4 = 15,27 cm
H_6 = 13,04 cm
H_8 = 5,77 cm

The period of the H_2 component was estimated to be T_2 = 12h25 min, this being the period of the main component M_2 in this group. For the other groups periods of 1/2, 1/3 and 1/4 of T_2 were used. The variance of all hourly levels measured at the Terneuzen station for 1983 was Var $\underline{y} = (145{,}3)^2$ cm^2. The standard error of measurement was assessed as $\varepsilon = 2{,}5$ cm.

Subsequently, eq (6-29) was fitted by trial and error to the values given in Table 6-2, choosing a combination of T_o, D and c values that minimized the squares of the sum of the differences between the results of eq (6-1), after substituting into it the correlation coefficients obtained using eq (6-27), and the data of Table 6-2 for the six cases with $\Delta t \neq 0$. This was found to occur for

T_o = 6300 s = 1 h.45 min,
D = 139 km,
c = 12,5 m/s.

These values seem realistic for the case considered and therefore were adopted for further use.

Eq (6-29) could be applied to other, non measured cases. In Fig. 6-18, curves are shown of the relation between $\sigma\Delta y$ halfway along the reach and the reach length, for Δt = 0,5 h, 1 h, 2 h and 3 h, as well as for simultaneous measurements only. All curves start at a value of about $\sigma\Delta y$ = 3 cm, which corresponds to

$$\sigma\Delta y = \sqrt{\varepsilon^2 + (\tfrac{1}{2}\varepsilon)^2 + (\tfrac{1}{2}\varepsilon)^2} = \varepsilon\sqrt{3/2}, \tag{6-31}$$

which becomes

$\sigma\Delta y$ = 3,06 cm for ε = 2,5 cm.

Eq (6-31) is based on the relation

$$\hat{y} = \tfrac{1}{2}\underline{x}_1 + \tfrac{1}{2}\underline{x}_2 \tag{6-32}$$

which approximately holds in the close vicinity of the site considered. If for all three terms the same standard error of measurement ε is assumed, the following can be stated for the difference between the measured and the calculated value:

$$\Delta y = \underline{y} - \hat{y} = \underline{y} - \tfrac{1}{2}\underline{x}_1 - \tfrac{1}{2}\underline{x}_2 \tag{6-33}$$

from which, assuming independence between the measurement errors, eq (6-29) can be derived.

For increasing reach lengths all Δt-curves appear to increase to a maximum at a distance of

$$z = c\,\Delta t, \tag{6-34}$$

which is the distance that the wave has travelled during time Δt. In this case the correlation between the water levels at the network stations is optimal, which means that only a small amount of information is acquired by using two strongly correlated stations. Consequently, the standard error $\sigma\Delta y$ at the site under examination attains a maximum value. Maximum values of $\sigma\Delta y$ are found for (see Fig. 6-18):

$\Delta t = 1$ h at $z = 12{,}5$ m/s $\times$ 3600s = 45 000 m.
$\Delta t = 2$ h at $z = 12{,}5$ m/s $\times$ 7200s = 90 000 m.
$\Delta t = 3$ h at $z = 12{,}5$ m/s $\times$ 10800s = 135 000 m.

These maxima are only a little lower than the values obtained from the case that only simultaneous measurements were made at the distance concerned. Consequently in this case only a little information is gained by using additional, time shifted measurements.

In the case where the network station distance is twice the distance a wave travels during Δt, so

$$z = 2c\,\Delta t, \qquad (6\text{-}35)$$

the standard error $\sigma\Delta y$ shows a minimum. The reason for this minimum is that the correlation between each of the network stations and the site under examination is now a maximum.

The values presented in Table 6-2 are also plotted in Fig. 6-18. The differences between these values and the corresponding curves are seen to be only minor.

If the requirement for the network configuration is that the standard error $\sigma\Delta y$ should nowhere exceed the value of 3,5 cm, given that only entire hourly values are sampled, then the 2h-curve provides the best solution. The criterion is attained at a reach length of 28,9 km. However, a better solution can be found using 0,5 hourly samples. For the 42 km Vlissingen-Hansweert reach a $\sigma\Delta y$-value of only 3,2 cm is found. The station distance might even be increased up to 51,3 km before $\sigma\Delta y$ attains a value of 3,5 cm.

Fig. 6-19 shows for the Vlissingen-Hansweert reach the graphs of the Lagrange and propagation functions as well as the standard errors $\sigma\Delta y$, $\sigma\Delta\hat{y}$ and $\varepsilon\sqrt{A^TA}$ for Δt=0,5h, 1h and 2h. This demonstrates once again that better results are obtained if a sampling interval $\Delta t = 0{,}5$h is used.

In summary it can be noted that the use of time shifted measurements can lead to a considerable reduction of the standard error $\sigma\Delta y$, subject to the premise that a feasible combination of reach length and time shift t is chosen. If this is not the case then the use of time shifted measurements will only introduce superfluous information.

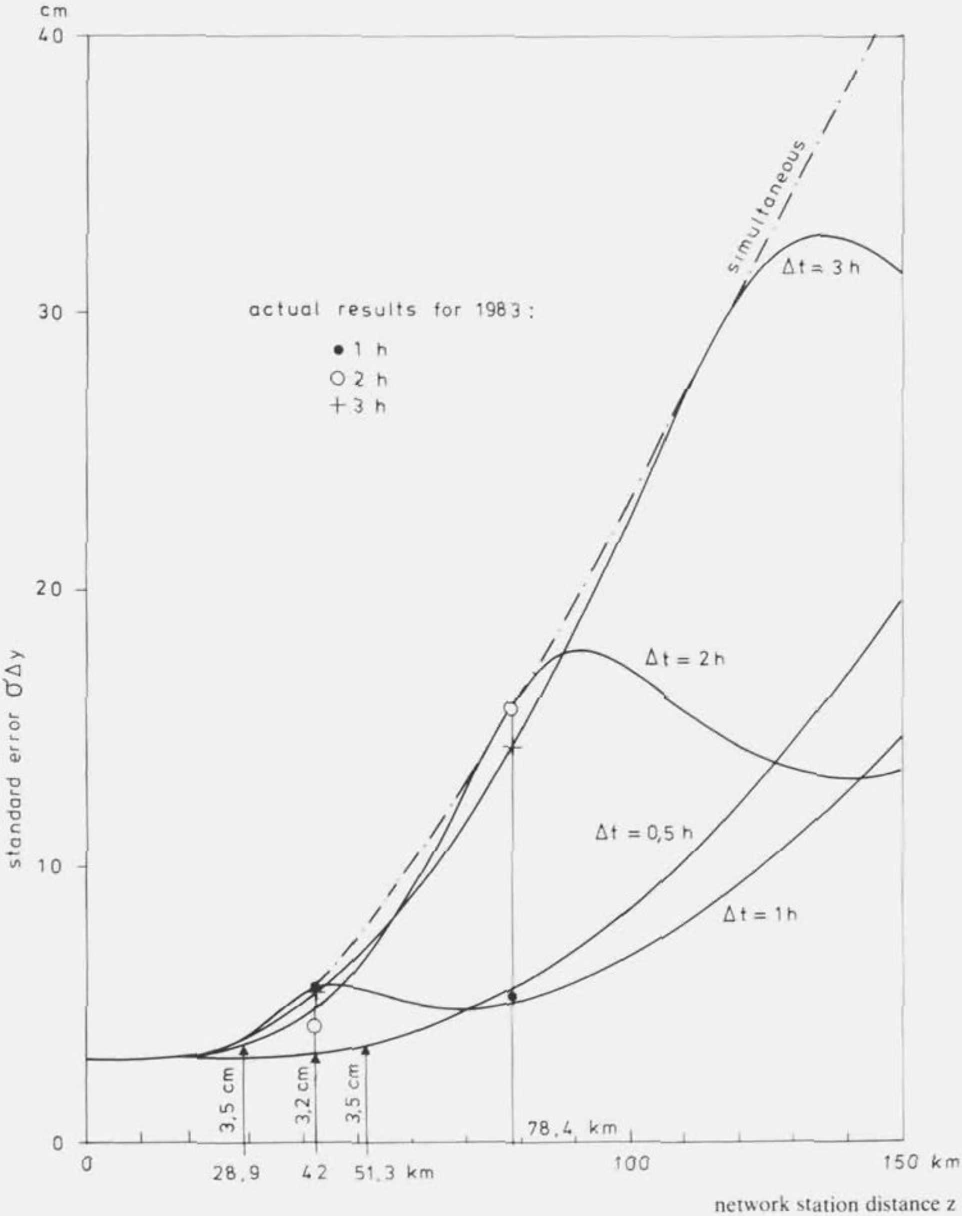


Fig. 6-18 Curves of halfway standard errors σΔy vs. gauging station distance for time intervals of 0,5h, 1h, 2h and 3h (Western Scheldt estuary)

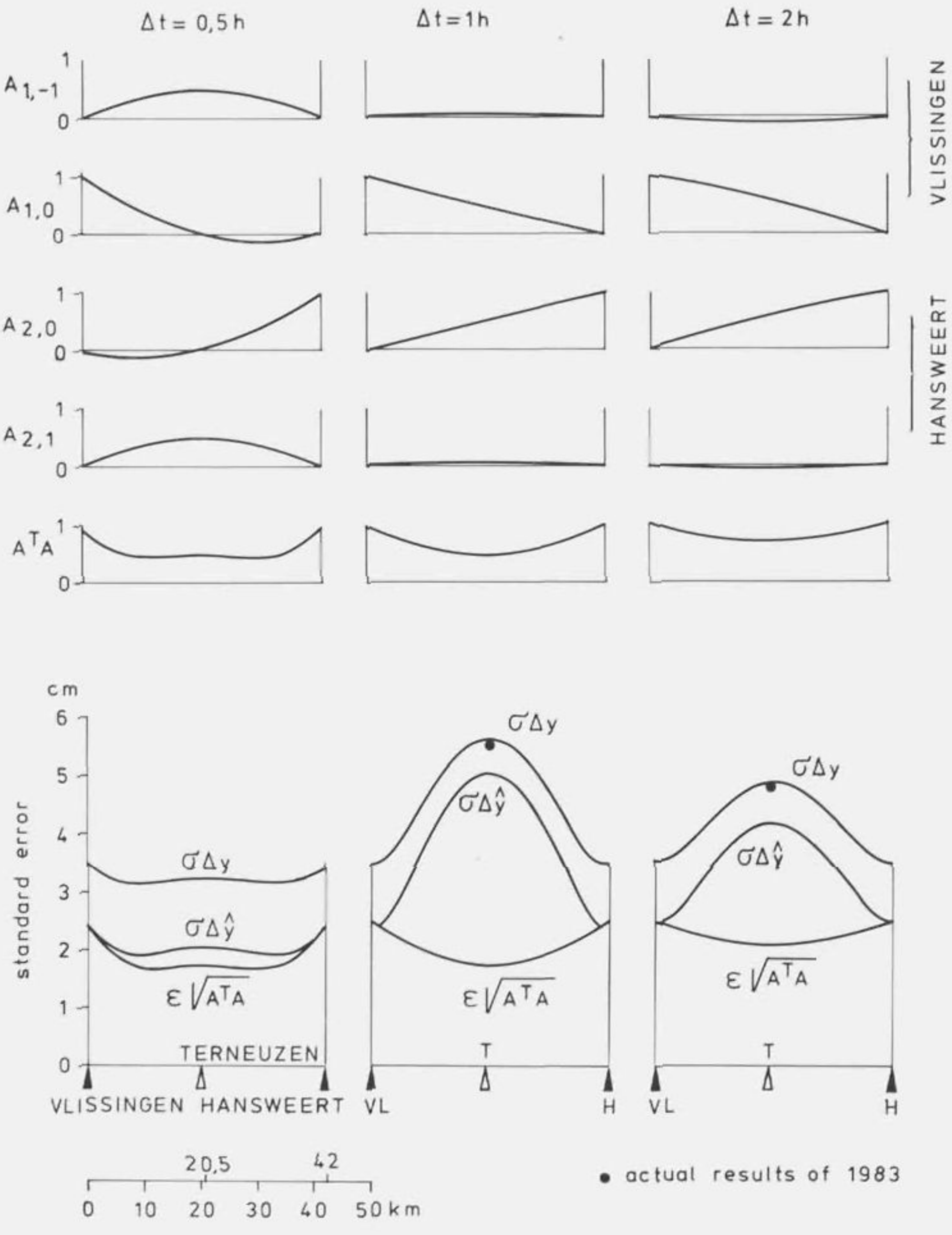


Fig. 6-19 Lagrange and propagation functions and standard errors along the Western Scheldt reach for time intervals of 0,5h, 1h and 2h

6.7. Application to an areal network

The above examples all relate to one dimensional cases in which gauging stations were located according to a line along a river or tidal reach. However, it is also possible to apply the method to two-dimensional cases where the stations are located over a certain area. For cases where only values for earlier measured sites were required (i.e. case b of Section 5.1.), an example has already been presented, namely the western part of the Wadden Sea in the Netherlands. (Section 5.7.3).

If water levels over a plane are required, then the correlation structure should be known in order to determine the correlation coefficients at all points on the plane with respect to the gauging stations. This means that there is no fundamental difference with the one-dimensional case. However, as well as the distance between two sites, the direction of the line of connection can also be of influence.

If a gaussian shape for the correlation structure is assumed, as for the one-dimensional examples (e.g. eq (6-14)), the following relation expressed in polar coordinates may be applied

$$\varrho(r, \varphi) = \varrho(0).\ \exp\left[-r^2 \left\{ \frac{\cos^2(\varphi-\alpha)}{D_1^2} + \frac{\sin^2(\varphi-\alpha)}{D_2^2} \right\}\right] \qquad (6\text{-}36)$$

where:
- r = the distance between the two sites considered.
- φ = the direction of their line of connection with respect to a fixed zero direction, e.g. the east.
- α = the direction of the strongest correlation.
- D_1 = the correlation scale in the α direction.
- D_2 = the correlation scale perpendicular to the α direction.

This structure implies the correlation ellipse a shown in Fig. 6-20.

The ellipse itself can be expressed as follows:

$$\frac{1}{r_e^2} = \frac{\cos^2(\varphi-\alpha)}{D^2} + \frac{\sin^2(\varphi-\alpha)}{D_2^2} \qquad (6\text{-}37)$$

where r_e =the radius of the ellipse, corresponding with a direction φ.

In all directions the correlation coefficients display gaussian curves, and the whole structure is of gaussian bellshaped form. For $\varrho(0)$ in eq (6-36) the correlation coefficient of a waterlevel 'with itself' can be substituted, as given by eq (6-18):

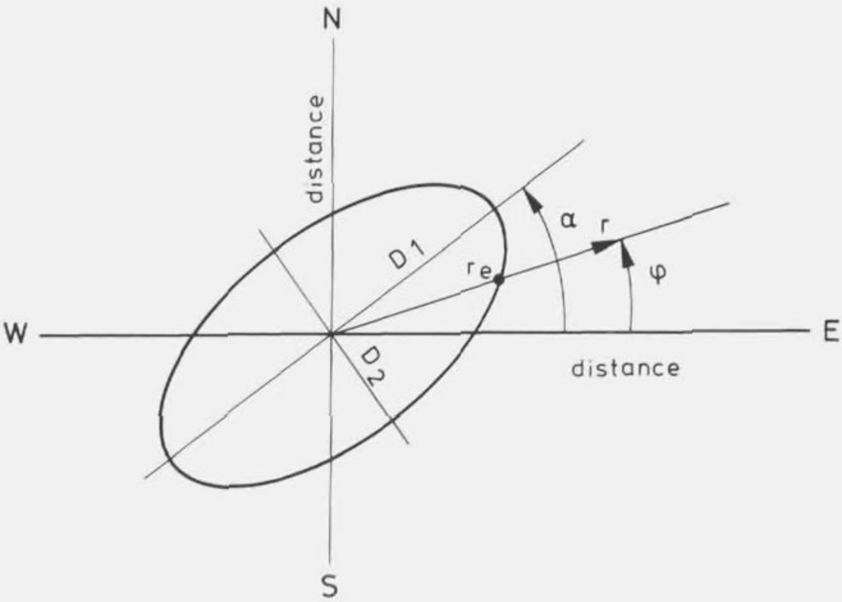


Fig. 6-20 Ellipse of the areal correlation structure.

$$\varrho(0) = 1 - \frac{\varepsilon_x^{\,2}}{\text{Var } \underline{x}} \qquad (6\text{-}38)$$

Now consider, as an application, a rectangular lake along which three gauging stations are located according to an equally sided triangle, as shown in Fig. 6-21. The sides of the lake have lengths of a and $\frac{1}{2}a\sqrt{3}$. Further the following values are adopted:

- standard deviation of the water levels: $\sigma\underline{y} = 1$ m.
- standard error of measurement: $\varepsilon = 0{,}05$ m.

First the case examined was where the correlation scales were equal and amounted to $D_1 = D_2 = 4a$. Thus the correlation ellipse transformed into a circle and, consequently, the α direction played no role. For the grid points, indicated in Fig. 6-21 the standard error $\sigma\Delta y$ was calculated according to eq (6-1). Subsequently isolines were drawn, yielding the result given in Fig. 6-22.The area, within which

$$\sigma\Delta y < \varepsilon\sqrt{2} \qquad (6\text{-}39)$$

holds or, equivalently (according to eq (6-2)),

$$\sigma\Delta\hat{y} < \varepsilon, \qquad (6\text{-}40)$$

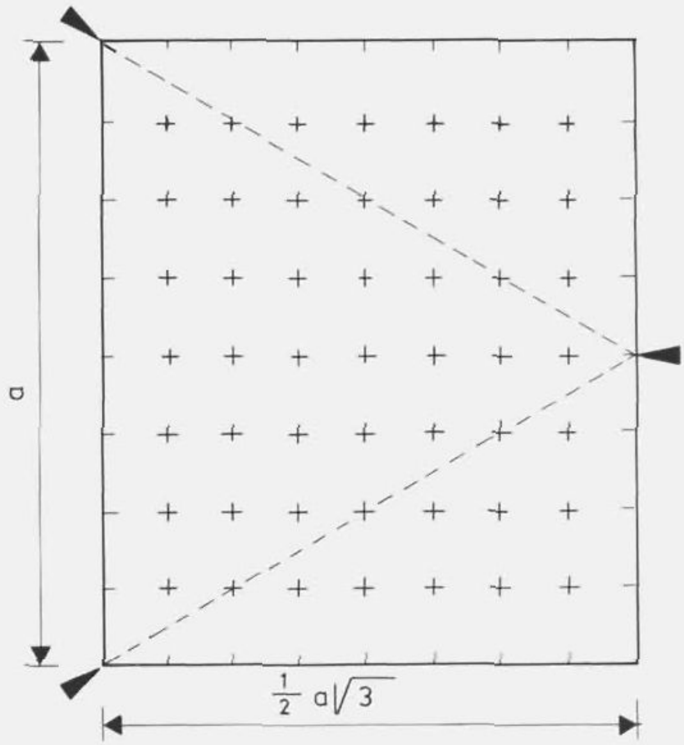


Fig. 6-21 Rectangular lake with three gauging stations

is shaded. It follows that this holds for the area within the triangle of the gauging stations, as well as for a small boundary area. Near the non gauged corners the standard error $\sigma\Delta y$ is considerably greater.

Now consider the case where one of the correlation scales is reduced to half of the other, thus yielding a real correlation ellipse. The situation was examined for angles α of 0°, 22,5°, 45°, 67,5° and 90°. The results are shown in Fig. 6-23, where the influence of the direction of strongest correlation appears clearly. The shaded area for which the condition of eq (6-39) holds is highly influenced by the direction of preference.

Of course, many cases can be examined in this way with various gauge locations and correlation structures. This example is only intended to demonstrate the behaviour of the standard error $\sigma\Delta y$ under the influence of varying correlation structures over a two-dimensional area.

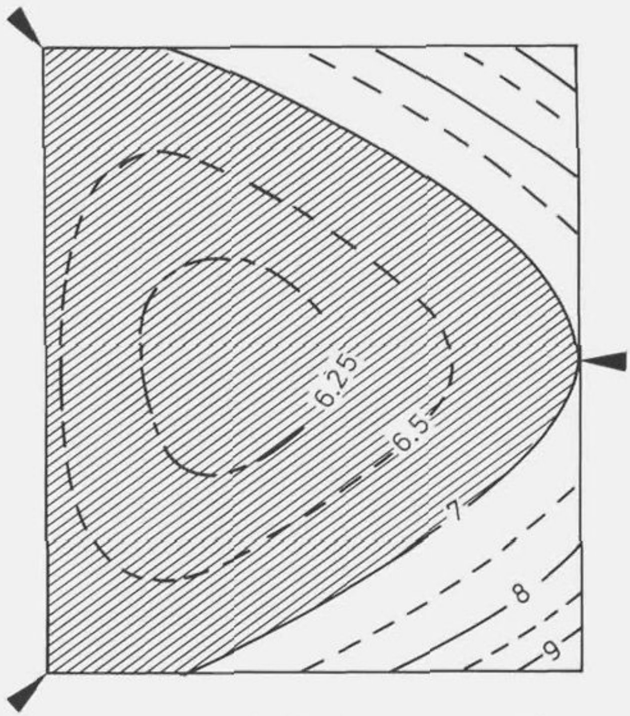


Fig. 6-22 Isolines of equal standard errors $\sigma\Delta y$ (cm) for a homogeneous correlation structure in the lake of Fig. 6-21

6.8. The Lake Grevelingen network

The areal approach, as introduced in Section 6.7, was applied to one of the lakes in the Delta region of the Netherlands. The lake in question, the Lake Grevelingen, originated from one of the former tidal inlets. The general situation is shown in Fig. 6-24. In 1983 there were four gauging stations around the lake, namely Brouwershaven, Brouwersdam, Ouddorp and Grevelingen. The question was whether these stations were adequate or could they be reduced to three or even two stations.

On the basis of 8h water level data for the year 1980 the values given in Table 6-3 were derived.

In order to assess the network, isolines of the standard error $\sigma\Delta y$ over the lake area were to be constructed, in the same way as was done for the hypothetical case of Section 6.7. It was necessary first to determine the correlation structure of the water levels. The correlation structure was based on eq (6-36), which includes 4 unknown parameters: $\varrho(0)$, D_1, D_2 and α. It is assumed that these parameters have the same value over the whole area considered.

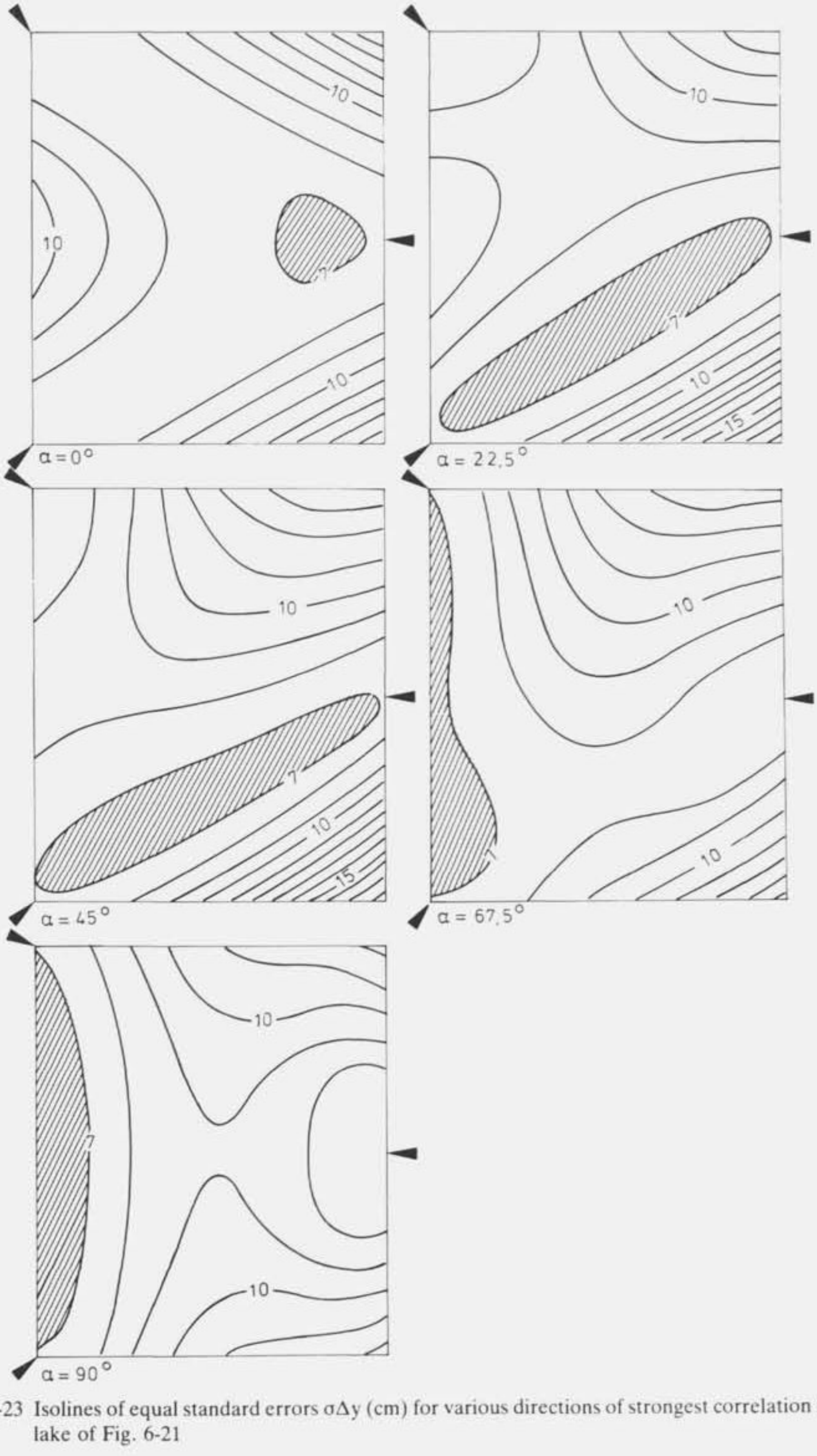


Fig. 6-23 Isolines of equal standard errors $\sigma\Delta y$ (cm) for various directions of strongest correlation in the lake of Fig. 6-21

station	mean level $\bar{x}$ (NAP + cm)	standard deviation σx (cm)	coordinates (m)	
			west-east	south-north
Brouwershaven	- 17,15	7,45	5500	2600
Brouwersdam	- 19,60	7,79	300	3900
Ouddorp	- 21,00	7,41	7500	9200
Grevelingen	- 19,67	7,25	18400	- 5400

Correlation coefficients

	Brouwersdam	Ouddorp	Grevelingen
Brouwershaven	0,977734	0,974363	0,962466
Brouwersdam		0,963627	0,934912
Ouddorp			0,939384

Table 6-3 Data from the gauging stations around Lake Grevelingen.

The parameters were determined by fitting eq (6-36) to the correlation coefficients given in Table 6-3, at the same time using the coordinates also given in that table. The values of r and φ were, on the basis of these coordinates, determined for each pair of stations. In this way 6 combinations could be considered, giving 6 equations with 4 unknows. These were calculated using the least squares principle. This was done in the following way. First eq (6-36) was rewritten in $\log_e$ notation, so

$$\log_e \varrho(r, \varphi) = \log_e \varrho(0) - \frac{r^2 \cos^2(\varphi-\alpha)}{D_1^2} - \frac{r^2 \sin^2(\varphi-\alpha)}{D_2^2} \qquad (6\text{-}41)$$

If the angle α had been known this equation could be solved by the solution of 3 linear equations with the unknowns $\log_e \varrho(0)$, $1/D_1^2$ and $1/D_2^2$ from which the parameters concerned could be derived easily. In this way, for given α the value

$$\sum_{i=1}^{6} \delta_{\alpha,i}^2 = \sum_{i=1}^{6} \{\underset{\text{(actual)}}{\log_e \varrho(r_i, \varphi_i)} - \underset{\text{(calculated)}}{\log_e (r_i, \varphi_i)}\}^2 \qquad (6\text{-}42)$$

was minimized. The next question was what value of α would lead to the absolute minimum of the minima thus found. This was solved by calculating a series of such minima from which the required α-value, could be selected. Fig. 6-25 shows a curve of $\Sigma\delta_{\alpha,i}^2$ against α. A minimum is found for $\alpha = 56{,}5°$. The corresponding values of the other parameters are

$\varrho(0) = 0{,}985146$,
$D_1 = 58679\text{m}$,
$D_2 = 91061\text{m}$.

The value of $\varrho(0)$ gives an indication of the standard error of measurement, and according to eq (6-38) is

$$\varrho(0) = 1 - \frac{\varepsilon_x^2}{\mathrm{Var}\underline{x}} .$$

Since the mean value of the standard deviations given in Table 6-3 amounts to 7,48 cm, it can be calculated that $\varepsilon_x^2 = (7{,}48)^2\ (1-0{,}985146) = 0{,}830\ \text{cm}^2$ and $\varepsilon_x = 0{,}911$ cm, *or 1 cm approximately. This is somewhat lower than the corresponding values* found for river and tidal stations, but bearing in mind the rather quiet character of the lake this is quite acceptable.

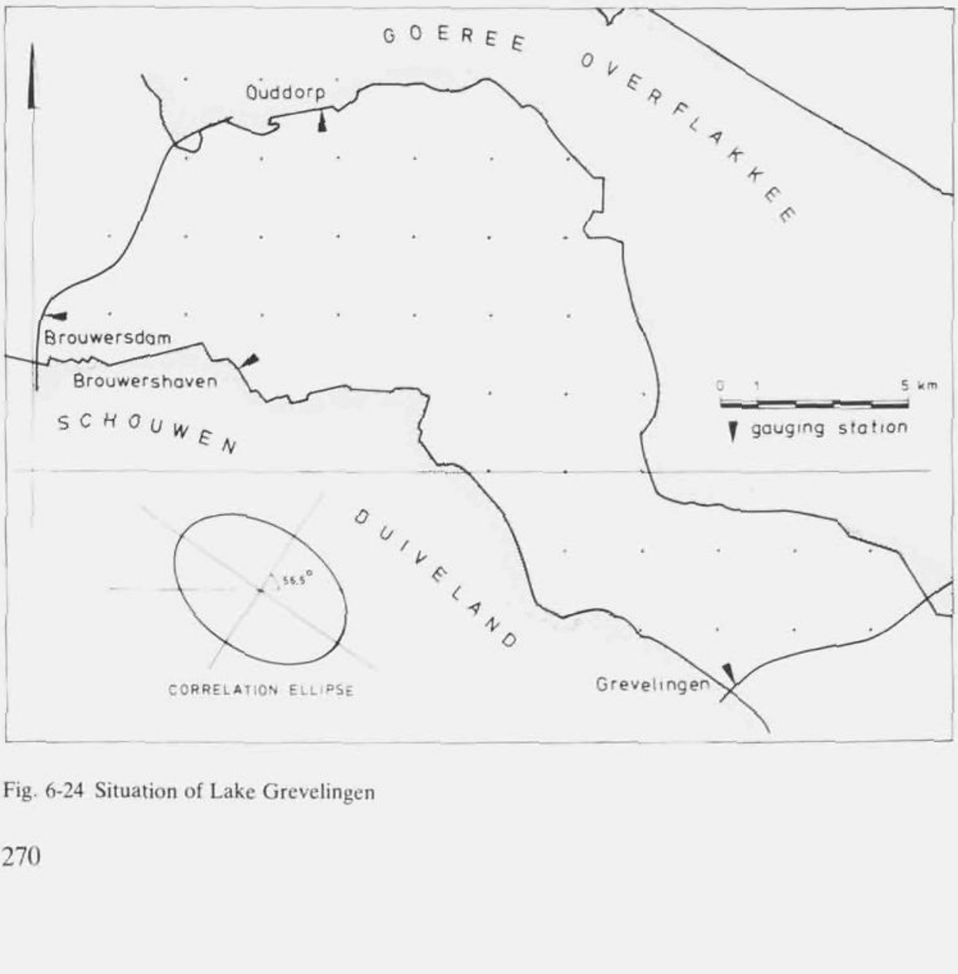


Fig. 6-24 Situation of Lake Grevelingen

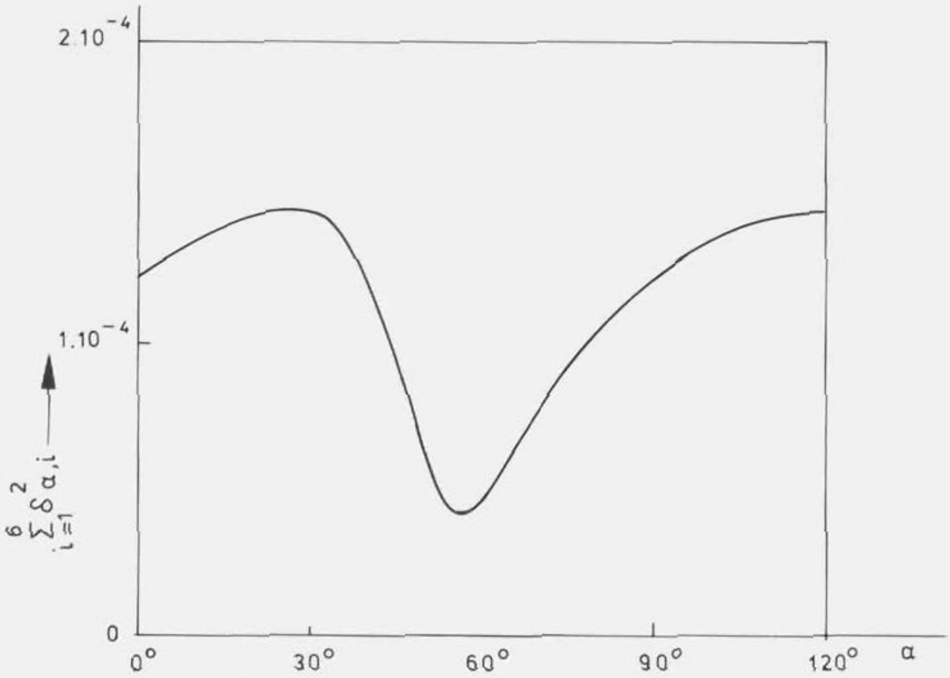


Fig. 6-25 Minimum of $\sum_{i=1}^{6} \delta_{\alpha,i}^2$ vs. angle α

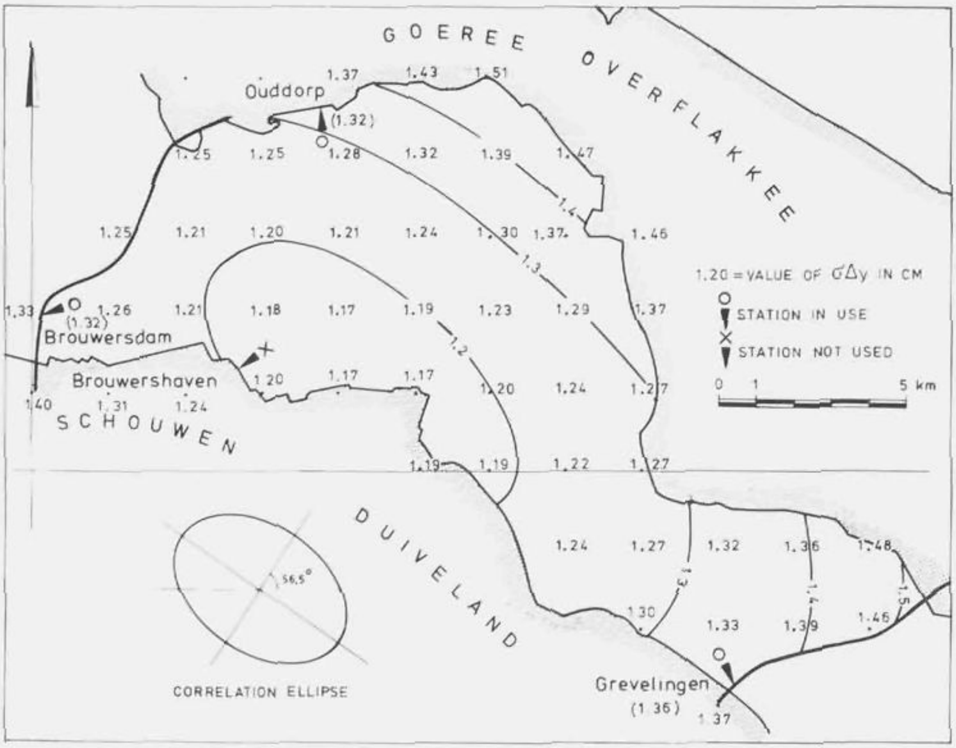


Fig. 6-26 Isolines of σΔy for gauging stations at Brouwersdam, Ouddorp and Grevelingen

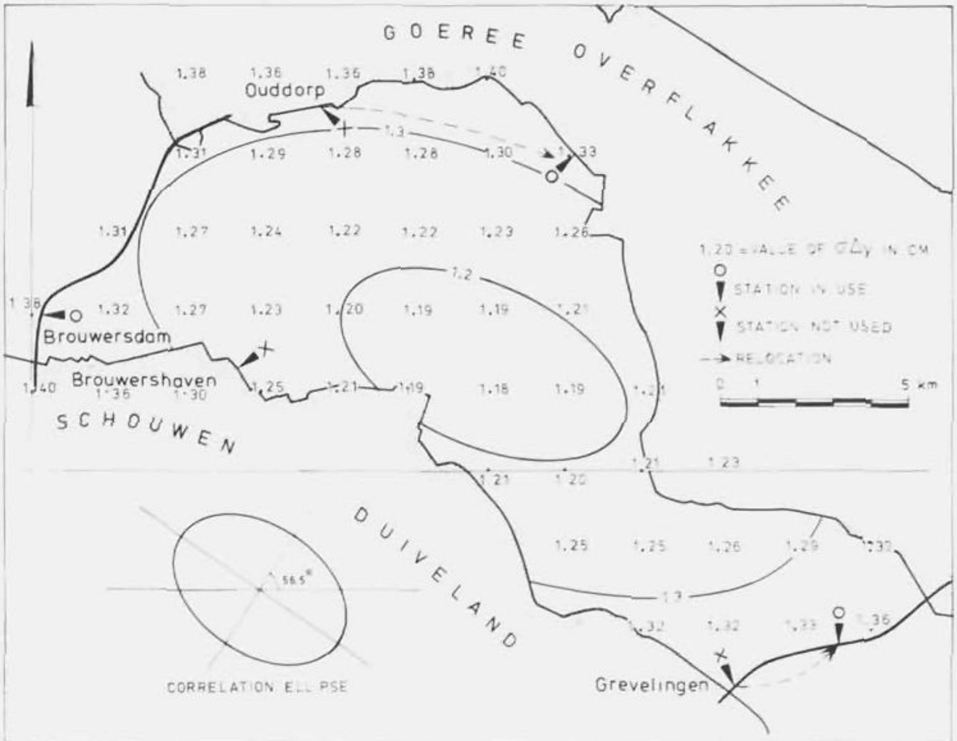


Fig. 6-27 Isolines of σΔy for gauging stations at Brouwersdam and relocated stations at Ouddorp and Grevelingen.

Having determined the correlation structure, it is now possible to calculate values of σΔy for each site within the lake. Isolines can then be constructed using a set of grid values as a basis. Fig. 6-26 shows isolines obtained using the three gauging stations Brouwersdam, Ouddorp and Grevelingen. If the design criterion is assessed such that the standard error σΔy may not exceed the limit $\varepsilon\sqrt{2}$, which for $\varepsilon = 1$ cm amounts to 1,4 cm, it appears that some areas in the east and the northeast of the lake do not fulfil this criterion. This could be avoided by relocation of the Ouddorp and Grevelingen stations, as shown in Fig. 6-27.

However, a criterion of 1,4 cm is very strict, and is not in line with criteria assessed for other waters in the country. As a rule, a maximum value of 3,5 cm is used. In this case only two stations at Lake Grevelingen are adequate. In Fig. 6-28 the isolines of σΔy are shown for stations at Brouwersdam and Grevelingen only. The maximum value of σΔy is found in the north of the lake and it amounts to 2,2 cm.

These two stations were finally selected for the network in Lake Grevelingen.

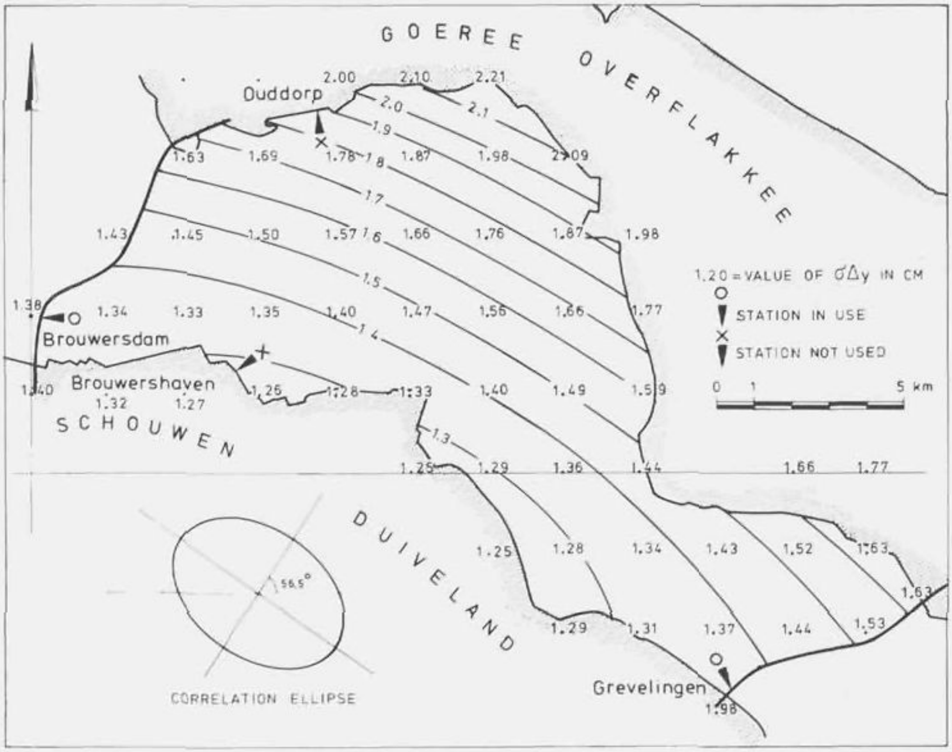


Fig. 6-28 Isolines of σΔy for gauging stations at Brouwersdam and Grevelingen.

7. Calculation of intermediate water levels usingg physically-based mathematical methods

7.1 Physically-based methods

In the preceding chapters the calculation of water levels along intermediate reaches between gauging stations was based on mathematical and statistical interpolation methods. These calculations produce the data needed for design and feasibility testing of the network. In the methods discussed, no or hardly any use is made of the physical relations between the variables i.e. the water levels. For the mathematical interpolation methods a water surface profile was assumed following specified mathematical functions (Chapter 4), which, in the case of approximated interpolation, are fitted by statistical methods. For the pure statistical interpolation methods (Chapters 5 and 6), such as optimum interpolation, use is made of the correlation structure derived from actual data series. Only in special cases can the correlation structure be reconstructed on the basis of physical phenomena. One such case is the tidal river where use can be made of the harmonic components. This was illustrated in the example of the Western Scheldt estuary.

Justification for the mathematical interpolation methods lies in the continuous shape of the curve which is adopted, since nature is also assumed to react continuously. In many cases the result may indeed be satisfactory.

The statistical methods have a firmer base since the results are based on the behaviour of the variables considered. Their behaviour in the past is assumed to persist unchanged into the future. However, in this statistical approach, the conditions which occur most frequently, such as the water levels under average conditions, will dominate the result. One is never sure whether under extreme circumstances, as in floods, stormsurges or very low discharges, the same relations hold as under average conditions. Nevertheless, if the derived relation is applied for any case whatsoever, great deviations from the observed conditions may be expected. One might try to derive special relations which hold in particular for extreme conditions. However, due to the insufficient number of data for such conditions, this is hardly possible.

It may be expected that an approach, which takes into account the physics of the phenomena considered will be better attuned to all possible conditions. However, one should realize that the mathematical formulation of physical phenomena will include a number of parameters or coefficients which are not known previously and which have to be derived from measurements. Here also, statistical methods are applied to estimate the values of the parameters. In fact, the physically-based mathematical methods are really mixed physical-statistical methods, but with a strong emphasis on the physical background.

This chapter will examine how water levels at intermediate sites between gauging stations can be derived using the basic laws of hydrodynamics i.e. the equation of continuity and the equation of motion. Models based on this principle have to be applied to a schematized channel system which is an imitation of the real situation. This imitation can range from very approximate to very detailed.

For the purpose of network design and operation a simple schematization is to be preferred, since rather short calculation times are required so that the method can compete with the methods described in the preceding chapters, such as the linear regression method which can operate very quickly and which does not need long and extensive calculations. If the computer time for the physical model is much longer, its application can only be justified if it produces better quality data.

It should be stressed here that a long computer run also implies an expensive one. Thus, from the viewpoint of cost reduction, which is of particular importance for developing countries, short computer times should be sought. Also, if the channel system schematization does not consist of too many elements, relatively little memory capacity will be required, and a smaller computer can be used.

In the following, a very simple channel schematization consisting of a small number of linked reaches (three only) will be discussed. Measurements are assumed to be carried out at both ends of the channel, whereas water levels are calculated for the places where the reaches join. These are the intermediate sites considered. The extent to which the calculated values correspond to measured values will be examined, serving to indicate whether measurements at these sites are necessary or not for obtaining the locally required accuracy and reliability.

It should be noted that the results will depend strongly on the dimensions of the streamflow profile and on the flow resistance which is applied. Since the applied schematization of the stream channel will generally be a simplification of the real situation, it is uncertain whether the actual channel dimensions will produce the best estimates of water levels and velocities. Therefore it might be desirable to apply dimensions which differ somewhat from the actual values, but which produce estimates which approach the water levels and velocities optimally. In this way, errors introduced by the schematization become compensated by modified dimensions.

An objection to the physical methods in comparison to methods described in the preceding chapters, is that the dimensions, the shape and the flow resistance of the streamflow profiles have to be taken into account separately. In the methods described before, only the water level data were of importance. The channel characteristics were included in the correlation structure, but its physical background is often is too unclear to allow for a mathematical description. Only for tidal rivers, as was discussed in the example of the Western Scheldt (see Section 6.6), can some physical elements be taken into account through the use of harmonic tidal components.

When combining physically-based methods with statistical techniques such as the Kalman filter the statistical element is again introduced. This can reduce the errors of the pure physical approach.

In Section 7.2 a physical model will be introduced, and it will be explained through some examples. Chapter 8 will give a description of the application of the Kalman filter technique to this model.

7.2 A physically-based mathematical model

As previously mentioned the schematization considered is a very simple one. It concerns a straight river reach with regular measurements at fixed time intervals at both ends. Within the reach two additional points are considered for which the water levels have to be calculated.

Two intermediate points are considered instead of simply one because of the application of the Kalman filter. This technique requires the availability of at least one additional measurement. By measuring one of the two intermediate points the result of the other can be adapted using the additional and mathematically redundant information.

Returning to the specific form of the mathematical model, the streamflow conditions can be described by the two basic equations of hydrodynamics:

1. the equation of continuity

$$\frac{\partial y}{\partial t} = - \frac{1}{W_s} \cdot \frac{\partial Q}{\partial z} \qquad (7\text{-}1)$$

where

y = the water level with respect to the horizontal reference level (e.g. NAP)
t = time
W_s = the width of the channel over which the water is stored (storage width)
Q = discharge
z = horizontal length along the streamflow direction.

2. The equation of motion

For the one-dimensional case under consideration this equation can be written as follows:

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial z} = F \qquad (7\text{-}2)$$

where:

v = the streamflow velocity
F = the combined forces acting on a unit mass of water. Thus, in fact, these represent accelerations.

The forces F consist of a number of elements from which the following are of importance only:

- the acceleration of the horizontal component of the bottom pressure. This amounts to gS_b, where S_b denotes the bottom slope; that is

$$S_b = - \frac{\partial b}{\partial z}$$

so

$$gS_b = - g \frac{\partial b}{\partial z} \qquad (7\text{-}3)$$

where

g = acceleration due to gravity (= 9,8 m/s^2)
b = bottom level with respect to the horizontal reference level

- the acceleration of the horizontal component of the water pressure, which is

$$gS_w = - g \frac{\partial (y-b)}{\partial z} \qquad (7\text{-}4)$$

- the retardation caused by the resistance force, amounting to gS_r, where, according to Chézy:

$$S_r = - \frac{v\,|v|}{C^2\,(y-b)}, \tag{7-5}$$

thus

$$gS_r = - g\,\frac{v|v|}{C^2\,(y-b)} \tag{7-6}$$

where

C = flow resistance coefficient or bottom friction coefficient of Chézy (expressed in $m^{1/2}/s$).

The direction of the above resistance slope, S_r, is always opposite to the flow direction. This has been implemented numerically by using the absolute value of v in eqs (7-5) and (7-6).

The acceleration of the wind forces will not be considered here, bearing in mind the limited nature of the model. Further the acceleration caused by surface tension is neglected. Transversal forces like Coriolis, and centrifugal forces play no role in this one-dimensional model. At the end of Section 7.2.9. their possible influence on the results will be discussed. For reasons of simplicity the accelerations acting on a unit mass are assumed to be constant over the cross section, so that no integration is required.

Substituting eqs (7-3), (7-4) and (7-6) into eq (7-2), and taking the second left hand term of eq (7-2) to the right hand side, leads to

$$\frac{\partial v}{\partial t} = - g\,\frac{\partial v}{\partial z} - g\,\frac{v|v|}{C^2\,(y-b)} - v\,\frac{\partial v}{\partial z}\,. \tag{7-7}$$

Application of eqs (7-1) and (7-7) allows the water levels at the intermediate sites to be calculated.

7.3 The numerical finite difference equations

Now consider Fig. 7-1 in which a schematization of the river reach is shown. The water levels will be calculated for the points 1 and 2 which divide the river reach into the sub reaches d_{01}, d_{12} and d_{23}. The velocities will be considered for points halfway between these sub reaches.

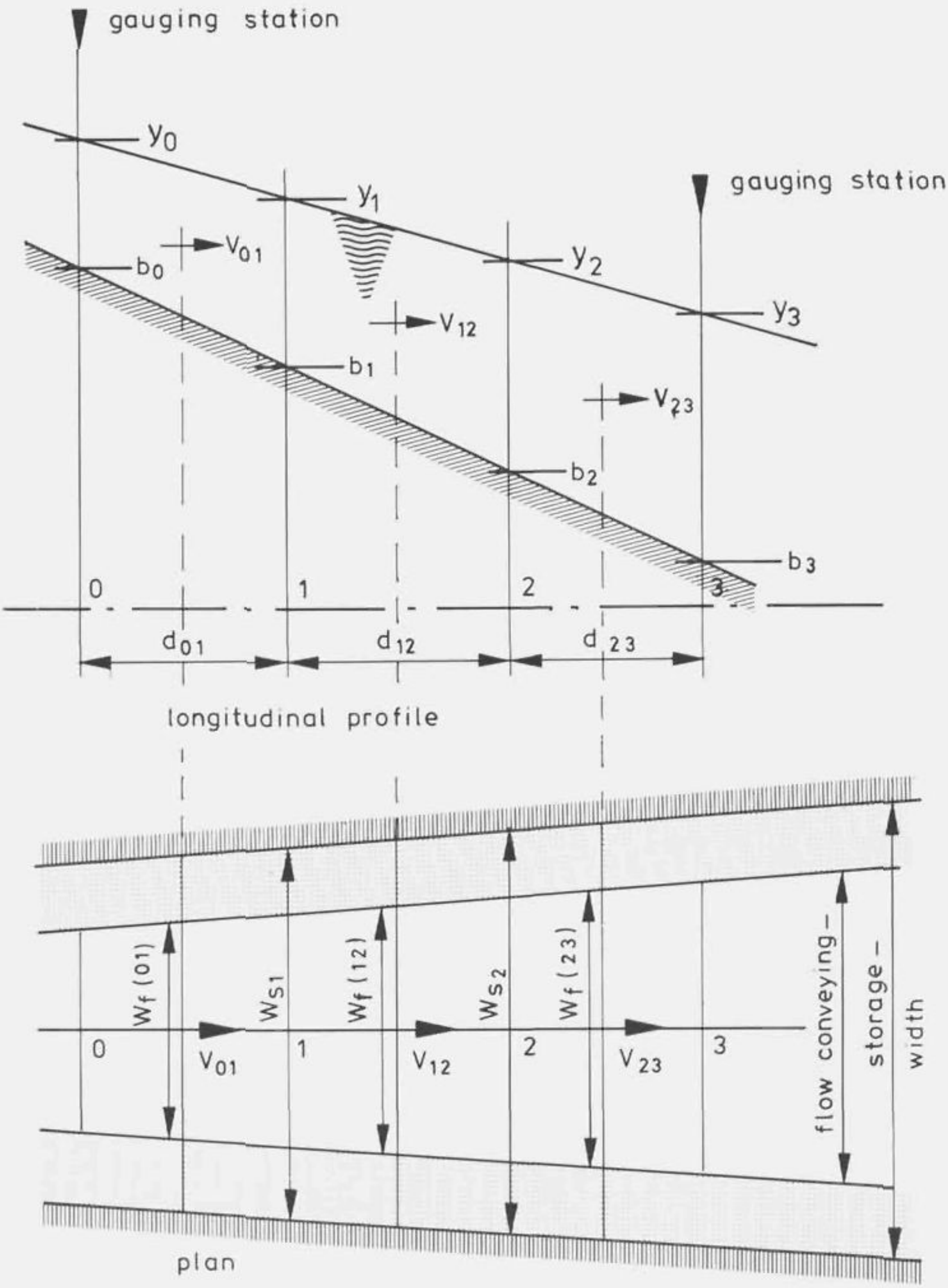


Fig. 7-1 Schematization of a river reach

A rectangular shape was used for the cross section of the streamflow profile with a fixed flow conveying width W_f. Let A_f be the area of the streamflow profile. This area is proportional to the water depth (y-b), according to

$$A_f = W_f\,(y\text{-}b). \tag{7-8}$$

In a natural streamflow profile with a more or less equal water depth in the transversal direction, such that the width is large compared with the depth, this profile may be approximated by a rectangle with depth (y-b) and width

$$W_f = \frac{A_f}{y\text{-}b} \tag{7-9}$$

This width will be called the flow conveying width (see Fig. 7-2).

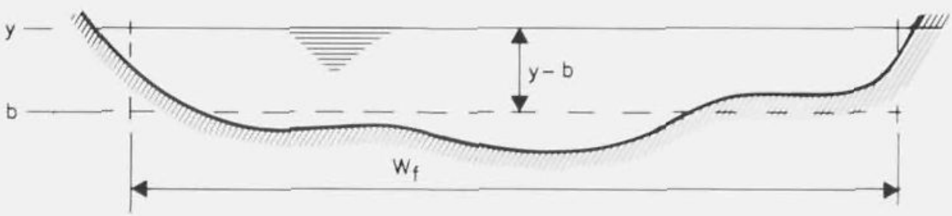


Fig. 7-2 Approximation of a streamflow profile by a rectangle.

The calculation scheme is shown in Fig. 7-3. The measurements are carried out at t_0 - ΔT, t_0, $t_0 + \Delta T$ etc. where ΔT denotes the measurement interval. The water levels y_i (i = 1,2) and velocities $v_{i,i+1}$ (i = 0, 1, 2) will be calculated for $t_0 + \Delta T$, using the measured and calculated data at t_0. The measurement interval ΔT will be divided into n time steps Δt. The times for which intermediate calculations are carried out are indicated by t.

On the basis of this scheme the two basic equations (7-1) and (7-7) will be transformed into numerical expressions. For the symbols to be used reference is made to Fig. 7-1 once more.

The transformation of the equation of continuity (7-1) will read:

$$\frac{\Delta y_i}{\Delta t} = -\frac{1}{W_{si}} \quad \frac{Q_{i,i+1} - Q_{i-1,i}}{\frac{1}{2}\,(d_{i-1,i} + d_{i,i+1})} \quad (i = 1,2) \tag{7-10}$$

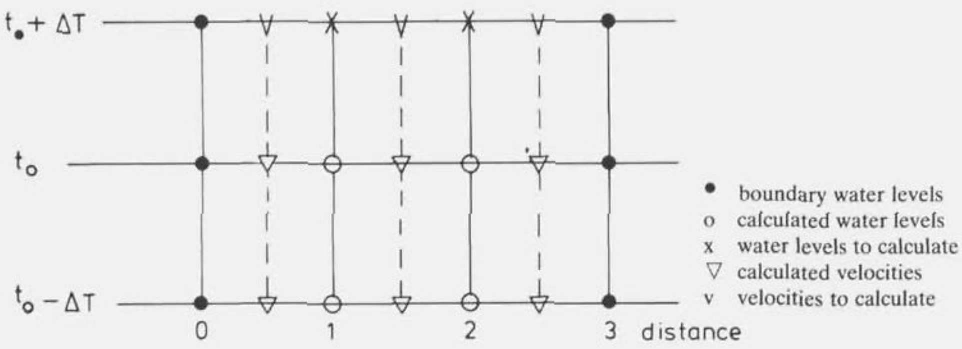


Fig. 7-3 Calculation scheme

The discharges, Q, in eq (7-10) were derived according to

$$Q_{i,i+1} = W_{f(i,i+1)}).\ {}^1\!/\!_2(y_i - b_i + y_{i+1} - b_{i+1})\ v_{i,i+1}.\ (i = 0, 1, 2); \qquad (7\text{-}11)$$

here W_f denotes the flow conveying width of the channel.

The transformation of the equation of motion, eq (7-7), is as follows:

$$\frac{\Delta v_{i,i+1}}{\Delta t} = -g\,\frac{y_{i+1} - y_i}{d_{i,i+1}} - g\,\frac{v_{i,i+1}|v_{i,i+1}|}{C^2.{}^1\!/\!_2(y_i - b_i + y_{i+1} - b_{i+1})} - v\,\frac{\partial v}{\partial z}\ (i = 0, 1, 2) \qquad (7\text{-}12)$$

The third right hand term is the so called Bernouilli term and requires special attention. Since eq (7-12) contains velocities which are calculated for points halfway between the water level measurement points, the gradients of v in the third right hand term of eq (7-12) have to be given over a length in which the point considered is located in the centre. Therefore the velocity values in the adjacent subreaches have to be used, which can be done, provided the lengths of the subreaches do not differ too much, using (see Fig. 7-4).

$$\frac{\partial v_{i,i+1}}{\partial z} = \frac{v_{i-1,i} - v_{i+1,i+2}}{{}^1\!/\!_2(d_{i-1,i} + 2d_{i,i+1} + d_{i+1,i+2})} \qquad (7\text{-}13)$$

However, in the case considered, this method can be applied only for i = 1, since for other values of i some of the required data do not exist because these would fall beyond the limits of the model. This problem is solved in the following way: the gradients of the velocities in the first and third (last) reach were derived by approximating the longitudinal water velocity profile by a parabola (see Fig. 7-5).

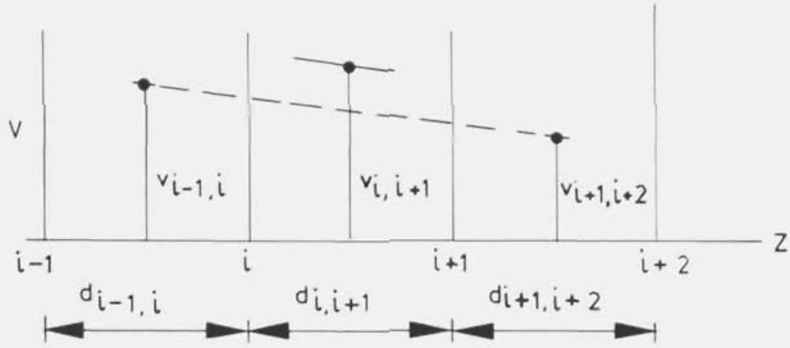


Fig. 7-4 Derivation of the velocity gradient: general case

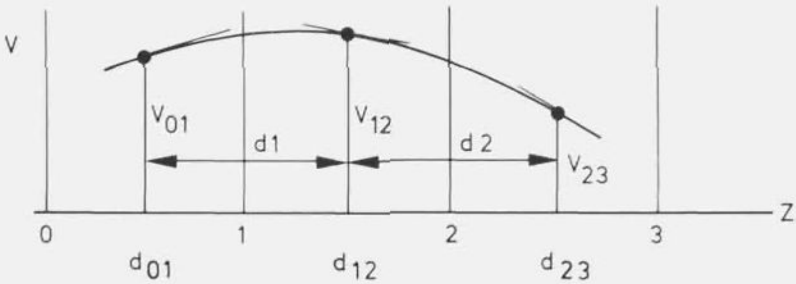


Fig. 7-5 Derivation of the velocity gradient in the outer reaches of a 3-reach model.

Let

$$\left.\begin{aligned} d_1 &= 1/2\,(d_{01} + d_{12}) \\ d_2 &= 1/2\,(d_{12} + d_{23}). \end{aligned}\right\} \qquad (7\text{-}14)$$

For the parabola

$$v = Az^2 + Bz + C, \qquad (7\text{-}15)$$

and for its gradients:

$$\frac{\partial v}{\partial z} = 2Az + B.^{*}) \qquad (7\text{-}16)$$

* Since the parabola varies with time, A, B and C are also functions of time. For this reason partial differentiation is indicated.

Shifting the origin of the z-axis to 1/2 d_{01} leads to:

$$A = \frac{(v_{12}-v_{01})\ (d_1+d_2)-(v_{23}-v_{01})d_1}{-d_1d_2(d_1+d_2)} \tag{7-17}$$

$$B = \frac{-(v_{12}-v_{01})\ (d_1+d_2)^2+(v_{23}-v_{01})d_1^2}{-d_1d_2(d_1+d_2)}$$

By elaboration of eq (7-16) for z=0, d_1 and (d_1+d_2) successively, using eq (7-17), the values of the derivatives concerned and subsequently the values of

$$\left(v\frac{\partial v}{\partial z}\right)_{i,i+1} \quad (i=0,\ 1,\ 2)$$

can easily be found, and substituted into eq (7-12). It should be noted that for the central point ($z=d_1$), eq (7-16) transforms after some algebra into

$$\frac{\partial v_{12}}{\partial z} = \frac{v_{12}(d_1^2-d_2^2)-v_{23}d_1^2+v_{01}d_2^2}{-d_1d_2(d_1+d_2)}\ . \tag{7-18}$$

If $d_1=d_2=d$ then

$$\frac{\partial v_{12}}{\partial z} = \frac{v_{23}-v_{01}}{2d} \tag{7-19}$$

which, in fact, corresponds to eq (7-13).

7.4 Boundary conditions

The date series of water levels y_0 and y_3 are applied as boundary conditions. If the time step Δt used in the calculations is smaller than the sampling interval ΔT of the measurements, a square degree (3-point) interpolation in time between the measured values is carried out. This interpolation makes use of the preceding, the actual and the next measurement of each boundary level. The data required at time $t_0 + \Delta t$ are y_0 and y_3 measured at $t_0-\Delta T$, t_0 and $t_0 + \Delta T$ (see Fig. 7-6).

The relation used for the interpolation is

$$y_i(t_0+k\Delta t) = \frac{1}{2} y_i(t_0+\Delta T) \frac{k}{n}\left(\frac{k}{n}+1\right) - y_i(t_0)\left(\frac{k}{n}-1\right)\left(\frac{k}{n}+1\right) +$$

$$+ \frac{1}{2} y_i(t_0-\Delta T) \frac{k}{n}\left(\frac{k}{n}-1\right) \quad (i=0 \text{ and } 3) \qquad (7\text{-}20)$$

where:

Δt = calculation time step
ΔT = measurement interval
n = number of time steps within a sampling interval
k = serial order of the time interval Δt concerned ($k = 1 \ldots n$)

This relation is a 2nd-degree function in k, which takes the following values for values of k, equal to -n, 0 and n:

$k=-n \qquad y_i=y_i(t_0-\Delta T)$
$k=0 \qquad y_i=y_i(t_0)$
$k=n \qquad y_i=y_i(t_0+\Delta T)$

This is made clear if the above values of k are substituted into eq (7-20).

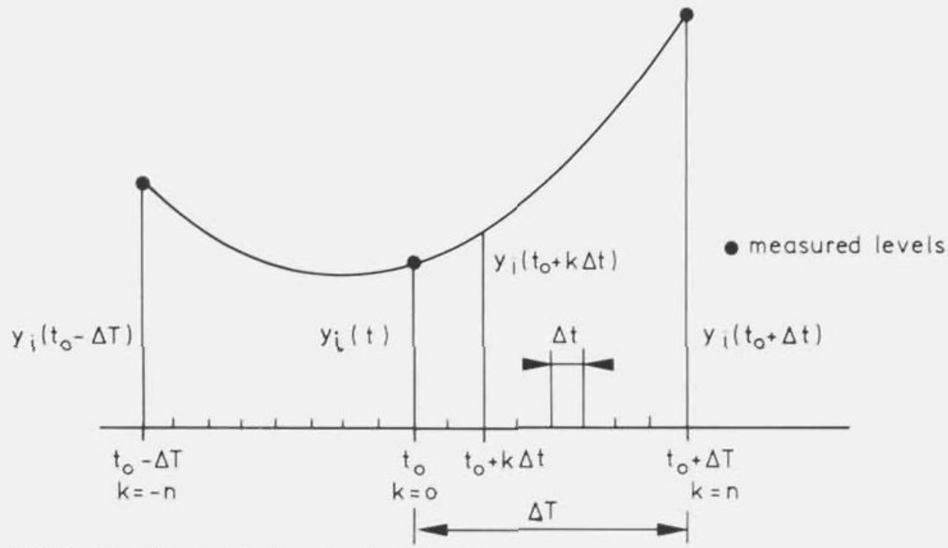


Fig. 7-6 Interpolation of the boundary levels y_0 and y_3.

7.5 Initial conditions

In order to begin calculations for $t_0 + \Delta T$, initial values for t_0 must be assigned to variables in the model. The model variables concerned are the water levels at sites 0, 1, 2 and 3 and the velocities in reaches 0-1, 1-2 and 2-3. Furthermore, in carrying out interpolation of the boundary levels as described in Section 7.2.3., the water levels at sites 0 and 3 should be used for the intervals ΔT earlier and later than t_0.

Water levels at boundaries 0 and 3 can be taken from actual measurements. If incidental or regular measurements are available at the intermediate sites 1 and 2, these should be used as initial values. If no data are available realistic looking values should be chosen. The starting time is the only moment that measured levels at the intermediate sites are used in the calculation.

As a rule no values are available for the initial velocities, and a realistic assumption has to be made. If the model is stable (see Section 7.2.6), the velocity values as well as the water level values will move to values adapted to the conditions of the model after a number of runs.

7.6 Calculation procedure

Values of y_i (i=1, 2) and $v_{i,i+1}$ (i=0, 1, 2) will be calculated for time t+ Δt from available data, (measured or calculated) at time t. For this purpose the following derivation is used:

$$y_i(t+\Delta t) = y_i(t) + \theta \left(\frac{\Delta y_i}{\Delta t}\right)_{t+\Delta t} . \Delta t + (1-\theta \left(\frac{\Delta y_i}{\Delta t}\right)_t \Delta t \tag{7-21}$$

and

$$v_{i,i+1}(t+\Delta t) = v_{i,i+1}(t) + \theta \left(\frac{\Delta v_{i,i+1}}{\Delta t}\right)_{t+\Delta t} \Delta t \quad + (1-\theta) \left(\frac{\Delta v_{i,i+1}}{\Delta t}\right)_t \Delta t. \tag{7-22}$$

The coefficient θ is a weight coefficient which is introduced to reduce the numerical errors. A value of $\theta = 1/2$ is recommended which results in the average value of the derivatives at t and t+ Δt being used.

The values of the derivatives at t are known (eqs (7-10) and (7-12)), but not those at t+ Δt. This problem is tackled as follows. A provisional calculation of y_i (t+Δt) and $v_{i,i+1}$ (t+Δt) is initially carried out according to:

$$y_i(t+\Delta t)^* = y_i(t) + \left(\frac{\Delta y_i}{\Delta t}\right)\Delta t \tag{7-23}$$

$$v_{i,i+1}(t+\Delta t)^* = v_{i,i+1}(t) + \left(\frac{\Delta v_{i,i+1}}{\Delta t}\right)\Delta t \tag{7-24}$$

Here the asterisks denote provisional values.

Boundary levels y_0 $(t+\Delta t)$ and $y_3(t+\Delta t)$ are also calculated according to eq (7-20) with $t=t_0+(k-1)\Delta t$. i.e. for the k^{th} time step after a measurement. Values are at this stage available for all variables. Next provisional values for the derivatives at $t+\Delta t$ are calculated, again using eqs (7-10) and (7-12), and these values are employed in eqs (7-21) and (7-22) to find revised values of $y_i(t+\Delta t)$ and $v_{i,i-1}$ $(t+\Delta t)$.

These values were adopted as final values, although some small errors remain on account of provisional values being used for the derivatives at $t+\Delta t$, (the third right hand terms in eqs (7-21) and (7-22)), based on provisional values of $y_i(t+\Delta t)$ and $v_{i,i-1}(t+\Delta t)$. The errors might be reduced by repeating the calculation of the derivatives for $t+\Delta t$ by applying eqs (7-21) and (7-22) once more. The iteration process could be continued until consecutive derivatives no longer showed differences. To save computer time the iteration process was not applied, and in the test example, described in Section 7.2.6, acceptable results were obtained.

In the programme the following were calculated successively:

- $y_i(t+\Delta t)^*$
- $\left(\frac{\Delta y_i}{\Delta t}\right)^*_{t+\Delta t} \Delta t$
- $\hat{y}_i(t+2\Delta t)^* = y_i(t+\Delta t)^* + \left(\frac{\Delta y_i}{\Delta t}\right)^*_{t+\Delta t}\Delta t$ (7-25)
- $y_i(t+\Delta t) = \frac{1}{2}\left\{ y_i(t) + y_i(t+2\Delta t)^* \right\}$ (7-26)

Eq (7-26) implies that "$\theta=1/2$" is taken into account. This may be shown as follows. Substitution of eq (7-25) into (7-26) yields

$$\hat{y}_i\,(t+\Delta t)+\ \frac{1}{2}\left\{ y_i(t)+y_i(t+\Delta t)^* + \left(\frac{\Delta y_i}{\Delta t}\right)^*_{t+\Delta t} \Delta t \right\}$$

Further substitution of eq (7-23) gives:

$$\hat{y}_i(t+\Delta t)=y_i(t)+\ \frac{1}{2}\left(\frac{\Delta y_i}{\Delta t}\right)^*_{t+\Delta t}\Delta t + \frac{1}{2}\left(\frac{\Delta y_i}{\Delta t}\right)_t \Delta t \qquad (7\text{-}27)$$

The above procedure is made clear by reference to Fig. 7-7. The calculation of velocities follows a similar procedure.

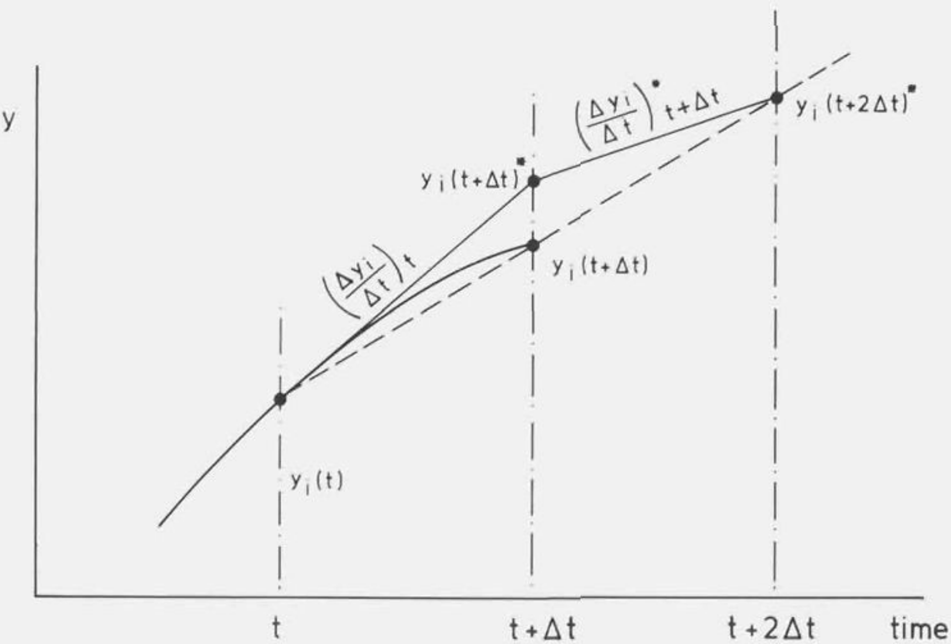


Fig. 7-7 Graphical representation of the calculation of $y_i(t+\Delta t)$.

7.7 Further evaluation of the model

Model evaluation should include tests of

- **Consistency**, i.e. whether the difference equations become equal to the differential equations for $\Delta t \rightarrow 0$ and $\Delta z \rightarrow 0$.
- **Stability**, i.e. whether disturbances are not amplified during successive time steps.

- **Convergence**, i.e. whether the solution of the difference equations corresponds to that of the differential equations for $\Delta t \to 0$ and $\Delta z \to 0$.
- **Accuracy**, i.e. whether the deviations of the results of the numerical equations from those of the results of the analytical equations remain within acceptable limits. This depends on the values of Δt and Δz that are used, and is essentially a matter of programme economy.

Consistency

The model is consistent if eqs (7-21) and (7-22) can be transformed into eqs (7-1) and (7-2) respectively for $\Delta t \to 0$ and $\Delta z \to 0$. This will first be proved for the *equation of continuity*, eq (7-1) and later on for the equation of motion, eq (7-7).

From eq (7-21) it follows that

$$\frac{y_i(t+\Delta t)-y_i(t)}{\Delta t} = \theta \left(\frac{\Delta y_i}{\Delta t}\right)_{t+\Delta t} + (1-\theta) \left(\frac{\Delta y_i}{\Delta t}\right)_t \qquad (7\text{-}28)$$

For further elaboration of the right hand side of eq (7-28) consider eq (7-10):

$$\frac{\Delta y_i}{\Delta t} = -\frac{1}{W_{si}} \frac{Q_{i,i+1} - Q_{i-1,i}}{1/2\ (d_{i-1,i} + d_{i,i+1})} \quad (i = 1,2)$$

Expanding the discharges as a Taylor series gives

$$Q_{i,i+1} = Q_i + \frac{\partial Q_i}{\partial z} \cdot \frac{1}{2} d_{i,i+1} + \frac{1}{2} \frac{\partial^2 Q_i}{\partial z^2} \cdot \frac{1}{4} d^2_{i,i+1} \dots$$

$$Q_{i-1,i} = Q_i - \frac{\partial Q_i}{\partial z} \cdot \frac{1}{2} d_{i-1,i} + \frac{1}{2} \frac{\partial^2 Q_i}{\partial z^2} \cdot \frac{1}{4} d^2_{i-1,i} \dots$$

Subtraction of the last series from the first and dividing by $W_{si} . 1/2\ (d_{i-1,i} + d_{i,i+1})$ yields

$$\frac{\Delta y_i}{\Delta t} = -\frac{1}{W_{si}} \left\{ \frac{\partial Q_i}{\partial z} + \frac{1}{2} \frac{\partial^2 Q_i}{\partial z^2} \frac{1}{2} . (d_{i,i+1} - d_{i-1,i}) \right\} \qquad (7\text{-}29)$$

Further the following Taylor series is developed:

$$\left(\frac{\Delta y_i}{\Delta t}\right)_{t+\Delta t} = \left(\frac{\Delta y_i}{\Delta t}\right)_t + \frac{\partial}{\partial t}\left(\frac{\Delta y_i}{\Delta t}\right)_t \Delta t + \frac{1}{2}\frac{\partial^2}{\partial t^2}\left(\frac{\Delta y_i}{\Delta t}\right)_t \Delta t^2 \qquad (7\text{-}30)$$

Substitution of eq (7-30) into eq (7-28) yields

$$\theta \left(\frac{\Delta y_i}{\Delta t}\right)_{t+\Delta t} + (1\text{-}\theta)\left(\frac{\Delta y_i}{\Delta t}\right)_t =$$

$$\left(\frac{\Delta y_i}{\Delta t}\right)_t + \theta\frac{\partial}{\partial t}\left(\frac{\Delta y_i}{\Delta t}\right)_t \Delta t + \frac{1}{2}\theta\frac{\partial^2}{\partial t^2}\left(\frac{\Delta y_i}{\Delta t}\right)_t \Delta t^2 \qquad (7\text{-}31)$$

Now it can be derived from eq (7-29) that

$$\frac{\partial}{\partial t}\left(\frac{\Delta y_i}{\Delta t}\right)_t = \frac{1}{W_{si}}\left\{\frac{\partial^2 Q_i}{\partial z \partial t} + \frac{1}{2}\frac{\partial^3 Q_i}{\partial z^2 \partial t} \cdot \frac{1}{2}(d_{i,i+1} - d_{i-1,1})\right\} \qquad (7\text{-}32)$$

Substitution of eqs (7-29) and (7-32) into eq (7-31), and neglecting higher order terms, yields

$$\theta \left(\frac{\Delta y_i}{\Delta t}\right)_{t+\Delta t} + (1\text{-}\theta)\left(\frac{\Delta y_i}{\Delta t}\right)_t =$$

$$- \frac{1}{W_{si}}\left\{\frac{\partial Q_i}{\partial z} + \frac{1}{2}\frac{\partial^2 Q_i}{\partial z^2} \cdot \frac{1}{2}(d_{i,i+1} - d_{i-1,i}) + \theta\frac{\partial^2 Q_i}{\partial z \partial t}\Delta t\right\} \qquad (7\text{-}33)$$

Consideration of the left hand side of eq (7-28) gives

$$y_i(t+\Delta t) = y_i(t) + \left(\frac{\partial y_i}{\partial t}\right)_t \Delta t + \frac{1}{2}\left(\frac{\partial^2 y_i}{\partial t^2}\right)_t \Delta t^2, \qquad (7\text{-}34)$$

and thus

$$\frac{y_i(t+\Delta t) - y_i(t)}{\Delta t} = \left(\frac{\partial y_i}{\partial t}\right)_t + \frac{1}{2}\left(\frac{\partial^2 y_i}{\partial t^2}\right)_t \Delta t. \qquad (7\text{-}35)$$

Replacing the left hand side of eq (7-28) by eq (7-35), and its right hand side by eq (7-33), yields

$$\frac{\partial y_i}{\partial t} = \left\{ -\theta \frac{1}{W_{si}} \frac{\partial^2 Q_i}{\partial z \partial t} - \frac{1}{2} \frac{\partial^2 y_i}{\partial t^2} \right\} \Delta t - \frac{1}{W_{si}} \left\{ \frac{\partial Q_i}{\partial z} + \frac{1}{2} \frac{\partial^2 Q_i}{\partial z^2} \frac{1}{2} \cdot (d_{i,i+} - d_{i-1,i}) \right\} \tag{7-36}$$

For $\Delta t \rightarrow 0$ and $\Delta z = 1/2\ (d_{i,i-1} + d_{i-1,1}) \rightarrow 0$ this approximates

$$\frac{\partial y_i}{\partial t} = - \frac{1}{W_{si}} \frac{\partial Q_i}{\partial z}$$

which really corresponds to eq. (7-1).

For the *equation of motion*, first consider the differential form of eq (7-12). It will be proved that each of the three right hand terms differs from the corresponding terms of the continuous form, eq (7-7), only by terms of second order in Δz, or in this case in $d_{1,i+1}$.

The first right hand term of eq (7-12):

$$-g \quad \frac{y_{i+1} - y_i}{d_{i,i+1}}$$

will be elaborated by developing:

$$y_{i+1} = y_{i,i+1} + \frac{\partial y}{\partial z} \cdot \frac{1}{2} d_{i,i+1} + \frac{1}{2} \frac{\partial^2 y}{\partial z^2} \cdot \frac{1}{4} d^2_{i,i+1} + \frac{1}{6} \frac{\partial^3 y}{\partial z^3} \cdot \frac{1}{8} d^3_{i,i+1} \tag{7-37}$$

$$y_i = y_{i,i+1} - \frac{\partial y}{\partial z} \cdot \frac{1}{2} d_{i,i+1} + \frac{1}{2} \frac{\partial^2 y}{\partial z^2} \cdot \frac{1}{4} d^2_{i,i+1} - \frac{1}{6} \frac{\partial^3 y}{\partial z^3} \cdot \frac{1}{8} d^3_{i,i+1} \tag{7-38}$$

so that

$$-g \frac{y_{i+1}-y_i}{d_{i,i+1}} = -g\left(\frac{\partial y}{\partial z} + \frac{1}{3}\frac{\partial^3 y}{\partial z^3} \cdot \frac{1}{8} d^2_{i,i+1}\right). \tag{7-39}$$

In the second right hand term of eq (7-12) the denominator can be developed, using again eqs (7-37) and (7-38), as follows:

$$y_{i+1} + y_i - b_{i+1} - b_i =$$

$$2\,y_{i,i+1} + \frac{\partial^2 y}{\partial z^2} \cdot \frac{1}{4} d^2_{i,i+1} - 2b_{i,i+1} + \frac{\partial^2 b}{\partial z^2} \cdot \frac{1}{4} d_i{}^2{}_{,i+1} \tag{7-40}$$

in which no first order terms appear. If substituted into the term of eq (7-12) considered this will lead to an expression in which first order terms are missing as well.

The third right hand term of eq (7-12) is

$$v \frac{\partial v}{\partial z},$$

and is calculated according to eqs (7-16) and (7-17). First consider the case for $z=0$ in eq (7-16). This becomes

$$\frac{\delta v_{01}}{\delta z} = B = \frac{-(v_{12}-v_{01})(d_1+d_2)^2 + (v_{23}-v_{01})d^2}{-d_1 d_2 (d_1+d_2)} \;{}^{*)} \tag{7-41}$$

Now consider the Taylor series expansion:

$$v_{12} = v_{01} + \frac{\partial v_{01}}{\partial z} d_1 + \frac{1}{2}\frac{\partial^2 v_{01}}{\partial z^2} d_1{}^2 + \frac{1}{6}\frac{\partial^3 v_{01}}{\partial z^3} d_1{}^3$$

$$v_{23} = v_{01} + \frac{\partial v_{01}}{\partial z}(d_1+d_2) + \frac{1}{2}\frac{\partial^2 v_{01}}{\partial z^2}(d_1+d_2)^2 + \frac{1}{6}\frac{\partial^3 v_{01}}{\partial z^3}(d_1+d_2)^3 \tag{7-42}$$

Substitution of (7-42) into (7-41) gives, after some manipulation,

$$\frac{\delta v_{01}}{\delta z} = \frac{\partial v_{01}}{\partial z} - \frac{1}{2}\frac{\partial^3 v_{01}}{\partial z^3} d_1 (d_1+d_2) \tag{7-43}$$

*) δ indicates a 2nd degree approximation for ∂, just as Δ indicates a 1st degree approximation.

Here too no first order terms appear. For v_{12} and v_{23} similar expressions, also without first order terms, can be derived.

Now all right hand terms of eq (7-12) have proved not to include first order terms. Thus eq (7-12) is always accurate in second order terms.

Substituting eqs (7-39), (7-40) and (7-43) into eq (7-12) yields

$$\frac{\Delta v_{i,i+1}}{\Delta t} = -g \frac{\partial y_{i,i+1}}{\partial z} - g \frac{v_{i,i+1}|v_{i,i+1}|}{C^2\{(y-b)_{i,i+1} + \text{s.o.t.}(z)\}}$$

$$-\left(v \frac{\partial v}{\partial z}\right)_{i,i+1} + \text{s.o.t.}(z), \qquad (i=0,1,2) \tag{7-44}$$

where s.o.t. (z) = second and higher order terms in Δz. For $\Delta z \rightarrow 0$ these second order terms disappear.

Now consider eq (7-22). This can be developed as follows:

$$\frac{v_{i,i+1}(t+\Delta t)-v_{i,i+1}(t)}{\Delta t} = \theta \left(\frac{\Delta v_{i,i+1}}{\Delta t}\right)_{t+\Delta t} -(1-\theta) \left(\frac{\Delta v_{i,i+1}}{\Delta t}\right)_t . \tag{7-45}$$

The left hand term leads to (compare eqs (7-34) and (7-35))

$$\frac{v_{i,i+1}(t+\Delta t) - v_{i,i+1}(t)}{\Delta t} = \left(\frac{\partial v_{i,i+1}}{\partial t}\right)_t + \frac{1}{2}\left(\frac{\partial^2 v_{i,i+1}}{\partial t^2}\right)_t \Delta t, \tag{7-46}$$

and the right hand side to

$$\theta \left(\frac{\Delta v_{i,i+1}}{\Delta t}\right)_{t+\Delta t} - (1-\theta) \left(\frac{\Delta v_{i,i+1}}{\Delta t}\right)_t =$$

$$\left(\frac{\Delta v_{i,i+1}}{\Delta t}\right)_t + \theta \frac{\partial}{\partial t}\left(\frac{\Delta v_{i,i+1}}{\Delta t}\right)_t \Delta t + \frac{1}{2} \theta \frac{\partial^2}{\partial t^2}\left(\frac{\Delta v_{i,i+1}}{\Delta t}\right)_t \Delta t^2. \tag{7-47}$$

Substituting eqs (7-46) and (7-47) into (7-45) yields

$$\left(\frac{\partial v_{i,i+1}}{\partial t}\right)_t + \frac{1}{2}\left(\frac{\partial^2 v_{i,i+1}}{\partial t^2}\right)_t \Delta t = \left(\frac{\Delta v_{i,i+1}}{\Delta t}\right)_t + \theta \frac{\partial}{\partial t}\left(\frac{\Delta v_{i,i+1}}{\Delta t}\right)_t \Delta t$$

$$+ \frac{1}{2}\theta \frac{\partial^2}{\partial t^2} \left(\frac{\Delta v_{i,i+1}}{\Delta t} \right)_t \Delta t^2 \tag{7-48}$$

or

$$\left(\frac{\partial v_{i,i+1}}{\partial t} \right)_t = \left(\frac{\Delta v_{i,i+1}}{\Delta t} \right)_t + \frac{\partial}{\partial t} \left(\theta \frac{\Delta v_{i,i+1}}{\Delta t} - \frac{1}{2} \frac{\partial v_{i,i+1}}{\partial t} \right)_t \Delta t$$

$$+ \frac{1}{2}\theta \frac{\partial^2}{\partial t^2} \left(\frac{\Delta v_{i,i+1}}{\Delta t} \right)_t \Delta t^2 \tag{7-49}$$

For $\Delta t \rightarrow 0$ this becomes

$$\frac{\partial v_{i,i+1}}{\partial t} = \frac{\Delta v_{i,i+1}}{\Delta t}, \tag{7-50}$$

or, with regard to eq (7-44)

$$\left(\frac{\partial v_{i,i+1}}{\partial t} \right)_t = -g \frac{\partial y_{i,i+1}}{\partial z} - g \frac{v_{i,i+1}|v_{i,i+1}|}{C^2(y-b)_{i,i+1}} - \left(v \frac{\partial v}{\partial z} \right)_{i,i+1} \tag{7-51}$$

This is exactly eq (7-7) for t and i,i+1 with which also for the equation of motion the consistency is proven.

Stability

Stability means that possibly introduced disturbances will not be amplified. Whether stability can be expected or not depends on the following factors:

- the value $\sqrt{gH} / \frac{\Delta z}{\Delta t}$ or $\sqrt{gH}\frac{\Delta t}{\Delta z}$, i.e. the ratio between the velocity with which disturbances are propagated, $\sqrt{gH}$, and the finite difference quotient $\frac{\Delta z}{\Delta t}$*). For H is taken the mean water depth $H = \bar{y}-\bar{b}$.
- the weight coefficient θ, introduced in eqs (7-21) and (7-22).
- the importance of the bottom friction. A linear friction coefficient, a, is applied in this stability analysis, and it is defined as

*) In this connection the distance interval Δz is half of the distance between the water level measurement sites 0, 1, 2 and 3 (in Fig. 7-1), since velocities are considered halfway those sites.

$$a = g \frac{\overline{|v|}}{C^2H} \tag{7-52}$$

where $\overline{|v|}$ denotes the mean of the absolute values of the velocities, thus disregarding the flow direction.

As a rule stability will be improved by increase of θ and of "a" and by decrease of $\sqrt{gH}\frac{\Delta t}{\Delta z}$. A more detailed discussion of the stability problem is given in Annex V. The following condition for stability, related to the mathematical model concerned, is derived:

$$|\lambda| = \sqrt{\left\{1\text{-}\theta gH\left(\frac{\Delta t}{\Delta z}\right)^2 \text{-} a\Delta t\,\frac{1\text{-}\theta a\Delta t}{2}\right\}^2 + (1\text{-}a\Delta t)^2\left\{gH\left(\frac{\Delta t}{\Delta z}\right)^2 - \left(\frac{a\Delta t}{2}\right)^2\right\}} \leqslant 1 \tag{7-53}$$

This condition will be tested in a number of applications, to be described later on in Sections 7.9, 7.10 and 7.11. In many publications the problem of stability has been discussed (e.g. Vreugdenhil (1979)).

Convergence.

If consistency and stability are assured this will also be the case for convergence, as is confirmed by the Lax theorem. A special analysis is therefore not required.

Accuracy.

Model accuracy is considered acceptable if the two basic equations (7-1) and (7-7) can be approximated such that the difference between the exact and the numerical equations do not include linear terms in t and z, but do at least include square terms. If this is the case, then the model is called accurate to second order.

First the equation of continuity, eq (7-1), will be examined. From eq (7-36) it follows that

$$\frac{\partial y_i}{\partial t} = -\frac{1}{W_{si}}\frac{\partial Q_i}{\partial z} - \left(\frac{1}{2}\frac{\partial^2 y_i}{\partial t^2} + \theta\frac{1}{W_{si}}\frac{\partial^2 Q_i}{\partial z \partial t}\right)\Delta t$$

$$-\frac{1}{W_{si}} \cdot \frac{1}{2}\frac{\partial^2 Q_i}{\partial z^2} \cdot \frac{1}{2}(d_{i,i+1} - d_{i-1,i}) \tag{7-54}$$

From this it follows that

$$\frac{\partial^2 y_i}{\partial t^2} = -\frac{1}{W_{si}}\frac{\partial^2 Q_i}{\partial z \partial t} - \left(\frac{1}{2}\frac{\partial^3 y_i}{\partial t^3} + \theta\frac{1}{W_{si}}\frac{\partial^3 Q_i}{\partial z \partial t^2}\right)\Delta t$$

$$-\frac{1}{W_{si}} \cdot \frac{1}{2}\frac{\partial^3 Q_i}{\partial z^2 \partial t \,.} \cdot \frac{1}{2}(d_{i,i+1}-d_{i-1,1}). \qquad (7\text{-}55)$$

Substitution of eq (7-55) into eq (7-54) yields

$$\frac{\partial y_i}{\partial t} = \frac{1}{W_{si}}\frac{\partial Q_i}{\partial z} - \left(-\frac{1}{2}\frac{1}{W_{si}}\frac{\partial^2 Q_i}{\partial z \partial t} + \theta\frac{1}{W_{si}}\frac{\partial^2 Q_i}{\partial z \partial t}\right)\Delta t$$

$$-\frac{1}{W_{si}} \cdot \frac{1}{2}\frac{\partial^2 Q_i}{\partial z^2} \cdot \frac{1}{2}(d_{i,i+1}-d_{i-1,i}) + \text{s.o.t.} \qquad (7\text{-}56)$$

Provided the following conditions are fulfilled:

1) $d_{i,i+1}=d_{i-1,i}$ (i.e. equal reaches)

2) $\theta = 1/2$,

then eq (7-56) becomes

$$\frac{\partial y_i}{\partial t} = \frac{1}{W_{si}} \cdot \frac{1}{2}\frac{\partial Q_i}{\partial z} + \text{s.o.t.} \qquad (7\text{-}57)$$

Apart from the second and higher order terms this is equal to eq (7-1), the equation of continuity, so this equation is calculated to second order accuracy.

For the examination of the equation of motion consider eq (7-49). Writing an equation explicitly in $\left(\frac{\Delta v_{i,i+1}}{\Delta t}\right)_t$ gives

$$\left(\frac{\Delta v_{i,i+1}}{\Delta t}\right)_t = \left(\frac{\partial v_{i,i+1}}{\partial t}\right)_t - \frac{\partial}{\partial t}\ \theta\left(\frac{\Delta v_{i,i+1}}{\Delta t} - \frac{1}{2}\frac{\partial v_{i,i+1}}{\partial t}\right)\Delta t$$

$$+ \text{s.o.t. (t)}. \qquad (7\text{-}58)$$

Substituting this into eq (7-49) gives

$$- \left(\frac{\partial v_{i,i+1}}{\partial t}\right)_t = \left(\frac{\Delta v_{i,i+1}}{\Delta t}\right)_t - \frac{\partial}{\partial t}\left[\theta \left\{ \left(\frac{\partial v_{i,i+1}}{\partial t}\right)_t - \frac{\partial}{\partial t} \quad \theta \left(\frac{\Delta v_{i,i+1}}{\Delta t}\right) - \right.\right.$$

$$\left.\left. \frac{1}{2} \quad \frac{\partial v_{i,i+1}}{\partial t}\right) \Delta t \right\} - \frac{1}{2}\left(\frac{\partial v_{i,i+1}}{\partial t}\right)_t \Bigg] \Delta t + \text{s.o.t. (t)} =$$

$$= \left(\frac{\Delta v_{i,i+1}}{\Delta t}\right)_t -$$

$$\frac{\partial}{\partial t}\left(\theta - \frac{1}{2}\right)\left(\frac{\partial v_{i,i+1}}{\partial t}\right)_t \Delta t + \frac{\partial^2}{\partial t^2}\theta \left(\theta \frac{\Delta v_{i,i+1}}{\Delta t} - \frac{1}{2}\frac{\partial v_{i,i+1}}{\partial t}\right)_t \Delta t^2. \tag{7-59}$$

The first right hand terms can be replaced by eq (7-44), in which no first order terms appear. The remaining first order term in eq (7-59) disappears if $\theta = 1/2$, and then

$$\left(\frac{\partial v_{i,i+1}}{\partial t}\right)_t = -g \frac{\partial y_{i,i+1}}{\partial z} - g \frac{v_{i,i+1} |v_{i,i+1}|}{C^2 (y-b)_{i,i+1}} -$$

$$v_{i,i+1} \quad \frac{\partial v_{i,i+1}}{\partial z} + \text{s.o.t. (t)} + \text{s.o.t. (z)}.$$

This equation is accurate to second order, and it approximates eq (7-7).

In summary, for the system discussed both of the basic equations are accurate to *second order if*

1) $d_{i,i+1} = d_{i-1,i}$ (required forthe equation of continuity) (7-60)

2) $\theta = 1/2$ (required for both the basic equations). (7-61)

7.8 Assessment of model parameters

In order to carry out the calculations procedure, a number of streamflow profile parameters has to be specified. These parameters include the dimensions of the profile, such as the storage widths, the flow conveying widths and the bottom elevations at the points considered, as well as the roughness coefficient C. The time step duration, Δt, or the number of timesteps n within the sampling interval ΔT, has also to be chosen.

For the 3-subreach model considered the values of the following parameters are required:

- bottom elevation : b_0, b_1, b_2, b_3
- flow conveying widths : $W_{f(01)}$, $W_{f(12)}$, $W_{f(23)}$
- storage widths : W_{s1}, W_{s2}
- lengths between sites: : d_{01}, d_{12}, d_{23}
- roughness coefficient*) : C
- timestep : Δt
- number of timesteps : n

Values for the widths and lengths might be obtained from field surveys or detailed river maps, and the bottom elevations from longitudinal profiles or soundings. The roughness coefficient C is the parameter, which, as a rule, includes all uncertainties and model simplifications. This parameter is the most difficult one to specify. It should be derived from earlier experiences or, if no other sources are available, from handbooks. For the examples discussed later on a value of $C = 50\ m^{1/2}/s$ was applied in the first instance.

The timestep Δt can be chosen on the basis of stability considerations on the one hand and on computer time requirements on the other hand. The number of timesteps, n, directly follows from the sampling interval as

$$n = \frac{\Delta T}{\Delta t} \qquad (7\text{-}62)$$

Let $t_0 + \Delta T$ be the 'present' time for which water levels and velocities have to be derived on the basis of data available at time t_o. Starting with the known boundary data

$y_i\ (t_0 - \Delta T)$, $y_i(t_0)$ and $y_i\ (t_0 + \Delta T)$ (i=0 and 3)

and with the initial data, assessed as discussed in Section 7.2.3,

$y_i(t_0)$ (i = 1,2) and $v_{i,i+1}(t_0)$ (i = 0,1,2)

*) The same value of the roughness coefficient C was adopted for all sub reaches.

the procedure can be implemented as described in Section 7.2.5.

The whole procedure is repeated n times within the sampling interval ΔT in order to obtain the v and y values for the next measurement. If a period of N measurements (samples) is considered, the procedure described in Section 7.2.5. is carried out $n \times N$ times (see Fig. 7-8).

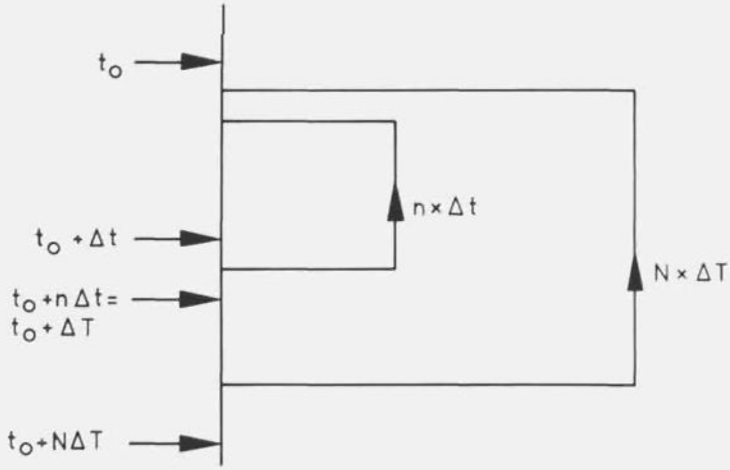


Fig. 7-8 Flow diagram of the sequence of calculations.

For the calculation of a tidal period of 13 h, hourly measurements and a time step of 6 min. are used, so

$$N = 13, \ n = 60/6 = 10.$$

In this case 130 executions of the procedure are required. For a case involving the calculations of 30 daily values in one month, each day was divided into 120 timesteps of 12 min duration, and 3600 executions were required.

7.9 Testing of the model: a hypothetical case

The model will now be applied to a hypothetical case for which an analytical solution exists and which corresponds to a certain extent to the practical example to be dealt with later in Section 7.2.9.

In order to make an analytical solution possible, some assumptions are made:

1. The water depth is great with respect to the vertical water level variations, i.e.
   $(y - \bar{y}) \ll (y - \bar{b})$ (7-63)

Then approximately

$$y - b \approx \bar{y} - b \tag{7-64}$$

2. There is no bottom friction in the streamflow channel so that

   $$C = \infty \tag{7-65}$$

   For reasons of model stability a weight coefficient $\theta = 0{,}6$ has been used.
3. The storage widths, W_s, the flow conveying widths, W_f, and the water depths are equal within each profile, and have the same values for all profiles. These values will be denoted W and $\bar{y}$-$\bar{b}$ respectively.

The consequences for the two basic equations (7-1) and (7-7) are as follows

For the discharge, Q, appearing in the equation of continuity, eq (7-1), the relation

$$Q = W_s\,(y-b)v \tag{7-66}$$

holds, or because of eq (7-64) and assumption 3

$$Q = W\,(\bar{y}-\bar{b})v, \tag{7-67}$$

so that

$$\frac{\partial Q}{\partial z} = W\,(\bar{y}-\bar{b})\,\frac{\partial v}{\partial z} \tag{7-68}$$

Substitution into eq (7-1) yields

$$\frac{\partial y}{\partial t} = -(\bar{y}-\bar{b})\,\frac{\partial v.}{\partial z} \tag{7-69}$$

In the equation of motion, eq (7-7), the second and third right hand terms can be deleted: the second (the friction term) because of assumption 2, and the third (the acceleration or Bernouilli term) because of the assumptions 1 and 3 which imply an almost constant flow profile in the various cross sections. The equation of motion becomes

$$\frac{\partial v}{\partial t} = -g\,\frac{\partial y.}{\partial z}. \tag{7-70}$$

A solution of eqs (7-69) and (7-70) is

$$y - \bar{y} = (y_0-\bar{y})\sin 2\pi\,(t/T-z/L) \tag{7-71}$$

$$v \;= (y_0-\bar{y})\,\sqrt{g/(\bar{y}-b)}.\,\sin 2\pi(t/T-z/L),$$

where

$y_0 - \bar{y}$ = the wave amplitude
T = the wave period
L = the wave length, for which $L = \sqrt{g(\bar{y}-\bar{b})}.\,T$ holds. (7-72)

The square root expression in eq (7-72) is the velocity of propagation of the wave, called the celerity c, so that

$$c = \sqrt{g(\bar{y}-\bar{b})}. \tag{7-73}$$

Consider a river reach, conforming to Fig. 7-1 with the following dimensions at all sites:

- bottom elevation: $b_0 = b_1 = b_2 = b_3 = -20$ m.
- mean water level: $\bar{y}_0 = \bar{y}_1 = \bar{y}_2 = \bar{y}_3 = 0$ m.
- widths W_s and W_f: 4000 m.

Assume a wave propagation time from site 0 to site 3 of 1h. Since the celerity amounts to

$$c = \sqrt{g(\bar{y}-b)} = \sqrt{9{,}8 \times 20} = 14 \text{ m/s}, \tag{7-74}$$

then the total length of the river reach is

$$d_{01} + d_{12} + d_{23} = cT = 14 \text{ m/s} \times 1 \times 3600 \text{ s} = 50{,}4 \times 10^3 \text{m}.$$

At equal distance intervals this means that $d_{01} = d_{12} = d_{23} = 16{,}8 \times 10^3$m. Further, the following values were assigned to the parameters in eqs (7-53) and (7-54):

$$y_o - \bar{y} = 2 \text{ m},$$
$$T = 12 \text{ h},$$

so that

$$L = 14 \text{ m/s} \times 12 \text{ h} \times 3600 \text{ s/h} = 604{,}8 \times 10^3 \text{ m}.$$

Using the model dimensions given above, the following solutions are found

$$y - 20 \text{ m} = 2 \sin 2\pi \left(\frac{t}{12} - \frac{z}{604{,}8 \times 10^3} \right), \tag{7-75}$$

$$v = 1{,}4 \sin 2\pi \left(\frac{t}{12} - \frac{z}{604{,}8 \times 10^3} \right). \tag{7-76}$$

Here t is expressed in h and z in m.

Table 7-1 shows the values of y_1 and y_2 produced by eq (7-75), in the columns headed by sin, indicating that the analytical sine function solution is used. It also shows the boundary levels y_0 and y_3. The results are given for entire hours. Table 7-2 shows in the same way the results for v_{01}, v_{12} and v_{23} found by eq (7-76), also in the columns headed by sin.

For the numerical calculations the bottom friction coefficient was given a value as high as C = 10 000 $m^{1/2}/s$, which could be considered equivalent to no bottom friction. The calculations were carried out using time steps of 180s, 360s and 720s respectively.

For testing the stability the following values are of importance:

$$\sqrt{g\,(\bar{y}-\bar{b})}\ \frac{\Delta t}{\Delta z} = \frac{14}{0{,}5 \times 16{,}8 \times 10^3}\ \Delta t,$$

$$a = \frac{\lceil v \rceil}{C^2(\bar{y}-\bar{b})} = 0 \text{ (see eq (7-52) and assumption 2)}$$

$$\theta = 0{,}6$$

Application of the condition of eq (7-53) leads to the following values for the criterion $|\lambda|$ (see also Annex V)

Δt	$\sqrt{g(\bar{y}-\bar{b})}\ \frac{\Delta t}{\Delta z}$	$\lvert\lambda\rvert$
180 s	0,3	0,992
360 s	0,6	0,987
720 s	1,2	1,208

These results imply that time steps of 180 s and 360 s will give a stable model, but a time step of 720 s will not.

Results of calculations using the three different time steps are also given in Table 7-1 for the water levels and in Table 7-2 for the velocities. These results confirm the instability of a model employing a time step of 720 s.

Fig. 7-9 shows a graphical representation of the sine curve form of the analytical solution. Deviations of the numerical solutions for water levels $\hat{y}_1$ and $\hat{y}_2$ from the

t h	y_0 cm	y_1 cm				y_2 cm				y_3 cm
h	sin	sin	180 s	360 s	720 s	sin	180 s	360 s	720 s	sin
0	0	-35	-35	-35	-35	-68	-68	-68	-68	-100
1	100	68	69	69	69	35	36	36	36	0
2	173	153	151	151	150	129	127	127	125	100
3	200	197	195	195	196	188	186	186	185	173
4	173	188	187	186	186	197	195	195	190	200
5	100	129	130	130	135	153	153	154	149	173
6	0	35	38	38	45	68	70	70	60	100
7	-100	-68	-65	-64	-51	-35	-32	-31	-43	0
8	-173	-153	-152	-152	-136	-126	-126	-127	-141	-100
9	-200	-197	-199	-198	-178	-188	-189	-189	-198	-173
10	-173	-188	-190	-190	-171	-197	-198	-198	-199	-200
11	-100	-129	-130	-130	-125	-153	-155	-154	-135	-173
12	0	-35	-33	-34	-35	-68	-67	-67	-23	-100
13	100	68	67	68	-9	35	34	34	123	0
14	173	153	153	153	29	129	128	127	259	100
15	200	197	194	195	219	188	184	185	300	173
16	173	188	188	187		197	196	195		200
17	100	129	129	129		153	152	153		173
18	0	35	39	38		68	72	71		100
19	-100	-68	-66	-65		-35	-32	-32		0
20	-173	-153	-151	-151		-129	-126	-126		-100
21	-200	-197	-199	-199		-188	-189	-189		-173
22	-173	-188	-190	-190		-197	-198	-198		-200
23	-100	-129	-129	-130		-153	-154	-154		-173
24	0	-35	-34	-34		-68	-68	-67		-100

Table 7-1 Results of comparing numerical and analytical models: water levels

sine curves are given in Fig. 7-10, for $\Delta t=180s$, as well as for $\Delta t=360s$. The deviations between the analytical and numerical solutions are caused by the following:

1. the finite values of the time and distance intervals;
2. the interpolation method, used for the boundary levels;
3. the assumptions made for the analytical approach:
   - no Bernouilli term, and
   - a constant water depth.

If exact sine values are used for the boundary levels instead of square degree interpolated values, or if the Bernouilli term is surpressed in the numerical calcula-

t h	v_{01} cm/s				v_{12} cm/s				v_{23} cm/s			
	sin	180 s	360 s	720 s	sin	180 s	360 s	720 s	sin	180 s	360 s	720 s
0	-12	-12	-12	-12	-36	-36	-36	-36	-59	-59	-59	-59
1	59	59	59	59	36	36	36	36	12	12	12	13
2	115	113	113	113	99	101	101	100	80	87	86	86
3	139	144	144	144	135	140	140	138	127	132	132	132
4	127	132	132	133	135	141	140	136	139	145	144	143
5	80	87	86	87	99	102	102	97	115	114	113	114
6	12	13	13	14	36	37	37	32	59	59	58	57
7	-59	-59	-60	-61	-36	-35	-37	-41	-12	-11	-11	-9
8	-115	-114	-115	-113	-99	-95	-95	-105	-80	-72	-73	-68
9	-139	-133	-133	-130	-135	-128	-129	-146	-127	-118	-118	-107
10	-127	-118	-118	-105	-135	-128	-128	-158	-139	-133	-133	-118
11	-80	-73	-73	-48	-99	-95	-95	-141	-115	-114	-114	-95
12	-12	-11	-11	30	-36	-36	-35	-105	-59	-60	-60	-32
13	59	58	58	120	36	36	37	-82	12	13	13	71
14	115	115	114	234	99	102	102	-138	80	86	86	203
15	139	144	144	517	135	140	141	-426	127	133	133	241
16	127	134	133		135	141	141		139	144	144	
17	80	86	86		99	102	102		115	115	114	
18	12	15	14		36	37	37		59	58	58	
19	-59	-59	-59		-36	-35	-35		-12	-10	-10	
20	-115	-113	-114		-99	-95	-95		-80	-73	-73	
21	-139	-134	-133		-135	-128	-128		-127	-117	-117	
22	-127	-117	-117		-135	-128	-128		-139	-134	-133	
23	-80	-74	-73		-99	-95	-94		-115	-113	-114	
24	-12	-10	-10		-36	-35	-35		-59	-61	-60	

Table 7-2 Results of comparing numerical and analytical models: velocities

tion the deviations will be smaller. However it was preferred to use the mathematical model under similar conditions as will be done later on in more practical cases. Anyhow the deviations remain within reasonable limits, in particular if the magnitude of the water level variations (+2 m to -2 m) is taken into account.

In Section 7.2.9 the same hypothetical example will be examined, but this time with bottom friction.

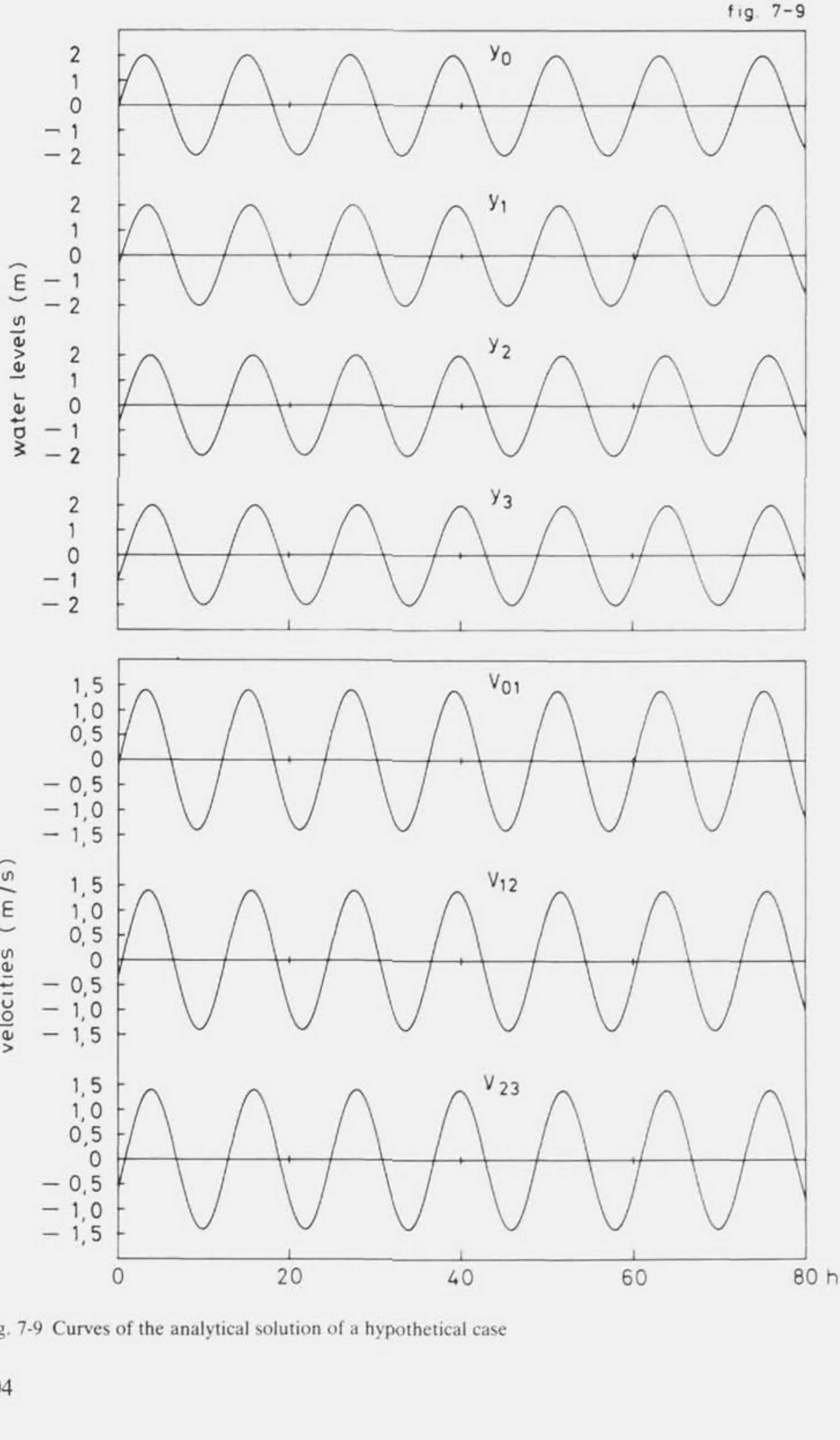


Fig. 7-9 Curves of the analytical solution of a hypothetical case

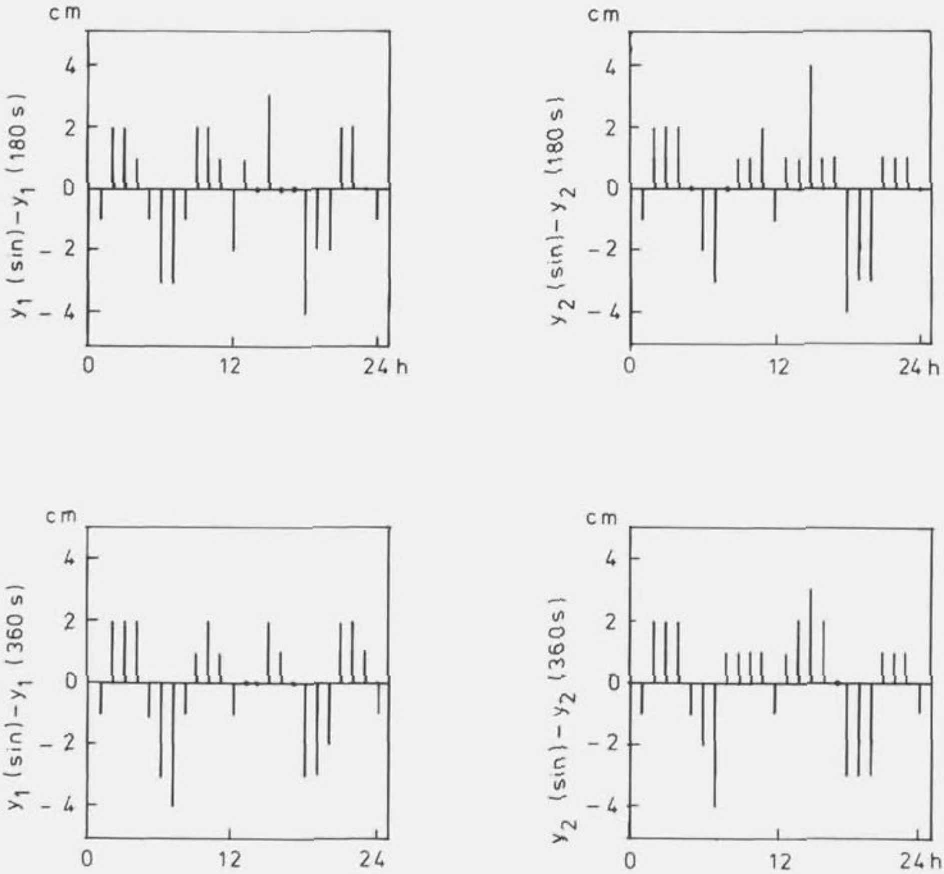


Fig. 7-10 Deviations of the numerical results of $\hat{y}_1$ and $\hat{y}_2$ from the analytical values

7.10 The starting period

The hypothetical example will also be used to test the influence of the initial conditions. Since these may include inaccuracies or disturbances it will take some time for their effect to disappear.

An improved approximation to an actual situation will be obtained this time by using a Chézy coefficient of $C=50\ m^{1/2}/s$. For reasons of accuracy the weight coefficient was put at $\theta = 0{,}5$. Three cases were calculated. The initial data are given in Table 7-3.

initial data		case 1	case 2	case 3
	y_0	0	0	0
cm	y_1	- 35	- 35	100
	y_2	- 68	- 68	100
	y_3	-100	-100	-100
	v_{01}	- 12	100	100
cm/s	v_{12}	- 36	100	100
	v_{23}	- 59	100	100

Table 7-3 Initial data used to test the influence of initial conditions

The data for case 1 correspond to the initial data for the sine curves, as given in the Tables 7-1 and 7-2 and used in the example without bottom friction. In case 2 all velocities are initially set at 1 m/s and in case 3 the water levels at the intermediate stations are set at 1 m. Of course the initial data for cases 2 and 3 are unrealistic, but for testing the model it is useful to examine if even improbable initial conditions become straightened out.

The results are shown in the Tables 7-4 and 7-5. For case 1 after 7h the situation has stabilized, since for t=19h (i.e. one wave period later) the same results are found. Case 2 coincides with case 1 after t = 10h and case 3 after t = 16h. These results show that a starting period of one day at the most is sufficient for this model to stabilize.

The resulting curves for the water levels and velocities are shown in Fig. 7-11. When compared with Fig. 7-9 the differences between the corresponding velocity curves are particularly noteworthy. These differences are caused by the influence of the bottom friction, which affects the velocities shown in Fig. 7-11, but not those of Fig. 7-9.

Although the results do not show any sign of instability, it seems useful to check the condition given by eq (7-53). The mean of the absolute velocity was, in view of the results, estimated at v = 0,5 m/s so that, according to eq (7-52)

$$a\,\Delta t = g \frac{\lceil v \rceil}{C^2(\bar{y}-\bar{b})}\,\Delta t = 9{,}8\,\frac{0{,}5}{(50)^2.20}\,360 = 0{,}0353$$

Further

$$g(\bar{y}-\bar{b})\,\frac{\Delta t}{\Delta z} = 14\,\frac{360}{0{,}5 \times 16{,}8 \times 10^3} = 0{,}36$$

and $\theta = 0{,}5$.

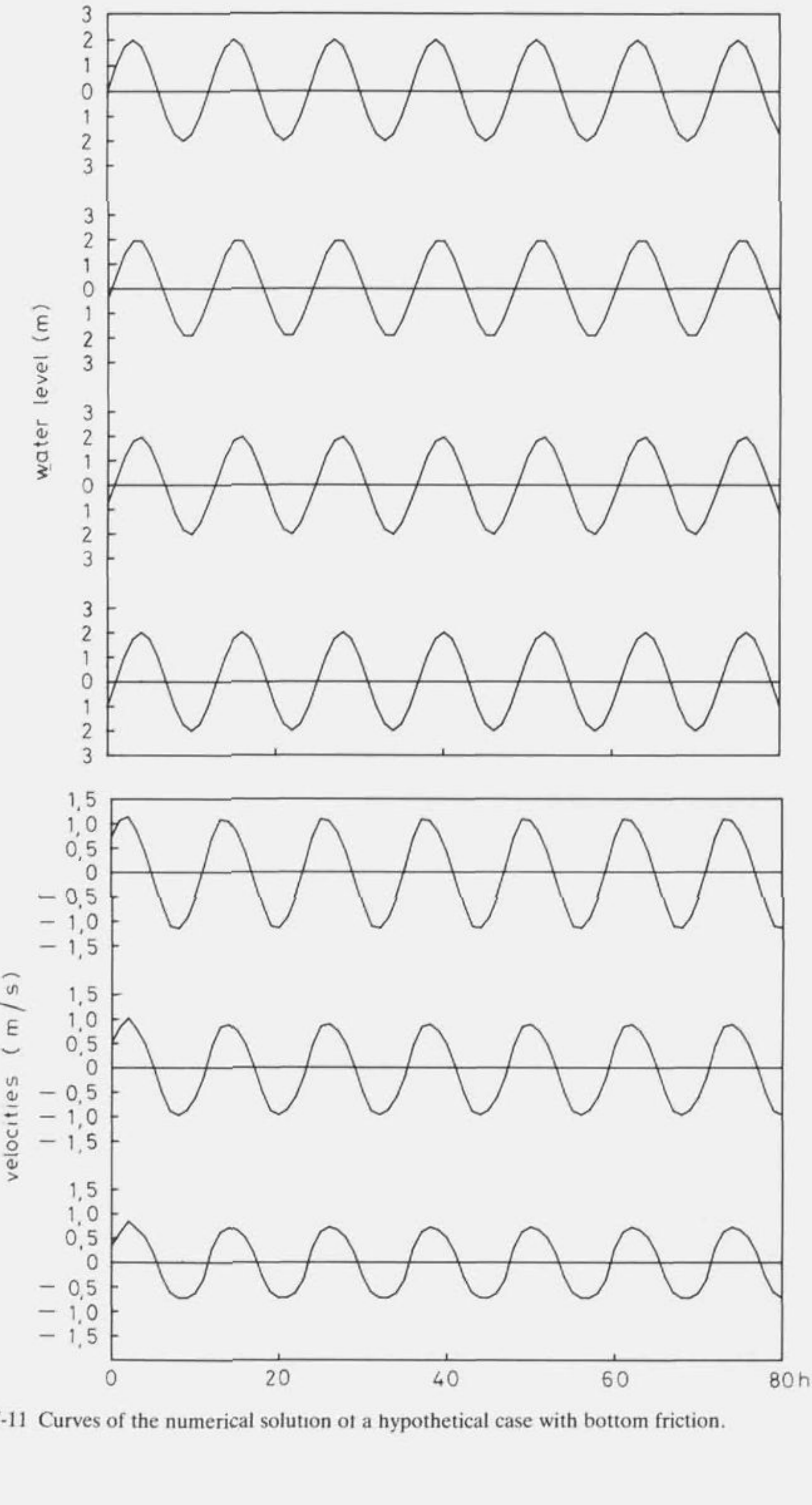


Fig. 7-11 Curves of the numerical solution of a hypothetical case with bottom friction.

t h	y_0 cm	y_1 cm			y_2 cm			y_3 cm
		case 1	case 2	case 3	case 1	case 2	case 3	
0	0	- 35	- 35	100	- 68	- 68	100	- 100
1	100	62	42	- 23	32	10	- 56	0
2	173	143	149	184	119	122	164	100
3	200	193	189	166	183	179	159	173
4	173	189	192	204	197	199	216	200
5	100	132	130	117	155	154	144	173
6	0	43	44	52	74	75	86	100
7	- 100	- 47	- 49	- 53	- 16	- 17	- 22	0
8	- 173	- 138	- 138	- 136	- 114	- 114	- 112	- 100
9	- 200	- 192	- 193	- 193	- 182	- 182	- 183	- 173
10	- 173	- 193	- 193	- 193	- 201	- 201	- 201	- 200
11	- 100	- 130	- 130	- 130	- 155	- 155	- 155	- 173
12	0	- 40			- 73			- 100
13	100	52			20			0
14	173	143			119			100
15	200	191			182			173
16	173	190			198			200
17	100	131			155			173
18	0	43			74			100
19	- 100	- 48			- 17			0
20	- 173	- 138			- 114			- 100
21	- 200	- 192			- 182			- 173
22	- 173	- 193			- 201			- 200
23	- 100	- 130			- 155			- 173
24	0	- 40			- 73			- 100

Table 7-4 Results of testing the model with different initial data: waterlevels.

Substitution of these values into eq (7-53) gives $|\lambda| = 0{,}996$, by which the requirement for stability is satisfied.

7.11 Example of the Western Scheldt tidal estuary.

The model described above is applied to the Western Scheldt tidal estuary for a 62 hour period i.e. 5 tidal cycles. The situation is depicted in Fig. 5-9. Water levels were calculated for the stations Terneuzen and Hansweert, using water levels measured at the Vlissingen and Bath gauging stations as boundary conditions. The results will be discussed for the period from 20 h, 22 January to 9 h, 25 January 1972. This period commences around the turn of the tide in the middle reach when the stream changed

t h	v_{01} cm/s			v_{12} cm/s			v_{23} cm/s		
	case 1	case 2	case 3	case 1	case 2	case 3	case 1	case 2	case 3
0	- 12	100	100	- 36	100	100	- 59	100	100
1	58	132	111	36	104	97	12	69	71
2	87	109	116	76	99	91	60	86	74
3	80	90	81	75	83	80	65	71	75
4	42	45	49	50	53	52	53	58	53
5	- 11	-8	- 13	7	9	8	21	23	25
6	- 70	- 69	- 68	- 48	- 46	- 47	- 29	- 26	- 32
7	- 106	- 105	- 107	- 81	- 80	- 79	- 56	- 56	- 54
8	- 104	- 104	- 104	- 84	- 84	- 84	- 62	- 62	- 63
9	- 79	- 79	- 79	- 71	- 71	- 71	- 58	- 58	- 57
10	- 36		- 35	- 45		- 45	- 48		- 49
11	20		19	-2		-2	- 22		- 22
12	72		72	52		52	35		34
13	110		110	88		88	64		65
14	109		109	95		95	78		78
15	87		87	82		82	73		73
16	46			53			56		
17	-9			9			23		
18	- 68			- 46			- 28		
19	- 106			- 81			- 56		
20	- 104			- 84			- 62		
21	- 79			- 71			- 58		
22	- 36			- 45			- 48		
23	20			-2			- 22		
24	72			52			35		

Table 7-5 Results of testing the model with different initial data: velocities.

from inflow to outflow. The calculations themselves started 500 h earlier, so the model could be considered to have well passed the introductory period required for the model to settle down (the period required is one day at the most,as was explained in section 7.2.9).

Approximate values of the dimensions were obtained from a hydrographic map which included depth curves. For the roughness coefficient a value $C = 60\ m^{1/2}/s$ was used. The sampling interval ΔT was 1h, which was divided into 10 timesteps Δt of 6 min = 360 s. The weight coefficient was put at $\theta = 0,5$.

The dimensions, roughness coefficient and timestep values adopted are summarized below:

– bottom elevations	:	b_0 = NAP – 20,00 m
		b_1 = NAP – 24,00 m
		b_2 = NAP – 16,00 m
		b_3 = NAP – 14,00 m
– flow conducting widths	:	$W_{f(01)}$ = 5000 m
		$W_{f(12)}$ = 3000 m
		$w_{f(23)}$ = 2000 m
– storage widths	:	W_{s1} = 5500 m
		W_{s2} = 4500 m
– lengths between sites*)	:	d_{01} = 20500 m
		d_{12} = 21500 m
		d_{23} = 19900 m
– roughness coefficient		C = 60 $m^{1/2}/s$
– timestep	:	Δt = 360 s
– number of timesteps	:	n = 10

Here again the stability was tested in view of the condition given by eq (7-53). This was carried out for a combination of mean dimensions and for a most unfavourable combination. The mean of the absolute velocity values was provisionally estimated at $\overline{|v|}$ = 0,5 m/s. Further the following values were used, respectively calculated:

	combination of mean values	unfavourable combination
$\bar{y} - \bar{b}$ (m)	18,5	24
Δz (m)	10 316,6	9950
$a\Delta t$	0,0265	0,0204
$\sqrt{g(\bar{y}-\bar{b}}. \dfrac{\Delta t}{\Delta z}$	0,4699	0,5549
$\lvert\lambda\rvert$	0,992	1,000

The $|\lambda|$-values are close to the limit of 1. However the results do not show any instabilities, and besides, a value $\overline{|v|}$ >0,5 m/s is found for the mean of the absolute velocity values. If $\overline{|v|}$ = 0,6 m/s is used, a value which seems more likely, then the following result is found:

	combination of mean values	unfavourable combination
$a\Delta t$	0,0317	0,0245
$\lvert\lambda\rvert$	0,988	0,997

The conclusion is that certainly stability can be expected. The results of this example are shown in Fig. 7-12.

*) The lengths of the three reaches do not differ too much. According to eq (7-60) the model was considered to be acceptable from the point of view of accuracy.

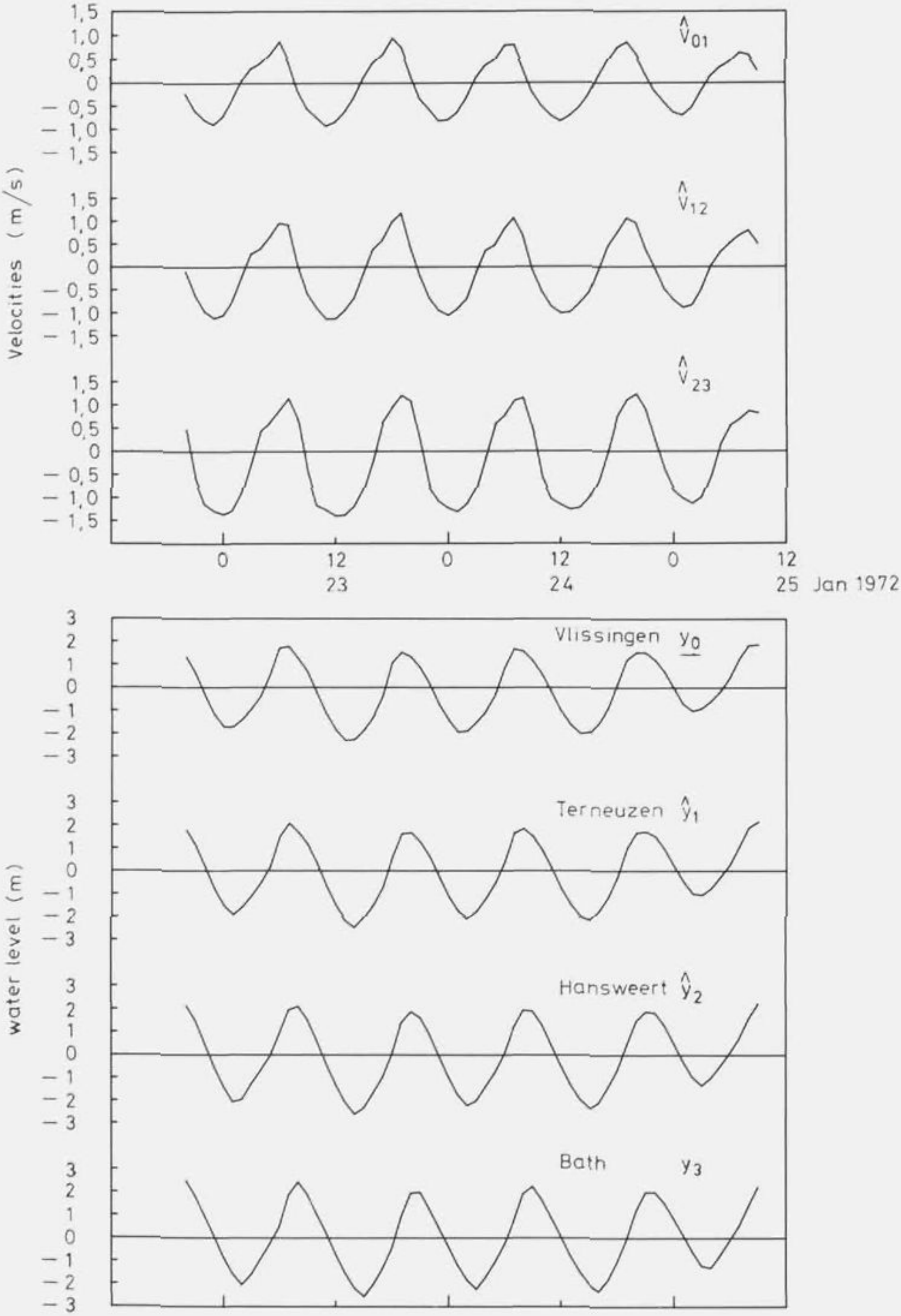


Fig. 7-12 Results of the mathematical model for the Western Scheldt, 23-25 January, 1972.

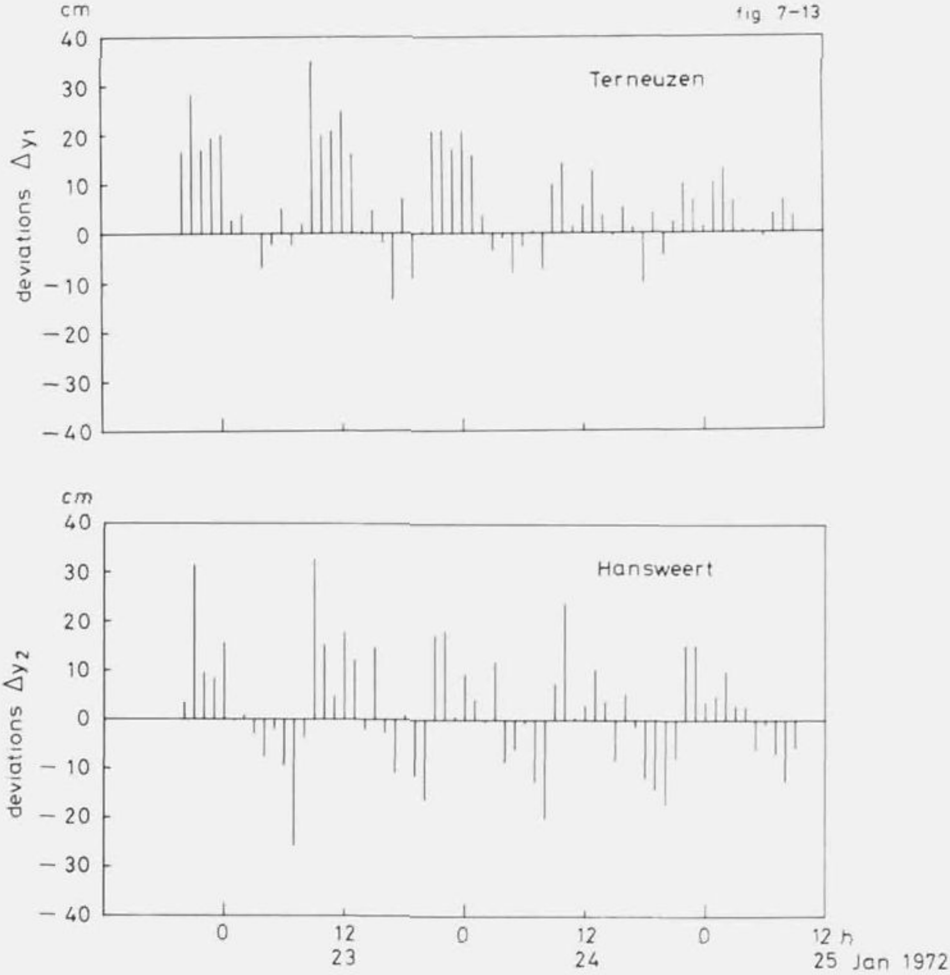


Fig. 7-13 Deviations between measured and calculated water levels (mathematical model).

The curves of the water levels y_0 and y_3 are the measured levels at the boundaries, whereas the curves of the water levels $\hat{y}_1$ and $\hat{y}_2$ and of the velocities v_{01}, v_{12} and v_{23} are calculated values for intermediate sites.

In the velocity curves it is possible to recognize the strong velocity increase shortly before high water. This high velocity corresponds to the rapid water level rise that occurs about 1h before high tide. The differences between measured and calculated values of the water levels Δy_1 and Δy_2, $\Delta y_i = y_i - \hat{y}_i$ $(i = 1,2)$ are given in Fig. 7-13. Summary statistics for these differences are as follows:

mean values:

Terneuzen $\bar{\Delta}\bar{y}_1$	= 6,326 cm
Hansweert $\bar{\Delta}\bar{y}_2$	= 1,643 cm
standard errors:	
Terneuzen $\sigma\Delta y_1$	= 10,099 cm
Hansweert $\sigma\Delta y_2$	= 11,842 cm
correlation coefficient:	
$\varrho\Delta y_1,\Delta y_2$	= 0,781

It is possible that the dimensions and roughness coefficient used could be modified so as to minimize the standard errors $\sigma\Delta y_1$ and $\sigma\Delta y_2$.

The resulting standard errors should be checked against the design criterion following from eq (2-23)
$\sigma\Delta y \leqslant \sqrt{\varepsilon^2 + E^2}$,
which for standard practice in the Netherlands gives the criterion
$\sigma\Delta y \leqslant 3{,}5$ cm

If this criterion is not fulfilled another series of sites should be selected, more closely located than those of the series considered above. If the result is still unsatisfactory, then this action should be repeated until the required result is obtained. In fact a similar procedure to that carried out for the regression methods of chapters 5 and 6 has to be followed.

Influence of transversal forces.

As was remarked in Section 7.2.1. transversal forces play no role in the model used. However, it is possible to estimate the order of magnitude of their influence on the errors over the cross section of the river. If a transversal force causes an acceleration f_t, there will be a surface gradient of f_t/g. At a river width of W_s, there is a difference in water level of

$$y_d = \frac{f_t . W_s}{g}. \tag{7-77}$$

For the example considered the surface width is about 5000 m. This means that approximately $y_d = 500 \times f_t$.

The most important transversal accelerations are the Coriolis acceleration and the centrifugal acceleration. For the Coriolis acceleration, which is caused by the earth's rotation, the following holds:

$f_{cor} = 2v\omega \sin\varphi$, (7-78)

where: v = flow velocity
ω = the earthls rotation velocity $2\pi/86400$ rad/s = 0,0000727 rad/s
φ = the local latitude. At 51 °N this yields $\sin\varphi = 0,777$.

At a velocity of $v = 1$ m/s the water level difference will be:

$y_d = 500 \times 2 \times 1 \times 0{,}0000727 \times 0{,}777 = 0{,}056$ m,

with the high level at the right hand side of the flow direction.

The centrifugal acceleration can be derived from

$f_{cent} = v^2/r$, (7-79)

where: r = the radius of the flow line.

At a radius of 5000 m, which is not unlikely in the example concerned, and with a flow velocity of $v = 1$ m/s, there will be a level difference of

$y_d = 500 \times 1^2/5000 = 0{,}010$ m.

The highest levels are at the outer side of the flow curve (i.e. always at the same bank of the river) whatever the flow direction, whereas the differences caused by the Coriolis forces are alternating.

The above differences give an impression of the degree to which water levels can be affected by transversal forces. It should be noted that these forces also influence the levels at the boundary stations, and therefore these levels should really be corrected before using them as input data. Further, the level differences given above concern a steady state condition, which will only be attained after some time. In the dynamic situation described this condition will not be reached.

The values found for the transversal differences were considered to be so small that they could be neglected bearing in mind other sources of inaccuracy affecting the model.

7.12 Comparison of the mathematical model with the linear regression method.

The performance of the simple physical method will now be compared with that of the regression method. Water levels for the same period as used above for the Terneuzen and Hansweert stations, were calculated using the water level data of the boundary stations Vlissingen and Bath. However, in order to use the same information, only the following data were used as input data:

Vlissingen	: $y_0\ (t_0\text{-}1h),\ y_0(t_0),\ y_0(t_0+1h)$
Bath	: $y_3\ (t_0\text{-}1h),\ y_3(t_0),\ y_3(t_0+1h)$

Linear relations were derived for Terneuzen, $y_1(t_0+1h)$ and for Hansweert, $y_2(t_0+1h)$. The coefficients of these relations were calculated from data, measured during the year 1971, which preceeds the periode considered.

In Fig. 7-14 the deviations from the measured tidal curve are shown for the gauges Terneuzen and Hansweert. The statistics of the differences Δy_i (i=1,2) are as follows:

mean values:	
Terneuzen $\bar{\Delta}\bar{y}_1$	= 0,731 cm
Hansweert $\bar{\Delta}\bar{y}_2$	= 1,508 cm
standard errors:	
Terneuzen $\sigma\Delta y_1$	= 6,104 cm
Hansweert $\sigma\Delta y_2$	= 7,537 cm
correlation coefficient:	
$\varrho\Delta y_1,\Delta y_2$	= 0,568

The standard errors found for the regression method are smaller than those of the mathematical model, at least in the example considered. In other cases the result may be the reverse, so for each case a special investigation of what method is to be preferred is desirable.

The following remarks should be made regarding the relative merits of the two methods:

1. The multiple linear regression method is easy to apply if the regression coefficients are known. For the determination of these coefficients a rather long period, preceding the test period, has to be analysed. For this method no knowledge of the geometric and hydraulic conditions is required.

2. The physically-based mathematical model requires a rather long and time consuming computer programme, even for the simple conditions assumed for the case considered. A reasonably good knowledge of the geometric and hydraulicconditions is required. However no data of a preceding period have to be used. This method also produces velocity and discharge data.

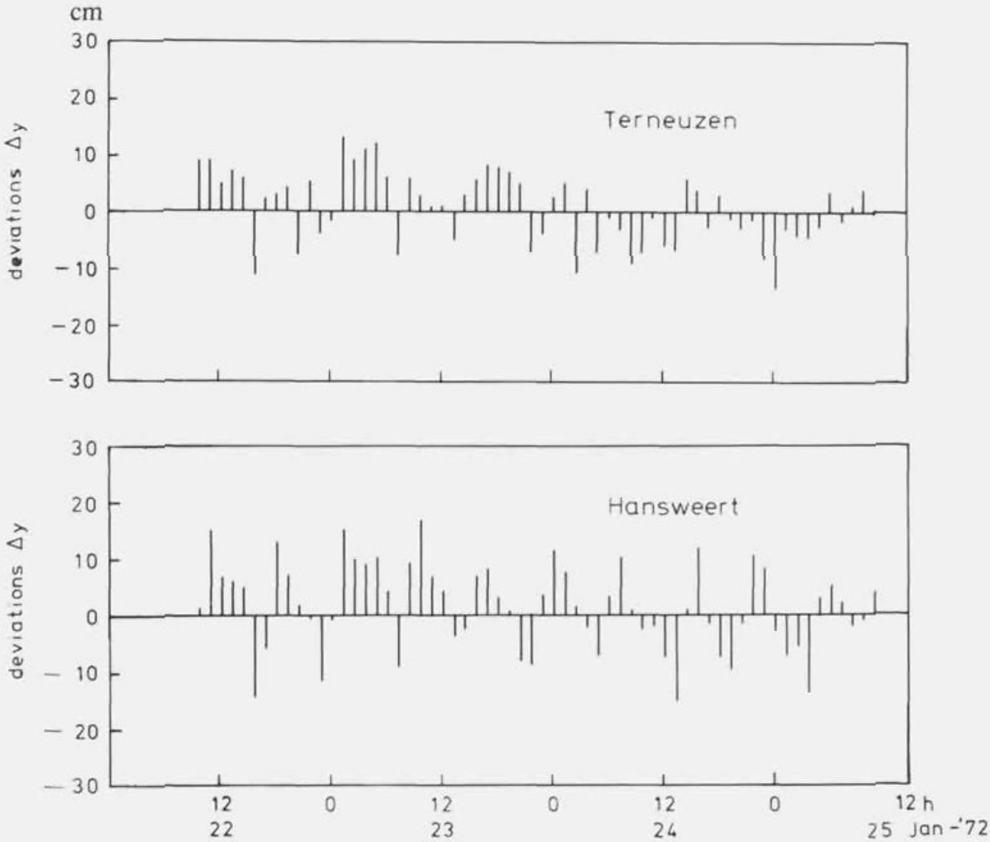


Fig. 7-14. Deviations between the measured and calculated water levels (linear regression method)

For use in network design, where the analyses are carried out only once, the best approach is to use both methods for various periods and to base the final design on the results of both. For current use, for instance for data checking, either one of the methods should be selected. In many cases the multiple linear regression method is acceptable and possibly preferable because of its simplicity. However, if it does not provide satisfactory results, a mathematical model can be considered.

8. Combination of a physically-based mathematical approach and a statistical approach

8.1 Application of linear regression to the deviations Δy.

The statistical approach, as described in chapter 5 was based on the correlation structure between the water levels in a certain area. This approach was applied to the whole range of water levels, in principle from the lowest to the highest. In the physically based mathematical approach as described in Chapter 7 a certain model was used which the water movement was assumed to follow. Because of the inadequacies of the model it was able to reconstruct the occuring phenomena within certain limits only.

In the following an attempt will be made to combine both approaches. The phenomena will be calculated by means of a physically based model, whereas further on the deviations between the model results and the measurements will be examined on their correlation structure. The thus found correlation structure can be used, after discontinuing some ofthe measurement stations, to calculate the possible corrections for the sites of these stations, on the base of the results of the remaining network stations.

In the model results, not only the inaccuracies of the water levels are of importance but also those of the velocities and those of the used parameters like the roughness coefficient and the channel dimensions. Besides there are the deviations due to the model itself, which is no more than a simplified imitation of what is really going on. These considerations were already discussed in chapter 2. In particular is referred here to eq (2-17).

Now consider the model of Section 7.2 in which the intermediate water levels y_1 and y_2 were derived, using the boundary levels y_0 and y_3. The model was applied to the example the Western Scheldt. This example led to mean values, standard errors and a correlation coefficient of the deviations with respect to the measured data at the two intermediate stations.

These results can be used if, besides of the two boundary levels y_0 and y_3, only one of the intermediate levels, e.g. y_1, is measured and the other is to be calculated. This can be done by the following procedure.

(1) measure $\underline{y}_0$, $\underline{y}_1$, and $\underline{y}_3$.

(2) calculate $\hat{y}_1$ and $\hat{y}_2$ using $\underline{y}_0$ en $\underline{y}_3$

(3) determine $\Delta y_1 = y_1 - \hat{y}_1$

(4) calculate Δy_2 according to the linear regression equation: (which is derived from previous measurements)

$$\Delta y_2 = \varrho\Delta y_1 y_2 \frac{\sigma\Delta y_2}{\sigma\Delta y_1} (\Delta y_1 - \bar{\Delta}\bar{y}_1) + \bar{\Delta}\bar{y}_2{}^{*)} \tag{8-1}$$

(5) correct $\hat{y}_2$ using Δy_2, yielding:

$$\hat{y}_2 (+) = \hat{y}_2 (-) + \Delta y_2{}^{*)} \tag{8-2}$$

or, because of eq (8-1):

$$\hat{y}_2 (+) = \hat{y}_2 (-) + \varrho\Delta y_1 y_2 \frac{\sigma\Delta y_2 (-)}{\sigma\Delta y_1} (\Delta y_1 - \bar{\Delta}\bar{y}_1) + \bar{\Delta}\bar{y}_2 \quad {}^{**)} \tag{8-3}$$

The coefficient, expressing the correction will be defined by k. Thus

$$k = \varrho\Delta y_1 y_2 \frac{\sigma\Delta y_2 (-)}{\sigma\Delta y_1}, \tag{8-4}$$

so that:

$$\hat{y}_2 (+) = \hat{y}_2 (-) + k (\Delta y_1 - \bar{\Delta}\bar{y}_1) + \bar{\Delta}\bar{y}_2 \tag{8-5}$$

After this procedure the standard error $\sigma\Delta y_2$ reduces to:

$$\sigma\Delta y_2 (+) = \sqrt{1 - \varrho^2\Delta y_1 y_2} \; . \; \sigma\Delta y_2 (-) \tag{8-6}$$

which follows from eq (5-36) if there are two variables: an independent Δy_1 (-) and a dependent Δy_2 (-). Then there is an unexplained variance Var Δy_2 (+).

If, for instance $\varrho\Delta y_1 y_2 = 0{,}5$ then

$$\sigma\Delta y_2 (+) = \sqrt{1 - (0{,}5)^2}. \; \sigma\Delta y_2 (-) = 0{,}866. \; \sigma\Delta y_2 (-).$$

Thus by carrying out additional measurements of y_1, besides those of y_0 and y_3 and

*) The (+) sign indicates the value after correction, the (-) sign the value before correction

**) $\varrho\Delta y_1 y_2$ stands for the correlation coefficient between Δy_1 and Δy_2

subsequently correcting $\hat{y}_2$ on the basis of linear regression between the deviations Δy the accuracy of $\hat{y}_2$ can be improved to an extent, depending on the correlation.

The above described procedure is a combination of a physically-based streamflow model and a statistical (regression) model. For the statistical component a similar discussion can be presented as was given in Chapter 5 for the variables as a whole.

The standard errors found for the physically based model are, just like those of the statistical model, built up of direct and propagated errors of the input data and of the model errors. The input data concern all variables and parameters used for this calculation, including the measured ones as well as those estimated by other means. In the model concerned these are water levels, velocities, roughness coefficient and channel dimensions. The uncertainties of all these elements influence not only the error of the final calculated values, but also their correlations.

8.2 Explanation in the light of a linear model

An improved insight into the relation between the error Δy and the input data may be gained by considering a special case where the physically-based model is linear.

Consider two variables y_1 and y_2, depending on two other variables x_1 and x_2 in the following way:

$$\left.\begin{aligned} \hat{y}_1 &= A\underline{x}_1 + B\underline{x}_2 \\ \hat{y}_2 &= C\underline{x}_1 + D\underline{x}_2 \end{aligned}\right\} \qquad (8\text{-}7)$$

Assume the means $\bar{\underline{x}}_1 = \bar{\underline{x}}_2 = 0$ for reasons of simplicity.

Initially consider the first equation of eqs (8-7). This can be transformed into (see also eq (2-4)):

$$\hat{y}_1 = A(x_{1t} + \Delta\underline{x}_1) + B\,(x_{2t} + \Delta\underline{x}_2) = Ax_{1t} + Bx_{2t} + A\,\Delta\underline{x}_1 + B\,\Delta\underline{x}_2 \qquad (8\text{-}8)$$

This equation does not produce the true value of y_{1t} because of the model error, to be indicated by r_{y1}.

The true value of y_1 may be expressed as

$$y_{1t} = Ax_{1t} + Bx_{2t} - r_{y1}. \qquad (8\text{-}9)$$

Subtraction of eq (8-9) from eq (8-8) yields:

$$\hat{y}_1 - y_{1t} = \Delta\hat{y}_1 = A\,\Delta\underline{x}_1 + B\,\Delta\underline{x}_2 + r_{y1}. \tag{8-10}$$

The difference $\Delta y_1 = \underline{y}_1 - \hat{y}_1$ may be expressed as (see eq (2-6))

$$\Delta y_1 = y_{1t} + \Delta\underline{y}_1 - (y_{1t} + \Delta\hat{y}_1) = \Delta\underline{y}_1 - \Delta\hat{y}_1, \tag{8-11}$$

so that, after substitut of eq (8-10) into eq (8-11), it follows that

$$\Delta y_1 = -\,A\,\Delta\underline{x}_1 - B\,\Delta\underline{x}_2 - r_{y1} + \Delta\underline{y}_1 \tag{8-12}$$

In the same way

$$\Delta y_2 = -\,C\,\Delta\underline{x}_1 - D\,\Delta\underline{x}_2 - r_{y2} + \Delta\underline{y}_2 \tag{8-13}$$

can be derived. Now consider the case that y_1 is measured but y_2 is not. Then the calculated value $\hat{y}$ from eq (8-7), now indicated by $\hat{y}_2$ (-) is to be corrected according to eq (8-5), in which:

$$k = \varrho\Delta y_1 y_2 \frac{\sigma\Delta y_2\,(-)}{\sigma\Delta y_1} = \frac{\mathrm{Cov}\,\Delta y_1\,\Delta y_2\,(-)}{\mathrm{Var}\,\Delta y_1}. \tag{8-14}$$

The correction coefficient k can be calculated when considering eqs (8-12) and (8-13). If no correlation is assumed between the corresponding terms of both equations, except for the model error terms r_{y1} and r_{y2} then the following expressions hold:

$$\begin{aligned} \mathrm{Var}\,\Delta y_1 &= A^2\,\varepsilon_{x1} + B^2\,\varepsilon_{x2}^2 + \mathrm{Var}\,r_{y1} + \varepsilon_{y1}^2 \\ \mathrm{Var}\,\Delta y_2 &= C^2\,\varepsilon_{x1}^2 + D^2\,\varepsilon_{x2}^2 + \mathrm{Var}\,r_{y2} + \varepsilon_{y2}^2 \\ \mathrm{Cov}\,\Delta y_1\,\Delta y_2 &= AC\,\varepsilon_{x1}^2 + BD\,\varepsilon_{x2}^2 + \mathrm{Cov}\,r_{y1}r_{y2} \end{aligned} \tag{8-15}$$

where ε_{x1} and ε_{x2} denote the standard errors of measurement for which hold, according to eq (2-7)

$$\varepsilon_{x1}^2 = \mathrm{Var}\,\Delta\underline{x}_1$$

$$\varepsilon_{x2}^2 = \mathrm{Var}\,\Delta\underline{x}_2$$

Substitution of the relevant expressions into eq (8-14) yields:

$$k = \frac{AC\,\varepsilon_{x1}{}^2 + BD\,\varepsilon_{x2}{}^2 + \mathrm{Cov}\,r_{y1}r_y}{A^2\,\varepsilon_{x1}{}^2 + B^2\,\varepsilon_{x2}{}^2 + \mathrm{Var}\,r_{y1} + \varepsilon_{y1}{}^2} \tag{8-16}$$

The variance of the corrected estimation error can be derived as follows. Subtraction of y_{2t} from both sides of (8-5) and substituting eq (8-11), yields

$$\Delta\hat{y}_2\,(+) = \Delta\hat{y}_2\,(-) + k\,(\Delta y_1 - \bar{\Delta}\bar{y}_1) + \bar{\Delta}\bar{y}_2. \tag{8-17}$$

The variance is calculated as follows:

$$\begin{aligned}\mathrm{Var}\,\Delta\hat{y}_2\,(+) = {} & \mathrm{Var}\,\Delta\hat{y}_2\,(-) + k^2\,\mathrm{Var}\,\Delta\underline{y}_1 + k^2\,\mathrm{Var}\,\Delta\hat{y}_1 \\ & + 2k\,\mathrm{Cov}\,\Delta\underline{y}_1\,\Delta\hat{y}_2\,(-) - 2k\,\mathrm{Cov}\,\Delta\hat{y}_1\,\Delta\hat{y}_2\,(-) \\ & - 2k^2\,\mathrm{Cov}\,\Delta\underline{y}_1\,\Delta\hat{y}_1.\end{aligned} \tag{8-18}$$

Assuming no covariance exists between errors in measured and calculated data, then eq (8-18) becomes

$$\mathrm{Var}\,\Delta\hat{y}_2\,(+) = \mathrm{Var}\,\Delta\hat{y}_2\,(-) + k^2\,\mathrm{Var}\,\Delta\underline{y}_1 + k^2\,\mathrm{Var}\,\Delta\hat{y}_1 - 2k\,\mathrm{Cov}\,\Delta\hat{y}_1\,\Delta\hat{y}_2\,(-). \tag{8-19}$$

Since, according to eq (2-10)

$$\mathrm{Var}\,\Delta y_1 = \mathrm{Var}\,\Delta\underline{y}_1 + \mathrm{Var}\,\Delta\hat{y}_1 \tag{8-20}$$

eq (8-19) may be transformed into

$$\mathrm{Var}\,\Delta\hat{y}_2\,(+) = \mathrm{Var}\,\Delta\hat{y}_2\,(-) + k^2\,\mathrm{Var}\,\Delta y_1 - 2k\,\mathrm{Cov}\,\Delta\hat{y}_1\,\Delta\hat{y}_2\,(-) \tag{8-21}$$

Because of eq (8-14) then

$$k\mathrm{Var}\,\Delta y_1 = \mathrm{Cov}\,\Delta y_1\,\Delta y_2\,(-). \tag{8-22}$$

and because of the assumed independence between measurement and calculation errors and also between measurement errors mutually one may write

$$\mathrm{Cov}\,\Delta\hat{y}_1\,\Delta\hat{y}_2\,(-) = \mathrm{Cov}\,\Delta y_1\,\Delta y_2\,(-). \tag{8-23}$$

Substitution of this into eq (8-21) yields

$$\mathrm{Var}\,\Delta\hat{y}_2\,(+) = \mathrm{Var}\,\Delta\hat{y}_2\,(-) - k\,\mathrm{Cov}\,\Delta y_1\,\Delta y_2\,(-) \tag{8-24}$$

Eq (8-24) shows that k is also a correction coefficient for the variance. Because of eq (8-14)

$$k \text{ Cov } \Delta y_1 \Delta y_2 (-) = \frac{\{\text{Cov } \Delta y_1 \Delta y_2 (-)\}^2}{\text{Var } \Delta y_1}, \quad (8\text{-}25)$$

which is always positive. Therefore the correction, given in eq (8-24), always implies a reduction of the variance of $\Delta\hat{y}_2$ (-), and thus an improvement in accuracy.

8.3 Principles of the Kalman filter

Linear regression leads to a minimum value of the variance. In the following the k-coefficient will be derived more directly than in the preceding section, but again aiming at a minimum of Var $\Delta y(+)$. The method is used for the description of the Kalman filter technique (Gelb, 1974). Applications of the Kalman filter to hydrology and hydraulics are described by Chao-Lin Chiu (1978) and most recently by Heemink (1986) and van Geer (1987).

Starting with eq (8-5); which gives the correction procedure:

$$\hat{y}_2 (+) = \hat{y}_2 (-) + k (y_1 - \hat{y}_1 - \bar{\Delta}\bar{y}_1) + \bar{\Delta}\bar{y}_2,$$

a similar expression for the differences is derived from eq (8-17):

$$\Delta\hat{y}_2 (+) = \Delta\hat{y}_2 (-) + k (\Delta y_1 - \Delta\hat{y}_1 - \bar{\Delta}\bar{y}_1) + \bar{\Delta}\bar{y}_2$$

Eq (8-10) reads

$$\Delta\hat{y}_1 = A \Delta \underline{x}_1 + B \Delta \underline{x}_2 + r_{y1} \quad (8\text{-}26)$$

whereas also

$$\Delta\hat{y}_2 = C \Delta \underline{x}_1 + D \Delta \underline{x}_2 + r_{y2} \quad (8\text{-}27)$$

Substitution of eqs (8-26) and (8-27) into (8-17) gives

$$\begin{aligned} \Delta\hat{y}_2 (+) &= C \Delta \underline{x}_1 + D \Delta \underline{x}_2 + r_{y2} + k (\Delta \underline{y}_1 - A \Delta \underline{x}_1 - B \Delta \underline{x}_2 - r_{y1} - \bar{\Delta}\bar{y}_1) + \bar{\Delta}\bar{y}_2 \\ &= (C\text{-}kA) \Delta \underline{x}_1 + (D\text{-}kB) \Delta \underline{x}_2 + r_{y2} - kr_{y1} - k(\Delta \underline{y}_1 - \bar{\Delta}\bar{y}_1) + \bar{\Delta}\bar{y}_2 \quad (8\text{-}28) \end{aligned}$$

Now calculate the variance, taking into account eq (2-7), gives

$$\begin{aligned} \text{Var } \Delta\hat{y}_2 (+) = {}& (C\text{-}kA)^2 \varepsilon_{x1}^2 + (D\text{-}kB)^2 \varepsilon_{x2}^2 + \text{Var } r_{y2} - 2k \text{ Cov } r_{y1} r_{y2} + k^2 \text{ Var } r_{y1} \\ & + k^2 \varepsilon_{y1}^2. \quad (8\text{-}29) \end{aligned}$$

This variance is to be minimized to obtain k. Differentation with respect to k yields

$$\frac{d\,\mathrm{Var}\,\Delta\hat{y}_2\,(+)}{dk} = -2A\,(C\text{-}kA)\,\varepsilon_{x1}^{\;2} - 2B(D\text{-}kB)\,\varepsilon_{x2}^{\;2} - 2\,\mathrm{Cov}\,r_{y1}\,r_{y2} + 2k\mathrm{Var}\,r_{y1} + 2k\,\varepsilon_{y1}^{\;2}, \tag{8-30}$$

This becomes 0 for

$$k = \frac{AC\,\varepsilon_{x1}^{\;2} + BD\,\varepsilon_{x2}^{\;2} + \mathrm{Cov}\,r_{y1}\,r_{y2}}{A^2\,\varepsilon_{x1}^{\;2} + B^2\,\varepsilon_{x2}^{\;2} + \mathrm{Var}\,r_{y1} + \varepsilon_{y1}^{\;2}}\,.$$

This k-value, according to the description of the Kalman filter technique appears to be the same as that found by linear regression of the differences Δy, which was given by eq (8-16).

It is also possible to derive a correction coefficient for a new estimator of $\Delta\hat{y}_1(+)$ itself. Following along similar lines as above this coefficient is found to be

$$k_1 = \frac{A^2\,\varepsilon_{x1}^{\;2} + B^2\,\varepsilon_{x2}^{\;2} + \mathrm{Var}\,r_{y1}}{A^2\,\varepsilon_{x1}^{\;2} + B^2\,\varepsilon_{x2}^{\;2} + \mathrm{Var}\,r_{y1} + \varepsilon_{y1}^{\;2}} \tag{8-31}$$

Note that, because of the first equation of (8-15):

$$k_1 = \frac{\mathrm{Var}\,\Delta y_1 - \varepsilon_{y1}^{\;2}}{\mathrm{Var}\,\Delta y_1} = 1 - \frac{\varepsilon_{y1}^{\;2}}{\mathrm{Var}\,\Delta y_1}\,. \tag{8-32}$$

This is in fact the correlation coefficient of the variable Δy_1 with itself. In this connection is referred for instance to eq (5-155).

The equations for the corrected values $\hat{y}_1(+)$ and $\hat{y}(+)$ can be combined as follows,

$$\hat{y}_1(+) = \hat{y}_1(-) + k_1(y_1 - \hat{y}_1(-) - \bar{\Delta}\bar{y}_1) + \bar{\Delta}\bar{y}_1 \tag{8-33}$$

$$\hat{y}_2(+) = \hat{y}_2(-) + k_2(y_1 - \hat{y}_1(-) - \bar{\Delta}\bar{y}_1) + \bar{\Delta}\bar{y}_2$$

where k_2 denotes the correction coefficient for $\hat{y}_2\,(+)$, i.e. k. The two correction coefficients k_1 and k_2 can be combined to a matrix

$$K = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \tag{8-34}$$

which is called the Kalman gain matrix.

8.4 Application to semi constant parameters

It is possible to consider model parameters as stochastic variables in such a way that these are corrected in the same way as conventional variables like water levels.

If the model reads for instance

$$\hat{y}_1 = \underline{x} + \underline{X} \tag{8-35}$$

$$\hat{y}_2 = \qquad \underline{X}$$

then, when compared with eq (8-7), it follows that

$$A = 1$$
$$B = 1$$
$$C = 0$$
$$D = 1$$

and $x_1 = x$
$x_2 = X$.

If, later on $\underline{y}_1$ is measured, but y_2 is not, then under the assumption, made for reasons of simplicity, that there is no correlation between possible model errors of y_1 and y_2 (Cov r_{y1} r_{y2} = 0), the correction coefficients k_1 and k_2 = k (eqs (8-31) and (8-16) respectively) become:

$$k_1 = \frac{\varepsilon_x^2 + \varepsilon_X^2 + \mathrm{Var}\ r_{y1}}{\varepsilon_x^2 + \varepsilon_X^2 + \mathrm{Var}\ r_{y1} + \varepsilon_{y1}^2} \tag{8-36}$$

and

$$k_2 = \frac{\varepsilon_X^2}{\varepsilon_x^2 + \varepsilon_X^2 + \mathrm{Var}\ r_{y1} + \varepsilon_{y1}^2} \tag{8-37}$$

The corrected values of $\hat{y}_1(+)$ and $\hat{y}_2(+)$ can then be derived according to eqs (8-33).

Example

A water level is measured with respect to a primary levelling mark 1 and, immediately thereafter, also to a temporary levelling mark 2. See fig. 8-1.

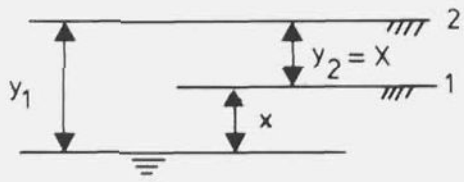


Fig. 8-1 Water level measurements with respect to two marks

Then:

x = the water level with respect to mark 1
$X = y_2$ = the difference between mark 2 and mark 1
y_1 = the water level with respect to mark 2.

Now let, for instance, the standard errors of measurement be

$$\varepsilon_X = 1 \text{ cm}$$
$$\varepsilon_x = 2{,}5 \text{ cm}$$
$$\varepsilon_{y1} = 2{,}5 \text{ cm}$$
$$\text{Var } r_{y1} = 1 \text{ m}^2,$$

If $\underline{y}_2 = \underline{X}$ was assessed at 48 cm and if the first measurement has produced $\underline{x} = 47$ cm and the additional measurement $\underline{y}_1 = 100$ cm, then according to eqs (8-36) and (8-37) it follows that

$$k_1 = \frac{(2{,}5)^2 + 1^2 + 1}{(2{,}5)^2 + 1^2 + 1 + (2{,}5)^2} = 0{,}569 \text{ and}$$

$$k_2 = \frac{1}{(2{,}5)^2 + 1^2 + 1 + (2{,}5)^2} = 0{,}069$$

Now eq (8-33) gives, if no systematic errors are assumed to occur

$$\hat{y}_1(+) = (47 + 48) + 0{,}569\,\{100 - (47 + 48)\} = 97{,}845 \text{ cm}$$
$$\hat{y}_2(+) = 48 + 0{,}069\,\{100 - (47 + 48)\} = 48{,}345 \text{ cm}$$

Thus the difference between the two levelling marks is now corrected to another value.

In the same way as in this simple example, model parameters can be treated like stochastic variables and can be adjusted after each measurement.

8.5 Non linear relations

If, instead of eq (8-7) the relations are not linear, but are of the general form

$$\left.\begin{aligned} \hat{y}_1 &= f(\underline{x}_1, \underline{x}_2) \\ \hat{y}_2 &= g(\underline{x}_1, \underline{x}_2) \end{aligned}\right\} \tag{8-38}$$

then a procedure can be followed in which eq (8-38) is linearized.

First substitute into eq (8-38) the following

$$\underline{x}_1 = x_{1t} + \Delta\underline{x}_1$$
$$\underline{x}_2 = x_{2t} + \Delta\underline{x}_2$$

Focussing on the first equation of eq (8-38) this yields:

$$\hat{y}_1 = f(x_{1t} + \Delta\underline{x}_1, x_{2t} + \Delta\underline{x}_2). \tag{8-39}$$

Because of eq (2-12) it holds also that

$$y_{1t} = f(x_{1t}, x_{2t}) - r_{y1} \tag{8-40}$$

so

$$\Delta\hat{y}_1 = \hat{y}_1 - y_{1t} = f(x_{1t} + \Delta\underline{x}_1, x_{2t} + \Delta\underline{x}_2) - f(x_{1t}, x_{2t}) + r_{y1} \tag{8-41}$$

Now defining

$$\Delta f = f(x_{1t} + \Delta\underline{x}_1, x_{2t} + \Delta\underline{x}_2) - f(x_{1t}, x_{2t}) \tag{8-42}$$

this may be approximated by

$$\Delta f = \frac{\partial f}{\partial x_{1t}} \Delta\underline{x}_1 + \frac{\partial f}{\partial x_{2t}} \Delta\underline{x}_2$$

$$= \frac{\partial f}{\partial \underline{x}_1} \cdot \frac{\partial \underline{x}_1}{\partial x_{1t}} \cdot \Delta\underline{x}_1 + \frac{\partial f}{\partial \underline{x}_2} \cdot \frac{\partial \underline{x}_2}{\partial x_{2t}} \Delta\underline{x}_2. \tag{8-43}$$

Since because of eq (2-4)

$$\underline{x} = x_t + \Delta\underline{x}$$

and assuming independence between x_t and $\Delta\underline{x}$

then

$$\frac{\partial \underline{x}}{\partial x_t} = 1 + \frac{\partial \Delta\underline{x}}{\partial x_t} = 1. \tag{8-44}$$

Applying eq (8-44) to eq (8-43) it follows that

$$\Delta f = \frac{\partial f}{\partial \underline{x}_1} \Delta\underline{x}_1 + \frac{\partial f}{\partial \underline{x}_2} \Delta\underline{x}_2. \tag{8-45}$$

Substitution into eq (8-41), and using eq (8-42), yields:

$$\Delta\hat{y}_1 = \frac{\partial f}{\partial \underline{x}_1} \Delta\underline{x}_1 + \frac{\partial f}{\partial \underline{x}_2} \Delta\underline{x}_2 + r_{y1}, \tag{8-46}$$

whereas, because of eq (2-9)

$$\Delta y_1 = - \frac{\partial f}{\partial \underline{x}_1} \Delta\underline{x}_1 - \frac{\partial f}{\partial \underline{x}_2} \Delta\underline{x}_2 - r_{y1} + \Delta\underline{y}_1. \tag{8-47}$$

Its variance is

$$\text{Var } \Delta y_1 = \left(\frac{\partial f}{\partial \underline{x}_1}\right)^2 \text{Var } \Delta\underline{x}_1 + \left(\frac{\partial f}{\partial \underline{x}_2}\right)^2 \text{Var } \Delta x_2 + \text{Var } r_{y1} + \varepsilon_{y1}^2 \tag{8-48}$$

Similar equations can be found for Var Δy_2 and for Cov $\Delta y_1 \Delta y_2$. These follow also by replacing in eqs (8-15) the coefficients A, B, C and D by the relevant partial derivatives.

If the differences $\Delta\underline{x}_1$, $\Delta\underline{x}_2$ and $\Delta\underline{y}_2$ are small enough this linearisation approximation can be accurate. It should be noted however that the coefficients are changing continuously, and are determined for each new calculation step making the linearisation a local one.

8.6 The use of matrix notations

Eq (8-33) can be written in matrix notation as follows:

$$\begin{bmatrix}\hat{y}_1(+)\\ \hat{y}_2(+)\end{bmatrix} = \begin{bmatrix}\hat{y}_1(-)\\ \hat{y}_2(-)\end{bmatrix} + \begin{bmatrix}k_1\\ k_2\end{bmatrix}[\underline{y}_1 - \hat{y}_1(-) - \bar{\Delta}\bar{y}_1] + \begin{bmatrix}\bar{\Delta}\bar{y}_1\\ \bar{\Delta}\bar{y}_2\end{bmatrix}. \qquad (8\text{-}49)$$

This is the case if only measurements of $\underline{y}_1$ are available, but not of y_2, but where $\hat{y}_2$ is calculated as well. Exactly which variables are measured and which are not is specified through the so called measurement matrix H, which defines the measurement proces as

$$\underline{y}_1 = H\begin{bmatrix}y_{1t}\\ y_{2t}\end{bmatrix} + \Delta\underline{y}_1. \qquad (8\text{-}50)$$

In the present case

$$H = \begin{bmatrix}1 & 0\end{bmatrix}. \qquad (8\text{-}51)$$

If there are n variables, with m variables measured, H is of the order m×n, with elements 1 on the positions corresponding with the measumed values and elements 0 elsewhere.

Now eq (8-49) can be written like:

$$\begin{bmatrix}\hat{y}_1(+)\\ \hat{y}_2(+)\end{bmatrix} = \begin{bmatrix}\hat{y}_1(-)\\ \hat{y}_2(-)\end{bmatrix} + \begin{bmatrix}k_1\\ k_2\end{bmatrix}\left\{\underline{y}_1 - H\left(\begin{bmatrix}\hat{y}_1(-)\\ \hat{y}_2(-)\end{bmatrix} + \begin{bmatrix}\bar{\Delta}\bar{y}_1\\ \bar{\Delta}\bar{y}_2\end{bmatrix}\right)\right\} + \begin{bmatrix}\bar{\Delta}\bar{y}_1\\ \bar{\Delta}\bar{y}_2\end{bmatrix}. \qquad (8\text{-}52)$$

In generalized form this can be written as

$$\hat{Y}(+) = \hat{Y}(-) + K\,\{\underline{Y} - H(\hat{Y}(-) + \bar{\Delta}\bar{Y})\} + \bar{\Delta}\bar{Y} \qquad (8\text{-}53)$$

For the linear equations (8-7) one may write:

$$\begin{bmatrix}\hat{y}_1(-)\\ \hat{y}_2(-)\end{bmatrix} = \begin{bmatrix}A & B\\ C & D\end{bmatrix}\cdot\begin{bmatrix}\underline{x}_1\\ \underline{x}_2\end{bmatrix}, \qquad (8\text{-}54)$$

or in generalized form

$$\hat{Y}(-) = \Phi\,\underline{X}. \qquad (8\text{-}55)$$

The matrix Φ is called the transformation matrix, which transforms $\underline{X}$ into $\hat{Y}(-)$. If the relation is not linear the matrix Φ may contain partial derivatives of $\hat{y}(-)$ with respect to the elements of $\underline{X}$.

The determination of the correction matrix K, generally known as the 'Kalman gain matrix' can be carried out by minimizing the variances of the errors with respect to the elements of matrix $\hat{Y}(+)$. This can be done as follows. Eq (8-53) is transformed by subtraction of the true values from the calculated and measured to

$$\Delta\hat{Y}(+) = \Delta\hat{Y}(-) + K\,[\Delta Y - H\,(\Delta\hat{Y}\,(-) + \bar{\Delta}\bar{Y})] + \bar{\Delta}\bar{Y}. \tag{8-56}$$

Further eq (8-55) is transformed into

$$\Delta\hat{Y}(-) = \Phi\Delta\underline{X} + r \tag{8-57}$$

which is equivalent to eq (8-10). Here r denotes the model error matrix, which is in this case:

$$r = \begin{bmatrix} r_{y1} \\ r_{y2} \end{bmatrix} \tag{8-58}$$

Substitution of eq (8-57) into eq (8-56) gives

$$\begin{aligned}\Delta\hat{Y}(+) &= \Phi\Delta\,\underline{X} + r + K\,[\Delta Y - H\,(\Phi\Delta\underline{X} + r + \bar{\Delta}\bar{Y})] + \bar{\Delta}\bar{Y} \\ &= (I - KH)\,(\Phi\Delta\underline{X} + r) + K\,(\Delta\underline{Y} - H\,\bar{\Delta}\bar{Y}) + \bar{\Delta}\bar{Y}.\end{aligned} \tag{8-59}$$

From this the covariance matrix of $\Delta\hat{Y}(+)$ can be derived:

$$P\,(+) = (I - KH)\,[\Phi\,E_x\,\Phi^T + R]\,(I - KH)^T + KE_yK^T, \tag{8-60}$$

where:

E_x, E_y = the measurement error variance matrices of $\Delta\underline{x}$ and $\Delta\underline{y}$;
R = the model error covariance matrix.

It is worth commenting on the composition of the measurement error variance matrices E_x and E_y. Assuming that the measurements of various variables are not correlated, it follows that only the measurement variances of the variables themselves occur in these matrices. These variances are located along the diagonal elements, so the matrix E_x for instance will look like:

$$E_x = \begin{bmatrix} \varepsilon_{x1}^2 & 0 \\ 0 & \varepsilon_{x2}^2 \end{bmatrix} \tag{8-61}$$

If there are more variables a similar diagonal matrix structure results.

For the model error variance matrix it is likely that non zero values occur also in the off-diagonal elements.

The covariance matrix of $\Delta\hat{Y}(-)$ follows from eq (8-57) as

$$P(-) = \Phi\, E_x\, \Phi^T + R, \tag{8-62}$$

so that, when substituting eq (8-62) into eq (8-60) it follows that

$$P(+) = (I-KH)P(-)\,(I-KH)^T + KE_yK^T. \tag{8-63}$$

Minimizing the trace (i.e. the sum of the elements of the main diagonal) of P(+) with respect to K yields:

$$\frac{\partial}{\partial K}\text{ trace } P(+) = -2(I-KH)\,P(-)\,H^T + 2KE_y = 0$$

or

$$-P(-)\,H^T + K\,[HP(-)\,H^T + E_y] = 0.$$

Thus

$$K = P(-)H^T[HP(-)H^T+E_y]^{-1}. \tag{8-64}$$

Further (8-63) can be elaborated as follows:

$$\begin{aligned} P(+) &= (I-KH)[P(-) - P(-)H^TK^T] + KE_yK^T \\ &= P(-) - P(-)H^TK^T - KHP(-) + KHP(-)H^TK^T + KE_yK^T \\ &= P(-) - P(-)H^TK^T - KHP(-) + K[HP(-)H^T+E_y]K^T \\ &= P(-) - P(-)H^TK^T - KHP(-) + P(-)H^TK^T \\ &= P(-) - KHP(-). \\ &= P(-)[I-KH] \end{aligned} \tag{8-65}$$

Eq (8-64) is the generalized form of eqs (8-31) and (8-16), eq (8-65) of eq (8-24).

It can be proved that not only the sum of the diagonal elements is minimized, but also each element separately. Introducting a matrix

$$M = [0 \ldots 0\ 1\ 0 \ldots 0] \tag{8-66}$$

and calculating $MP(+)M^T$, then only one diagonal element remains. Now suppose a gain matrix K^1 is chosen to minimize the variance of corresponding variable in the model. Only one line of K^1 is required, namely that, corresponding with the element concerned. This means that the variance of that element is to be minimized with respect to MK^1.

Elaboration of $MP(+)M^T$ using eq (8-63) gives

$$MP(+)M^T = M\,[(I-K^1H)P(-)(I-K^1H)^T+K^1E_yK^{1T}]\,M^T$$

$$= M(I-K^1H)P(-)(I-K^1H)^TM^T+MK^1E_yK^{1T}M^T$$

$$= (M-MK^1H)P(-)(M-MK^1H)^T+(MK^1)E_y(MK^1)^T. \qquad (8\text{-}67)$$

Minimizing to MK^1 yields

$$\frac{\partial}{\partial MK_1} MP(+)M^T = -2(M-MK^1H)P(-)H^T+2(MK^1)E_y = 0$$

or

$$-MP(-)H^T + MK^1H.P(-)H^T+MK^1E_y = 0$$

so that

$$MK^1[HP(-)H^T + E_y] = MP(-)H^T$$

and

$$MK^1 = MP(-)H^T[HP(-)H^T+E_y]^{-1} \qquad (8\text{-}68)$$

which, because of eq (8-64), can be written as

$$MK^1 = MK$$

or

$$K^1 = K. \qquad (8\text{-}69)$$

Thus the gain matrix does not change when minimizing one separate element only. Since this holds for all elements, the Kalman filter will minimize the variances of all elements concerned separately. This is very important because variables having different dimensions and orders of magnitude (water levels, velocities, bottom friction coefficient, bottom levels, flow conveying and storage widths) need not to be standardized to one unit standard error. All variables can be used in the computations with their own dimensions and magnitudes.

8.7 Application to time series

The Kalman filter is usually applied to time series. The variables at a time t_0 are transformed into corresponding variables at a time $t_0 + \Delta T$.*)
The transformations described by eq (8-7) can, for this case, be written like:

$$\hat{y}_1(-)(t_0 + \Delta T) = A\underline{y}_1(t_0) + B\,\underline{y}_2(t_0) \tag{8-70}$$

$$\hat{y}_2(-)(t_0 + \Delta T) = C\underline{y}_1(t_0) + D\,\underline{y}_2(t_0)$$

The values at time t_0 do not need to be measurements but can be estimated values after correction, at time t_0. Then

$$\hat{y}_1(-)(t_0 + \Delta T) = A\hat{y}_1(+)(t_0) + B\,\hat{y}_2(+)(t_0) \tag{8-71}$$

$$\hat{y}_2(-)(t_0 + \Delta T) = C\hat{y}_1(+)(t_0) + D\,\hat{y}_2(+)(t_0),$$

or in matrix notation:

$$\hat{Y}(-)(t_0 + \Delta T) = \Phi\,\hat{Y}(+)(t_0) \tag{8-72}$$

Here $\hat{Y}(-)(t_0 + \Delta T)$ and $\hat{Y}(+)(t_0)$ are called the 'state vectors' at $t_0 + \Delta T$ before correction and at t_0 after correction respectively. The variables of $\hat{Y}(-)(t_0 + \Delta T)$ may also be influenced by external influences denoted by the values $\underline{u}(t_0)$, for instance

$$\hat{y}_1(-)(t_0 + \Delta T) = A\hat{y}_1(+)(t_0) + B\hat{y}_2(+)(t_0) + L\underline{u}(t_0)$$

$$\hat{y}_2(-)(t_0 + \Delta T) = C\hat{y}_1\,(+)\,(t_0) + D\hat{y}_2\,(+)(t_0) + M\underline{u}(t_0), \tag{8-73}$$

or, in generalized form:

$$\hat{Y}(-)(t_0 + \Delta T) = \Phi Y(+)(t_0) + \Lambda U(t_0). \tag{8-74}$$

Here Φ denotes the interal transformation matrix and Λ the external transformation matrix

*) Capital T corresponds to the notations, used in Chapter 7.

In the non linear case, as described in Section 8.6 the transformation matrices may vary in time as well. Then eq (8-73) should be written:

$$\hat{Y}(-)(t_0 + \Delta T) = \Phi(t_0).\ \hat{Y}(+)\ (t_0) + \Lambda(t_0).\ U(t_0). \qquad (8\text{-}75)$$

This is the full transformation formula of $\hat{Y}(+)(t_0)$ into $\hat{Y}(-)(t_0 + \Delta T)$. It should be noted that for many models the transformation is not linear, and eq (8-75) will not be used for the transformation as such since this will be carried out using the original model equations. However knowledge of eq (8-75) is required anyhow for the calculation of the error propagation. In fact eq (8-75) expresses the sensitivity of the result $\hat{Y}(-)(t_0 + \Delta T)$ to each of the input data included in $\hat{Y}(+)(t_0)$ and $\underline{U}(t_0)$. For this reason in the procedure a linearized approximation of the physical model will be derived.

Now let

$P(+)(t_0)$ = the covariance matrix of $\Delta\hat{Y}(+)\ (t_0)$
$P(-)(t_0+\Delta T)$ = the covariance matrix of $\Delta\hat{Y}(-)\ (t_0 + \Delta T)$
$V(t_0)$ = the covariance matrix of $\underline{U}(t_0)$
$R(t_0)$ = the model error covariance matrix

Now, from the form of eq (8-75), it follows that

$$P(-)(t_0 + \Delta T) = \Phi(t_0)P(+)(t_0)\Phi(t_0)^T + \Lambda(t_0)V(t_0)\ \Lambda(t_0)^T + R(t_0). \qquad (8\text{-}76)$$

Further let the measured values at $t_0 + \Delta T$ be given by the vector $\underline{Y}(t_0 + T)$ with a measurement error matrix $E(t_0 + \Delta T)$. The Kalman gain matrix can, similarly to eq (8-64) be expressed by:

$$K(t_0 + \Delta T) = P(-)(t_0 + \Delta T)H^T(t_0 + \Delta T)\ [H(t_0 + \Delta T)\underline{P}(-)(t_0 + \Delta T)$$

$$H^T(t_0 + \Delta T) + E(t_0 + \Delta T)]^{-1} \qquad (8\text{-}77)$$

where: $H(t_0 + \Delta T)$ is the measurement matrix at $t_0 + \Delta T$.

In most cases $E(t_0 + \Delta T)$ and $H(t_0 + \Delta T)$ are invariant in time, so:

$$K(t_0 + \Delta T) = P(-)(t_0 + \Delta T).\ H^T\ [H.P(-)(t_0 + \Delta T).H^T + E]^{-1}. \qquad (8\text{-}78)$$

If there are n internal variables, of which m are measured, and if there are n_e external variables, then the different matrices have the following dimensions.

$\hat{Y}(+)(t_0)$; $\hat{Y}(-)(t_0 + \Delta T)$, $\hat{Y}(+)(t_0 + \Delta T)$	:	$n\times 1$
$\underline{Y}(t_0 + \Delta T)$	:	$m\times 1$
$\underline{U}(t_0)$	:	$n_e\times 1$
$\Phi(t_0)$	:	$n\times n$
$\Lambda(t_0)$	:	$n\times n_e$
$P(+)(t_0)$, $P(-)(t_0 + T)$, $R(t_0)$	:	$n\times n$
$V(t_0)$	:	$n_e\times n_e$
$H(t_0 + \Delta T)$	:	$m\times n$
$E(t_0 + \Delta T)$	:	$m\times m$
$\bar{\Delta}\bar{Y}$	:	$m\times 1$
$K(t_0 + \Delta T)$	:	$n\times m$

The correction of $\hat{Y}(-)(t_0 + \Delta T)$ to $\hat{Y}(+)(t_0 + \Delta T)$ takes place in accordance with eq (8-53), which now reads:

$$\hat{Y}(+)(t_0 + \Delta T) = \hat{Y}(-)(t_0 + \Delta T) + K(t_0 + \Delta T)$$
$$[\underline{Y}(t_0 + \Delta T)\text{-}H\{\hat{Y}(-)(t_0 + \Delta T) + \bar{\Delta}\bar{Y}\}] + \bar{\Delta}\bar{Y} \qquad (8\text{-}79)$$

The expression in brackets following H in eq (8-79), being of the order $m\times 1$, is transformed into a $n\times 1$-matrix after multiplication by the Kalman gain matrix. Thus the m differences, formed between the measured values and the calculated values of the same elements, become transformed into n corrections for all elements. Non measured values get corrections too.

The error variance matrix for the corrected data of *$\hat{Y}(+)(t_0 + \Delta T)$ is also influenced.* It follows from eq (8-65)

$$P(+)(t_0 + \Delta T) = [I\text{-}K(t_0 + \Delta T) \,.\, H(t_0 + \Delta T)] \,.\, P(-)(t_0 + \Delta T) \qquad (8\text{-}80)$$

The procedure of the Kalman filter is be summarized in the flow diagram of Fig. 8-2, which shows also the numbers of the equations, which are used to calculate the values concerned.

At time $t_0 + \Delta T$ the procedure starts again for the next time step.

8.8 Determination of the values of the elements of the matrices concerned.

From the above background description it follows that for the Kalman filter calculations knowledge of the following matrices is required:

- the (internal) transformation matrix : Φ
- the (external) transformation matrix : Λ

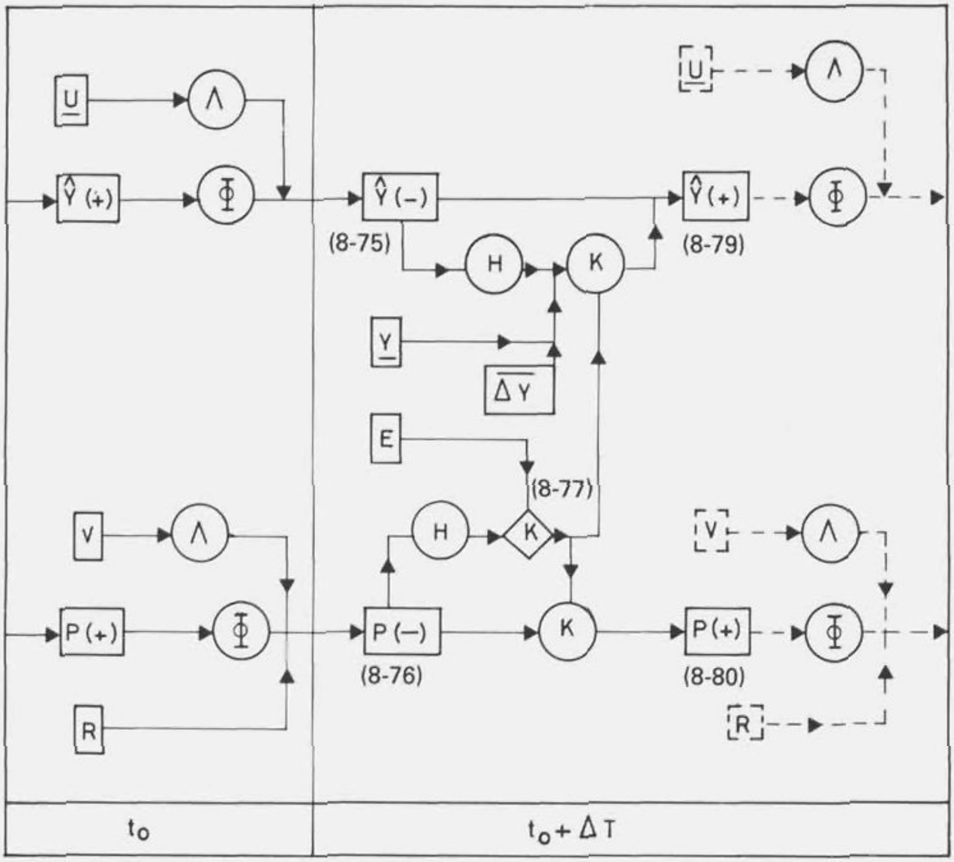


Fig. 8-2 Flow diagram of the Kalman filter

- the measurement matrix : H
- the measurement error variance matrix : E
- the model error variance matrix : R
- the mean difference matrix : $\bar{\Delta}\bar{Y}$
- the error variance matrix of the elements of $\hat{Y}(t_0)(+)$: $P(+)(t_0)$
- the error variance matrix of the elements of $\underline{U}(t_0)$: V.

In the following, first some general considerations will be discussed concerning the assessment of the elements of the various matrices. Then, in Section 8.9 attention will focus on an application to the example of Section 7.2.

The internal transformation matrix Φ

For a linear model the elements of this matrix can easily be found from the equations expressing the relations between the values at t_0 and those at $t_0 + \Delta T$. If, for instance

$$\left.\begin{aligned} \hat{y}_1(-)(t_0 + \Delta T) &= A\hat{y}_1(+)(t_0) + B\hat{y}_2(+)(t_0) \\ \hat{y}_2(-)(t_0 + \Delta T) &= C\hat{y}_1(+)(t_0) + D\hat{y}_2(+)(t_0), \end{aligned}\right\} \tag{8-81}$$

then:

$$\Phi = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{8-82}$$

If this transformation is carried out $\underline{n}$ times, in time steps Δt, for calculating the values at $t_0 + \Delta T$, where $\Delta T = \underline{n}\,\Delta t$, then the transformation matrix will be:

$$\Phi = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{\underline{n}} \tag{8-83}$$

If the model is not linear, no matrix operation will be applied for the calculation of $\hat{Y}(-)(t_0 + \Delta T)$. Instead use will be made of the model equations concerned, and having the general form of eqs (7-22) and (7-23). However, knowledge of the sensitivity of the results to the errors of the input variables (and parameters, if required) is needed in order to find the covariance matrix $P(-)(t_0 + \Delta T)$ required for further calculations. As was discussed in Section 8.5 this problem is tackled by linearization of the transformation and thus composing a transformation matrix Φ. Such a transformation matrix contains the partial derivatives as first order approximations. Note that as a rule the elements of Φ are not constant but change at each timestep. In the case of two variables the transformation in general can be written as

$$\left.\begin{aligned} \hat{y}_1(-)(t_0 + \Delta T) &= f\,\{\hat{y}_1(+)(t_0), \hat{y}_2(+)(t_0)\} \\ \hat{y}_2(-)(t_0 + \Delta T) &= g\,\{\hat{y}_1(+)(t_0), \hat{y}_2(+)(t_0)\}. \end{aligned}\right\} \tag{8-84}$$

The matrix, acting as transformation matrix, now holding for time t_0 only, will be

$$\Phi(t_0) = \begin{bmatrix} \dfrac{\partial f}{\partial \hat{y}_1(+)(t_0)} & \dfrac{\partial f}{\partial \hat{y}_2(+)(t_0)} \\[2ex] \dfrac{\partial g}{\partial \hat{y}_1(+)(t_0)} & \dfrac{\partial g}{\partial \hat{y}_2(+)(t_0)} \end{bmatrix}. \tag{8-85}$$

If a time interval ΔT consists of $\underline{n}$ time steps Δt the transformation matrix would follow from

$$\Phi(t_0) = \prod_{i=0}^{n-1} \Phi^*(t_0 + i\,\Delta t), \tag{8-86}$$

where:

$\Phi^*(t_0 + i\,\Delta t)$ = the transformation matrices over a time step Δt.

The objection to this procedure is that an accumulation of inaccurancies and errors can take place. Therefore the following approach will be used. Consider the transformation

$$\hat{Y}(-)(t_0 + \Delta T) = \Phi(t_0).\,\hat{Y}(+)(t_0) \tag{8-87}$$

but which is calculated using the model equations, as was explained in Section 8.7 in the text following eq (8-75)

Now repeat this calculation of $Y(-)\,(t_0+\Delta T)$, but this time the first variable of $\hat{Y}(+)(t_0)$, $\hat{y}_1(+)(t_0)$, is increased by a small amount $\Delta^* y_1(+)$ [+]). Then namely a new set of variables are found at $t_0 + \Delta T$ as follows:

$$\hat{Y}(-)(t_0 + \Delta T) + \Delta Y^* = \Phi(t_0) \begin{bmatrix} \hat{y}_1(+)(t_0) + \Delta^* y_1(+) \\ \hat{y}_2(+)(t_0) \\ . \\ . \\ . \\ \hat{y}_n(+)(t_0) \end{bmatrix}$$

$$= \Phi(t_0)\hat{Y}(+)(t_0) + \Phi(t_0) \begin{bmatrix} \Delta^* y_1(+) \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}. \tag{8-88}$$

Subtraction of eq (8-87) gives:

$$\Delta^* Y = \begin{bmatrix} \varphi_{11} \\ . \\ . \\ . \\ \varphi_{n1} \end{bmatrix} \Delta^* y_1(+) \tag{8-89}$$

+) The amount $\Delta^* y_1(+)$ should not be confused with $\Delta y_1(+)$ of eq (2-5). The distinction is expressed by an asterisk.

where the elements φ_{ij} (i = 1...n, j = 1...n) are elements of the matrix Φ. Further according to eq (8-89)

$$\begin{bmatrix} \varphi_{11} \\ . \\ . \\ . \\ \varphi_{n1} \end{bmatrix} = \Delta^*Y/\Delta^*y_1(+) \qquad (8\text{-}90)$$

By excecuting this procedure for all j (j =1...n), all columms of $\Phi(t_0)$ can be calculated, so that this matrix can be filled up entirely. The transformation is carried out n additional times, making this a time consuming procedure. It could be speeded up considerably using parallel computing technology.

Note that this transformation matrix of this non linear model only concerns the changes Δ^*Y, but not the whole values Y, thus

$$\Delta^*\hat{Y}(-)(t_0 + \Delta T) = \Phi(t_0)\, \Delta^*Y(+)(t_0). \qquad (8\text{-}91)$$

In the mathematical model of Chapter 7 the following 13 elements (variables and parameters) were considered to be possibly affected by transformations:

- 4 water levels : y_0, y_1, y_2, y_3
- 3 velocities : v_{01}, v_{12}, v_{23}
- bottom friction coefficient : C
- 3 flow conducting widths : $W_{f(01)}, W_{f(12)}, W_{f(23)}$
- 2 storage widths : W_{s1}, W_{s2}

This is in contrast to other publications in which only parameters like C were considered to be variable, but not dimensions, such as widths. The bottom elevations $b_0...b_3$ and the distances d_{01}, d_{12} and d_{23} were considered to be fixed. In the transformation matrix, these (semi)variables are given an element 1 on the diagonal of Φ, symbolically shown by eq (8-92)
The elements of $\Phi(t_0)$, which were used for the calculation of the real variables, located in the rows 2, 3, 5, 6 and 7 were calculated as indicated by eqs (8-88), (8-89) and (8-90). To the elements of $\hat{Y}(+)(t_0)$ the following increments were added in subsequent calculations

water levels	Δ^*y_i	= 0,01	m	(i=0...3)
velocities	$\Delta^*v_{i,i+1}$	= 0,01	m/s	(i=0...2)
flow conveying widths	$\Delta^*W_{f(i,i+1)}$	= 100	m	(i=0...2)
storage widths	Δ^*W_{si}	= 100	m	(i=1, 2)

$$
\Delta^* \begin{bmatrix} \hat{y}_0(-) \\ \hat{y}_1(-) \\ \hat{y}_2(-) \\ \hat{y}_3(-) \\ \hat{v}_{01}(-) \\ \hat{v}_{12}(-) \\ \hat{v}_{23}(-) \\ C \\ W_{f(01)} \\ W_{f(12)} \\ W_{f(23)} \\ W_{s1} \\ W_{s2} \end{bmatrix}_{(t_0+\Delta T)} = \begin{bmatrix} 0 \quad \cdots \quad 0 \\ \text{13 transf.functions for } y_1 \\ \text{13 transf.functions for } y_2 \\ 0 \quad \cdots \quad 0 \\ \text{13 transf.functions for } v_{01} \\ \text{13 transf.functions for } v_{12} \\ \text{13 transf.functions for } v_{23} \\ 1 \\ \quad 1 \\ \quad\quad 1 \\ \quad\quad\quad 1 \\ \quad\quad\quad\quad 1 \\ \quad\quad\quad\quad\quad 1 \end{bmatrix} .\Delta^* \begin{bmatrix} \hat{y}_0(+) \\ \hat{y}_1(+) \\ \hat{y}_2(+) \\ \hat{y}_3(+) \\ \hat{v}_{01}(+) \\ \hat{v}_{12}(+) \\ \hat{v}_{23}(+) \\ C \\ W_{f(01)} \\ W_{f(12)} \\ W_{f(23)} \\ W_{s1} \\ W_{s2} \end{bmatrix}_{(t_0)} \tag{8-92}
$$

The bottom friction coefficient was used in the programme in the form

$$
C^* = 10\,000/C^2 \tag{8-93}
$$

The dimension of C^* is $[t^2 l^{-1}]$ expressed in units $10^{-4}\ s^2m^{-1}$. To C^* an increment was added of $\Delta^*C^* = 0{,}01.\ 10^{-4}\ s^2m^{-1}$.

The above increments led to small changes in the calculated values of $y_i(-)(t_0 + \Delta T)$, $(i=1;2)$ and of $v_{i,i+1}(-)(t_0 + \Delta T)$, $(i=0...2)$. These changes, divided by the increments concerned, produced the relevant elements of $\Phi(t_0)$.

The boundary values $y_0(-)(t_0 + \Delta T)$ and $y_3(-)(t_0 + \Delta T)$ were considered not to be influenced by the internal transformation, but to act as external influences. For this reason the 1st and 4th row of $\Phi(t_0)$ are given zero values.

The external transformation matrix Λ

For this matrix a similar procedure can be carried out as for the internal transformation matrix Φ. This time the external variables, included in $\underline{U}(t_0)$ should be varied succesively. For the example considered the external transformation matrix was set up as follows.

$$
\Delta^* \begin{bmatrix} \hat{y}_0(-) \\ \hat{y}_1(-) \\ \hat{y}_2(-) \\ \hat{y}_3(-) \\ \hat{v}_{01}(-) \\ \hat{v}_{12}(-) \\ \hat{v}_{23}(-) \\ C \\ W_{f(01)} \\ W_{f(12)} \\ W_{f(23)} \\ W_{s1} \\ W_{s2} \end{bmatrix}_{(t_0+\Delta T)} = \begin{bmatrix} 1 & \ldots & 0 \\ 2 \text{ transf.functions for} & & y_1 \\ 2 \text{ transf.functions for} & & y_2 \\ 0 & \ldots & 1 \\ 2 \text{ transf.functions for} & & v_{01} \\ 2 \text{ transf.functions for} & & v_{12} \\ 2 \text{ transf.functions for} & & v_{23} \\ 0 & \ldots & 0 \\ 0 & \ldots & 0 \\ 0 & \ldots & 0 \\ 0 & \ldots & 0 \\ 0 & \ldots & 0 \\ 0 & \ldots & 0 \end{bmatrix} . \Delta^* \begin{bmatrix} y_0 \\ \\ y_3 \end{bmatrix}_{(t_0+\Delta T)} \tag{8-94}
$$

which is equivalent to

$$
\Delta^* \; \hat{Y}(-)(t_0 + \Delta T) = \Lambda\,(t_0).\; \underline{U}(t_0) \tag{8-95}
$$

for the external contribution to $\hat{Y}(-)(t_0 + \Delta T)$

The measurement matrix H

This follows readily from the selection of the m variables to the measured, from the n variables acting in the model.

In the example H is a (2×13) matrix

$$
H = \begin{bmatrix} 0\,1\,0\,0 & | & 0 \ldots 0 \\ 0\,0\,1\,0 & | & 0 \ldots 0 \end{bmatrix}. \tag{8-96}
$$

If only one of the levels y_1 or y_2 is measured H transforms into a simple one row matrix with a 1 at the position related to the element concerned.

The measurement error variance matrix E

As was remarked earlier (see eq (8-61)), this is a matrix, having diagonal elements made up of the variances of measurement of the m measured variables. The other matrix elements are 0, under the assumption that the measurement errors are uncorrelated.

For the present example this matrix only concerns the measurements errors of y_1 and y_2, since control measurements were carried out only for these two variables. Since the standard error of measurement was determined at 2,5 cm the variance is

$$\text{Var}\ \Delta y = 0{,}000625\ \text{m}^2$$

the matrix E is therefore

$$E = \begin{bmatrix} \text{Var}\ \Delta y_1 & 0 \\ 0 & \text{Var}\ \Delta y_2 \end{bmatrix} = \begin{bmatrix} 0{,}000625 & 0 \\ 0 & 0{,}000625 \end{bmatrix}. \tag{8-97}$$

The model error variance matrix R

This matrix should include the variances of the model errors, with the propagated measurement errors of the input data excluded. The variance matrix relates to all variables, and also to some of the parameters, e.g. roughness coefficient. An idea of the errors associated with the m measured variables, can be obtained by running the model without Kalman filter and comparing the calculated values $\hat{y}_i$ with the measured levels y_i ($i = 1,2$). The resulting variances and covariances of $\Delta y_i = y_i - \hat{y}_i$ give a first idea of what values should be assigned to the elements for Var Δy_1, Var Δy_2 and Cov Δy_1 Δy_2 in R. For the other variables and parameters the variances and covariances should be assessed, taking into account their accuracy.

Following the run without the Kalman filter a run should be carried out with the Kalman filter. This will produce in the matrix P(-) series of data for Var $\Delta\hat{y}_1(-)$, Var $\Delta\hat{y}_2(-)$ and Cov $\Delta y_1(-)$ $\Delta y_2(-)$. These data should be compared with corresponding data found by comparison of measured and calculated data. If the do not coincide satisfactory another run with Kalman filter should be carried out with modified elements of R. This is a trial and error procedure, to be continued until a satisfactory result is found.

If only some of the parameters are to be adjusted then a variance of 0 should be assigned to those elements which are considered to be fixed. For instance, if adjustments are to be made to the bottom friction coefficient and the flow conveying widths only, then the storage widths should be given zero variance.

In the example of the Western Scheldt Estuary (see Section 8.10) the following values were used for the first Kalman filter run.

$$\text{Var}\ \Delta\hat{y}_1 = \text{Var}\ \Delta\hat{y}_2 = 0{,}01\ \text{m}^2$$
$$\text{Var}\ \Delta\hat{v}_{01} = \text{Var}\ \Delta\hat{v}_{12} = \text{Var}\ \Delta\hat{v}_{23} = 0{,}01\ \text{m}^2/\text{s}^2$$

For the model parameters, which were not to be fixed the following values were applied:

$\text{Var } C^* \quad = 0{,}1 \ (m/s^2)^{-2}$
$\text{Var } W_f \quad = 5000 \ m^2$
$\text{Var } W_s \quad = 5000 \ m^2$

Between the water levels and between adjacent velocities a correlation coefficient 0,5 was used.

The above data specification led to the following matrix R in its most completed implementation.

$$R = \begin{bmatrix} 0 & & & & & & & & & \\ & \begin{bmatrix} 0{,}01 & 0{,}005 \\ 0{,}005 & 0{,}01 \end{bmatrix} & & & & & & \bigcirc & & \\ & & 0 & & & & & & & \\ & & & \begin{bmatrix} 0{,}01 & 0{,}005 & 0{,}0025 \\ 0{,}005 & 0{,}01 & 0{,}005 \\ 0{,}0025 & 0{,}005 & 0{,}01 \end{bmatrix} & & & & & & \\ & & & & 0{,}1^{*)} & & & & & \\ & & & & & 5000^{**)} & & & & \\ & & & & & & 5000^{**)} & & & \\ & \bigcirc & & & & & & 5000^{**)} & & \\ & & & & & & & & 5000^{**)} & \\ & & & & & & & & & 5000^{**)} \end{bmatrix} \tag{8-98}$$

*) In case 1 this value was set at zero (see S. 8.10.1).
**) In cases 1 and 2 these values were set at zero (see S.8.10.1).

The mean difference matrix $\bar{\Delta}\bar{Y}$

This is a one column matrix with possibly non zero values at the locations corresponding to the measured variables. The values are the mean differences between calculated and measured values. For the example considered these were $\bar{\Delta}\bar{y}_1$ and $\bar{\Delta}\bar{y}_2$. So the column matrix was as follows

$$\bar{\Delta}\bar{Y} = \left[\begin{array}{c} 0 \\ \bar{\Delta}\bar{y}_1 \\ \bar{\Delta}\bar{y}_2 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right] \qquad (8\text{-}99)$$

Because of the results, given in Chapter 7, the following values were assigned for the first run

$\bar{\Delta}\bar{y}_1 = 6{,}326$ cm
$\bar{\Delta}\bar{y}_2 = 1{,}643$ cm

The error variance matrix $P(+)(t_o)$ of the elements of $Y(t_o)$

This matrix is only required when starting the calculations for the first time step. For further calculations it is automatically derived from the preceding result, according to eq (8-80).

For the initial matrix $P(+)(t_0)$ the variances of measurement can be used, this time also including those for the external variables, y_0 and y_3. The variance of the model parameters, which are included in the Kalman filter procedure, should be estimated in the light of earlier measurements, e.g. of water levels, velocities, bottom profile soundings, mapping surveys etc. As a rule, only diagonal elements should be assigned a non zero value.

In the example of the Western Scheldt estuary (Section 8.10.1) the following values were assigned, where appropriate, to the diagonal elements of $P(+)(t_0)$

Var $\Delta\hat{y}_i$	$= 0{,}000625$	m^2	$(i = 0 \ldots 3)$
Var $\Delta\hat{v}_{i,i+1}$	$= 0{,}01$	m^2/s^2	$(i = 0 \ldots 2)$
Var C^*	$= 0{,}5$	$(m/s^2)^{-2}$	
Var $W_{f(i,i+1)}$	$= 9.10^4$	m^2	$(i = 0 \ldots 2)$
Var W_{si}	$= 9.10^4$	m^2	$(i = 1, 2)$

The error variance matrix V of the elements of $\underline{U}$ (t_0)

This matrix also needs to be assessed only once but it is used repeatedly at each timestep to introduce the errors of measurement of the boundary levels $\underline{y}_0$ and $\underline{y}_3$.

It looks like the matrix E having the form

$$\underline{U}(t_0) = \begin{bmatrix} \text{Var } \Delta\underline{y}_0 & 0 \\ 0 & \text{Var } \Delta\underline{y}_3 \end{bmatrix} = \begin{bmatrix} 0{,}000625 & 0 \\ 0 & 0{,}000625 \end{bmatrix}. \tag{8-100}$$

This matrix acts in eq (8-73).

Having assessed all the relevant matrices the calculations and its results can be considered.

8.9. Application to a hypothetical case

The technique deseribed above was applied to the hypothetical example, presented in Section 7.2.9. This example concerned a channel of uniform cross sections along the whole length. Sine waves acted as boundary conditions at both ends. The calculations produced water level and velocity data at intermediate sites. After a short starting period the situation had become stabilized in such a way that a certain, periodically recurring pattern of the water levels and velocity was formed having the same period as the external sine waves.

For this application the data of the recurring pattern were used as if they were measured input data for the intermediate sites. The water level and velocity data, given for t = 12 h in Tables 7-5 and 7-6, at which time the situation had become stabilized, were used as initial conditions.

The dimensions and other parameters of the model were as follows:

bottom elevations	: $b_0 = b = b_2 = b_3$	=	-20 m
flow conveying widths	: $W_{f(01)} = W_{f(12)} = W_{f(13)}$	=	4000 m

storage widths	: $W_{s1} = W_{s2}$	=	4000 m
reaches lengths	: $d_{01} = d_{12} = d_{23}$	=	16800 m
roughness coefficient	: C	=	50 m $^{1}/_{2}$/s
measurement interval	: ΔT	=	1 h
time step	: Δt	=	10 min
number of time steps	: n	=	6.

In this application a number of experiments was carried out. Some of the parameters were fixed, whereas others were considered to have a certain variability, expressed as a standard error or a variance. If one or more of the parameters were given a value, different from that in the earlier model which the input data had produced, this would lead to calculated water levels and velocities other than the given input data of these variables. The model, when adjusted to the input data, would be forced to change the parameters to values, better corresponding with those of the original model.

To examine this a number of experiments were carried out. The results obtained, indicated in Table 8-1 are described in the following.

experiment nr.	parameter C m $^{1}/_{2}$/s	$W_{f(01)}$ m	$W_{f(12)}$ m	$W_{f(23)}$ m	W_{s1} m	W_{s2} m
1	(60)	4000	4000	4000	4000	4000
2	(40)	4000	4000	4000	4000	4000
3	50	4000	(3000)	(5000)	4000	(5000)
4	(40)	4000	(3000)	(5000)	4000	(5000)

Table 8-1 Kalman filter experiments in a hypothetical case.

Those parameters not figuring in Table 8-1 were fixed in all experiments. Of the values in the table, those given in brackets were attached with noise and were given values,different from the original model values.

In the experiments where parameters were considered as variables, noise values (variances) were assigned, as given in Table 8-2.

variable; parameter	initial noise (matrix $P(+)(t_0)$)	model noise (matrix R; V)
all $\Delta\hat{y}$	0,0001 m^2	0,0001 m^2
all $\Delta\hat{v}$	0,0001 $(m/s)^2$	0,0001 $(m/s)^2$
$C^* = 10^4/C^2$	0,5 $(m/s^2)^{-2}$	0,1$(m/s^2)^{-2}$
W_f	9.10^4 m^2	5.10^3 m^2
W_s	25.10^4 m^2	5.10^3 m^2

Table 8-2 Noise values used in Kalman filter experiments for a hypothetical case.

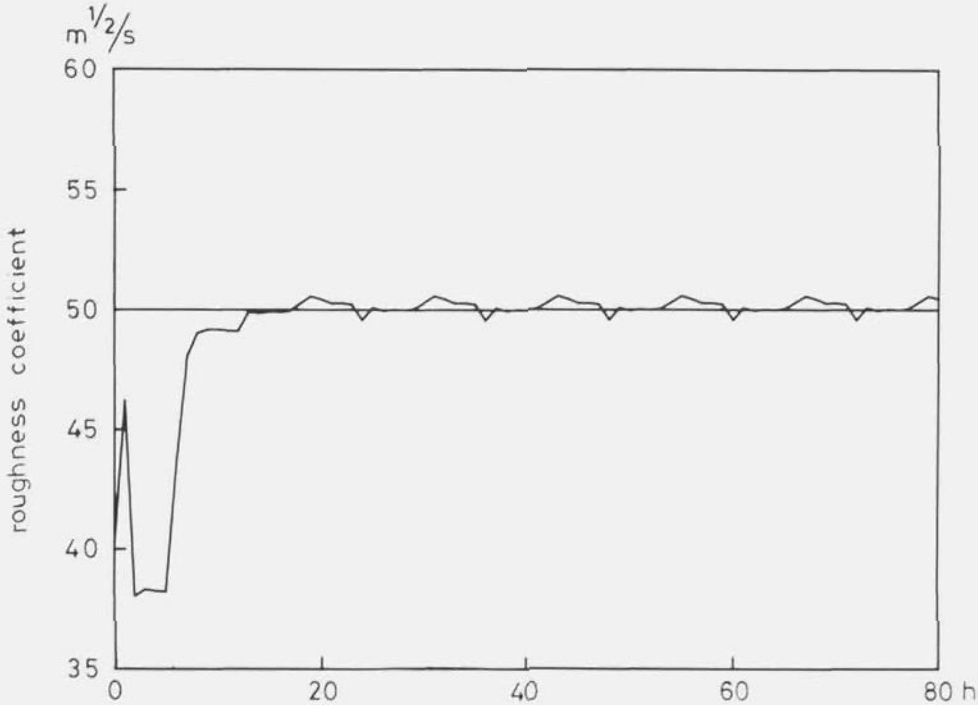


Fig. 8-3 Convergence of the roughness coefficient, starting with C = 40 m $^{1/2}$/s (hypothetical case)

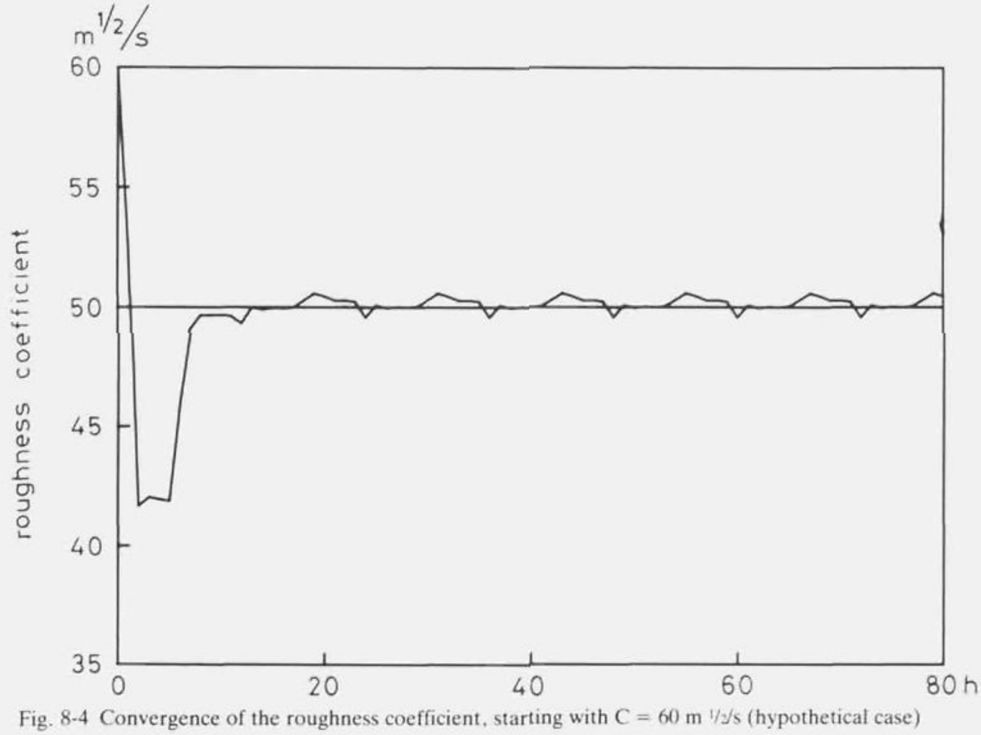
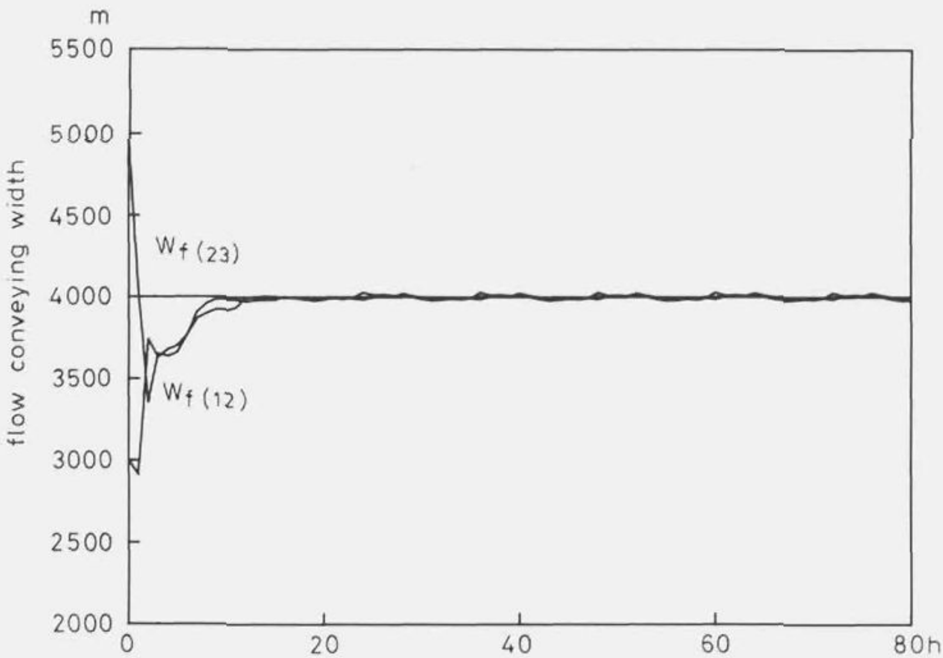


Fig. 8-4 Convergence of the roughness coefficient, starting with C = 60 m $^{1/2}$/s (hypothetical case)



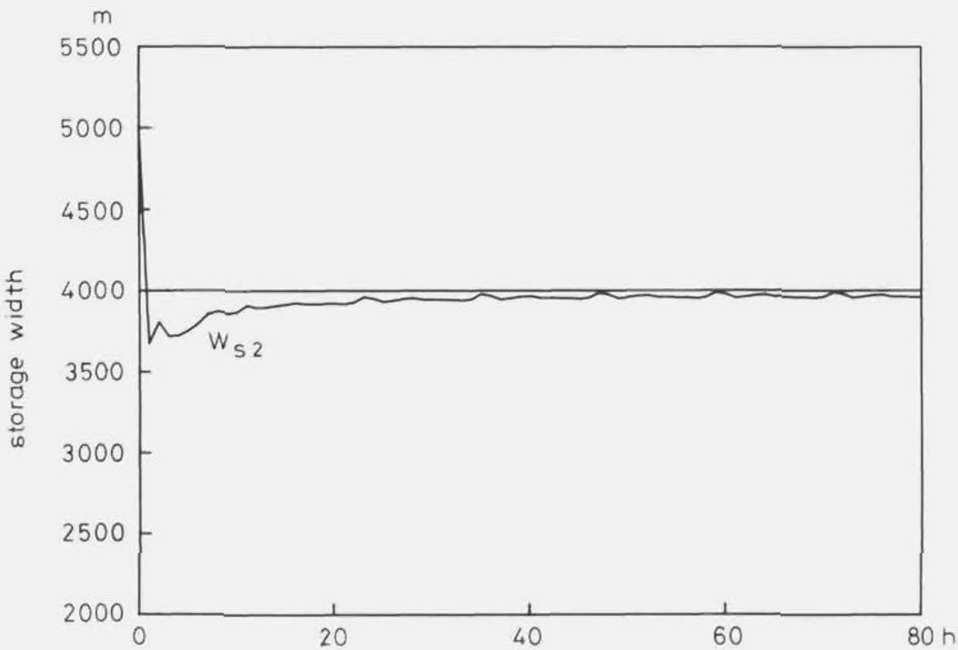


Fig. 8-5 Convergence of the flow conveying widths and the storage widths (hypothetical case)

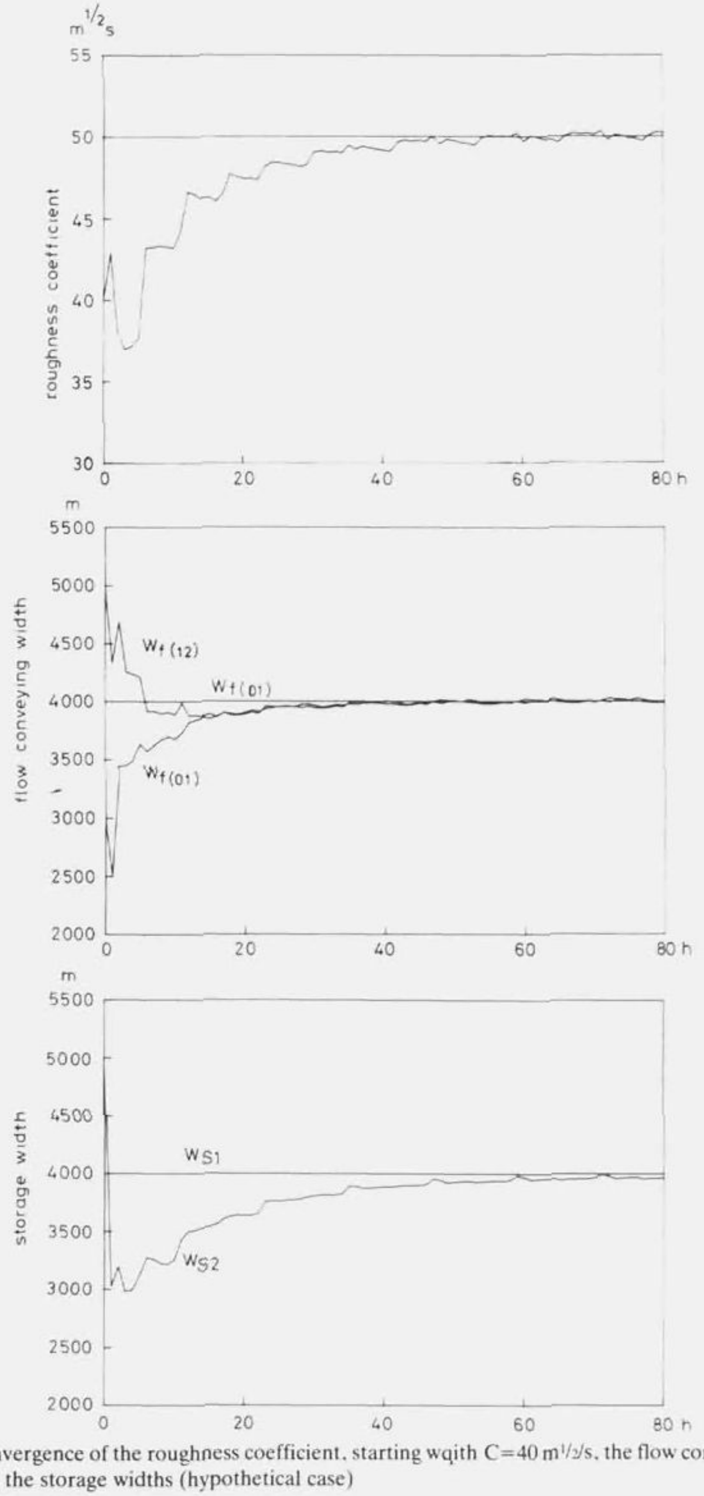


Fig. 8-6 Convergence of the roughness coefficient, starting wqith $C = 40\ m^{1/2}/s$, the flow conveying widths and the storage widths (hypothetical case)

The behaviour of the variable parameters is shown in Figs 8-3 ... 10. They show some strong variations in the starting period, followed by a gradual transition to the original model values.

In the experiments 1 and 2, where only the C-value is variable, the adjustment is rather quick, i.e. less than 10 h (Figs 8-3 and 8-4). In experiment 3 (Fig. 8-5), with two variable flow conveying widths ($W_{f(12)}$ and $W_{f(23)}$) and one storage width (W_{s2}), the adaption of the flow conveying widths took less than 20h, that of the storage width about 30h. If also the roughness coefficient is variable, like in experiment 4 (Fig. 8-6), it took about 50h before the right parameter values were attained.

Other experiments, with various combinations of variable parameters showed similar results, all leading to the true parameter values, which had produced the water level data, now given for the boundary and the intermediate stations. The experiments show that the Kalman filter technique is able to adjust parameters to their correct value, even if they are given a wrong initial value.

8.10. Application to the Western Scheldt estuary

The calculation technique, desceribed in the Sections 8.7 and 8.8 was applied to the Western Scheldt estuary, and in particular to the model, described in Section 7.11.

In the application two phases can be distinguished:
- determination of parameters and dimensions, making use of measured water level data at both intermediate stations.
- determination of water levels at one intermediate station where no measurements are carried out, making use of water level data at the other intermediate station.

8.10.1. Determination of parameters and dimensions

In this case a choice had to be made as to which parameters and dimensions were to be selected for adjustment. It is unwise to select too many since there is a danger of loosing the grip on the situation. If, for instance, all storage widths and all flow conveying widths are allowed to vary, the calculations would not lead to a solution. In the equation of continuity expressed in terms of velocities, the flow conveying width and the storage width only appear as their ratio, but not separately. Looking at eq (7-1)

$$\frac{\partial y}{\partial t} = - \frac{1}{W_s} \frac{\partial Q}{\partial z}$$

and substituting for Q (compare eq (7-11))

$$Q = W_f\,(y-b)\,v$$

it follows that

$$\frac{\partial y}{\partial t} = -\frac{W_f}{W_s} \cdot \frac{\partial (y-b)v}{\partial z} \qquad (8\text{-}101)$$

where the result depends on the ratio W_f/W_s. In the equation of motion, eq (7-7), the widths do not figure at all.

The foregoing means that all combinations of W_f and W_s which have the right ratio will produce the correct result, whatever the individual values of W_f and W_s. In this case the situation is unidentifiable.

When making a selection it was considered that in the seaward part of the estuary the dimension could be better determined than in the eastern, inland part with its many creeks and shallows. Therefore the flow conveying width $W_{f(01)}$ and the storage width W_{s1} were fixed at their values, used in Section 7.11, i.e.

$$W_{f(01)} = 5000 \text{ m}$$
$$W_{s1} = 5500 \text{ m}$$

The flow conveying widths $W_{f(12)}$ and $W_{f(23)}$ and the storage width W_{s2} were varied. Two alternatives were examined:

Case 1, with fixed value of $C = 60 \text{ m}^{1/2}/\text{s}$
Case 2, with variable C-value.

These cases correspond to the experiments for the hypothetical example in Section 8.9. In the calculations noise values were applied as indicated in Section 8.8. The fixed values were given a noise value of zero.

The calculations related to the period from 0h-2 January 1972 to 20h-22 January 1972, i.e. the period, preceding the period of the case, examined in Section 7.11, with an extension to 23h-30 January 1972. During these three weeks the model and the Kalman filter got ample opportunity to overcome the starting phenomena. The results of these calculations are shown first for case 1 in Figs 8- 7 to 8-12, and then for case 2 in Figs 8-13 to 8-20.

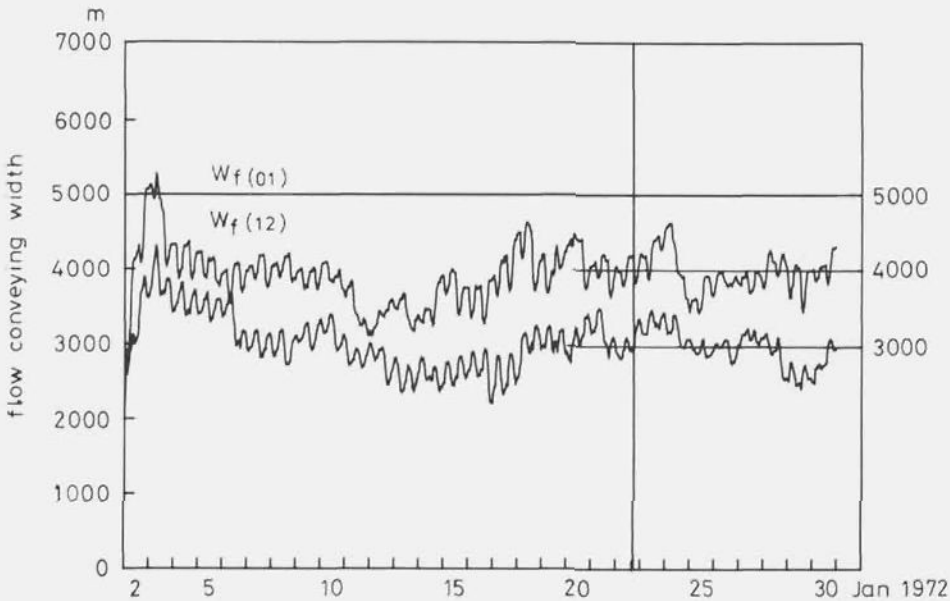


Fig. 8-7 Evolution of the flow conveying widths (Western Scheldt example, case 1)

Case 1.

Fig 8-7 shows the evolution of the flow conveying widths $W_{f(12)}$ and $W_{f(23)}$. Unlike in the hypothetical case they did not arrive at a certain constant value, but remained in oscillation with the daily and fortnightly tidal movements. On the basis of these trajectories final values were assessed as follows:

$$W_{f(12)} = 4000 \text{ m}$$
$$W_{f(23)} = 3000 \text{ m}$$

In Fig. 8-8 the evolution of the storage width W_{s2} is given. The variations are stronger than those of the flow conveying width, but a certain stabilization seems to be attained around $W_{s2} = 5500$ m, at which value this storage width was fixed for further examination. The decrease after 28 January is only of a transistory nature, and it will be recovered when continuing the series.

Fig 8-9 shows the course of the standard errors, before the Kalman filter adaption, of the water levels y_1 and y_2, Fig. 8-10 that of the velocities v_{01}, v_{12} and v_{23}. The following values were determined as means:

$$\sigma\Delta y_1 = 7 \text{ cm}$$
$$\sigma\Delta y_2 = 10 \text{ cm}$$
$$\text{all } \sigma\Delta v = 10 \text{ cm/s}$$

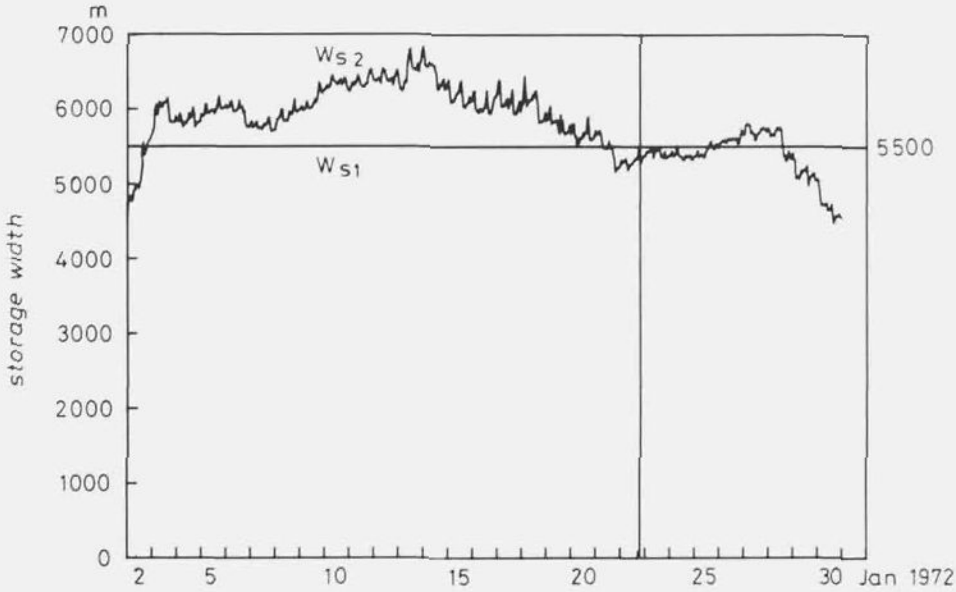


Fig. 8-8 Evolution of the storage widths (Western Scheldt example, case 1)

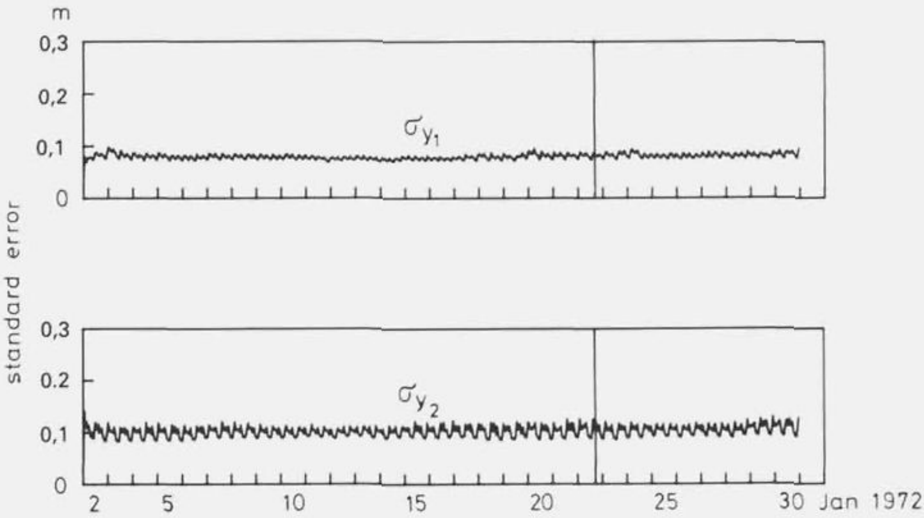


Fig. 8-9 Evolution of water level noise (Western Scheldt example, case 1)

In Fig. 8-11 the evolution of the standard errors of the flow conveying widths $W_{f(01)}$ and $W_{f(12)}$ is given. They vary around 250 m and are both moving along the same line with a fortnightly period corresponding to the spring- neap tidal cycle. The highest

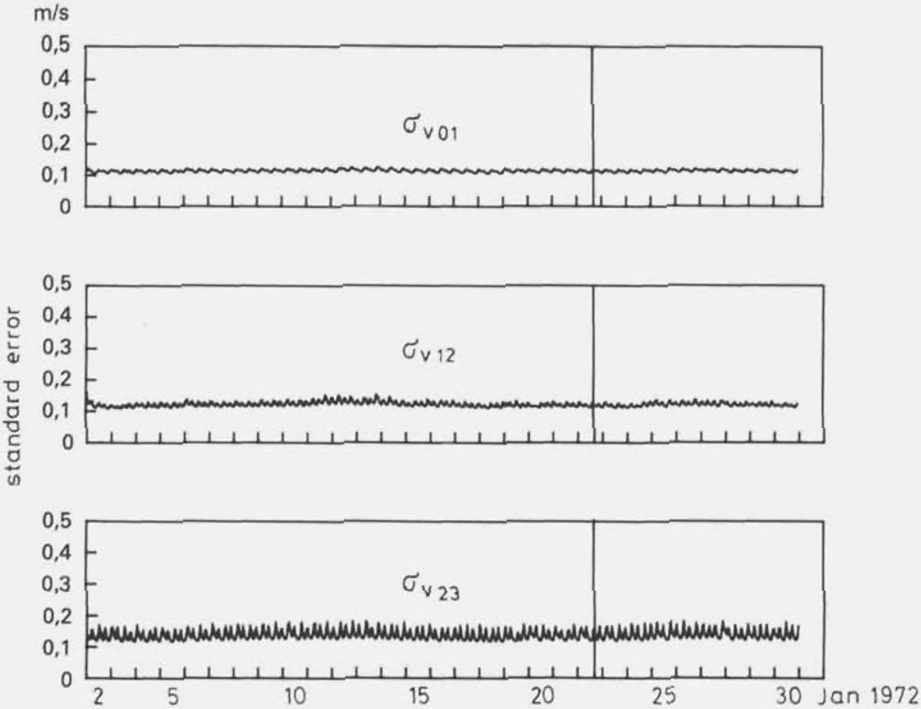


Fig. 8-10 Evolution of velocity noise (Western Scheldt example, case 1)

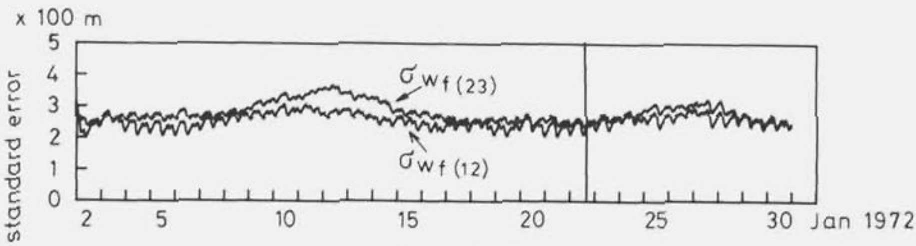


Fig. 8-11 Evolution of noise of flow conveying widths (Western Scheldt example, case 1)

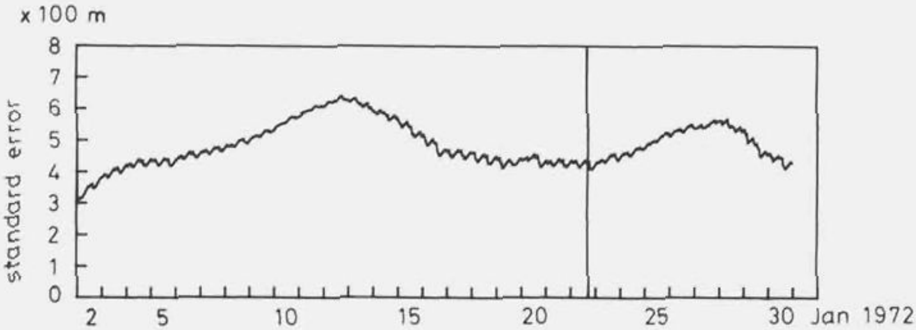


Fig. 8-12 Evolution of noise of storage width (Western Scheldt example, case 1)

values are at neap tide, which indicated less accurate results for that phase. This holds even stronger for the standard errors of the storage width W_{s2} (Fig. 8-12).

Case 2.

In case 2 the roughness coefficient C also was varied. Its evolution is shown in Fig. 8-13. Apparantly the initial value of $C = 60\ m^{1/2}/s$ was too high. The curve leads with important variations, to a value of around $C = 40\ m^{1/2}/s$.

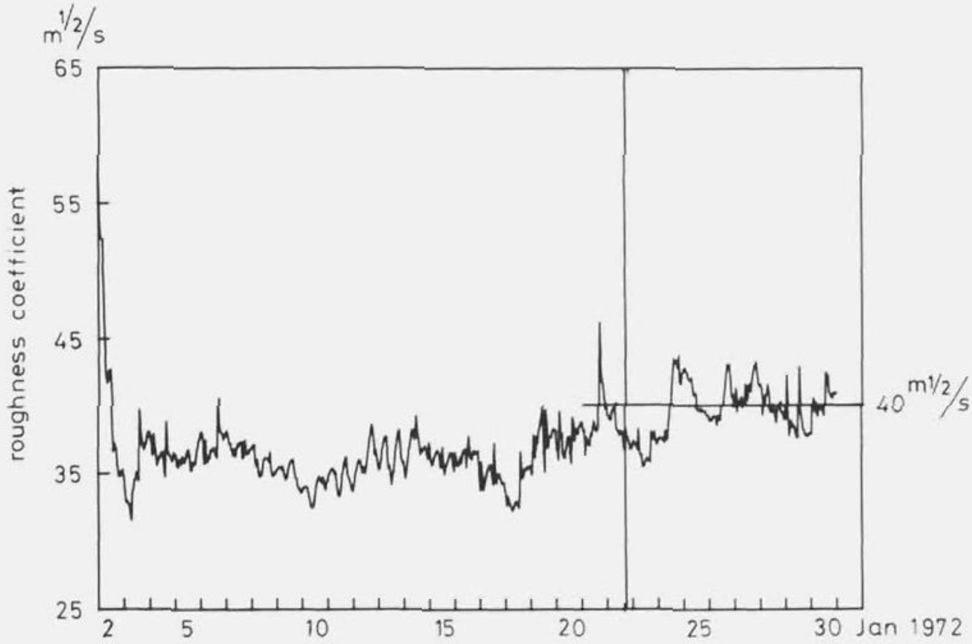


Fig. 8-13 Evolution of the roughness coefficient, starting with $C = 60\ m^{1/2}/s$ (Western Scheldt example, case 2)

The evolutions of the flow conveying widths $W_{f(12)}$ and $W_{f(23)}$ and of the storage width W_{s2} show some similarity with those of case 1 (Fig. 8-14 and 8-15 respectively), although leading to somewhat lower values. As final values were estimated:

$W_{f(12)} = 3300$ m
$W_{j(23)} = 2000$ m
$W_{s2} = 5000$ m

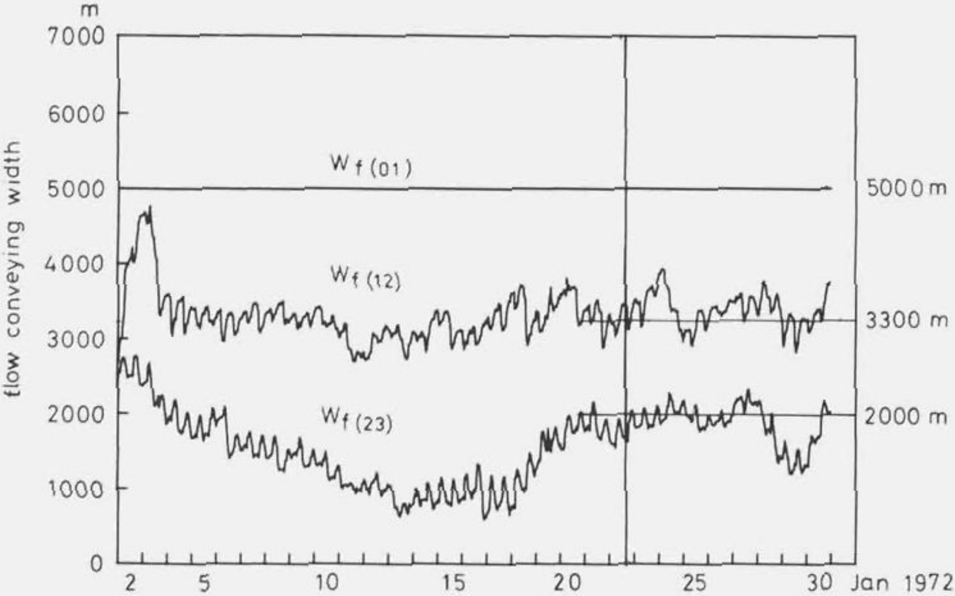


Fig. 8-14 Evolution of the flow conveying widths (Western Scheldt example, case 2)

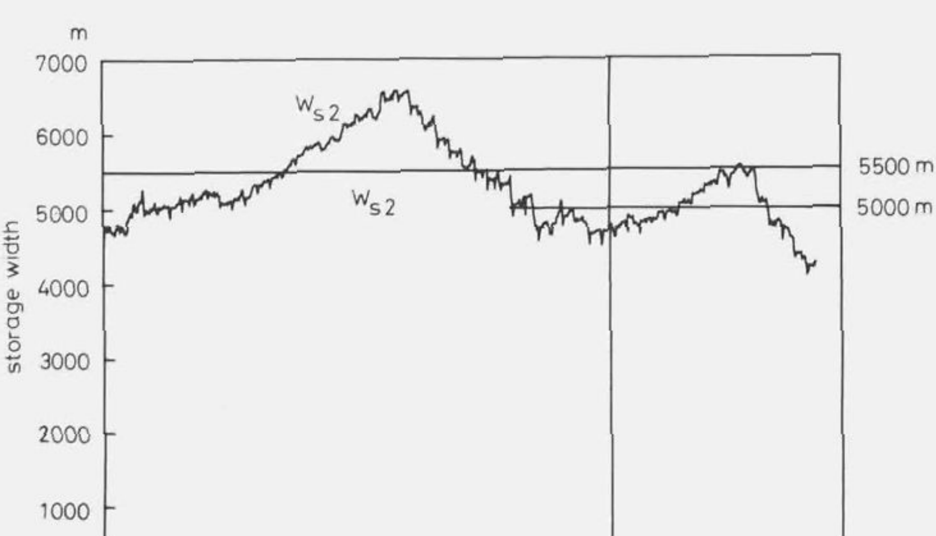


Fig. 8-15 Evolution of the storage widths (Western Scheldt example, case 2)

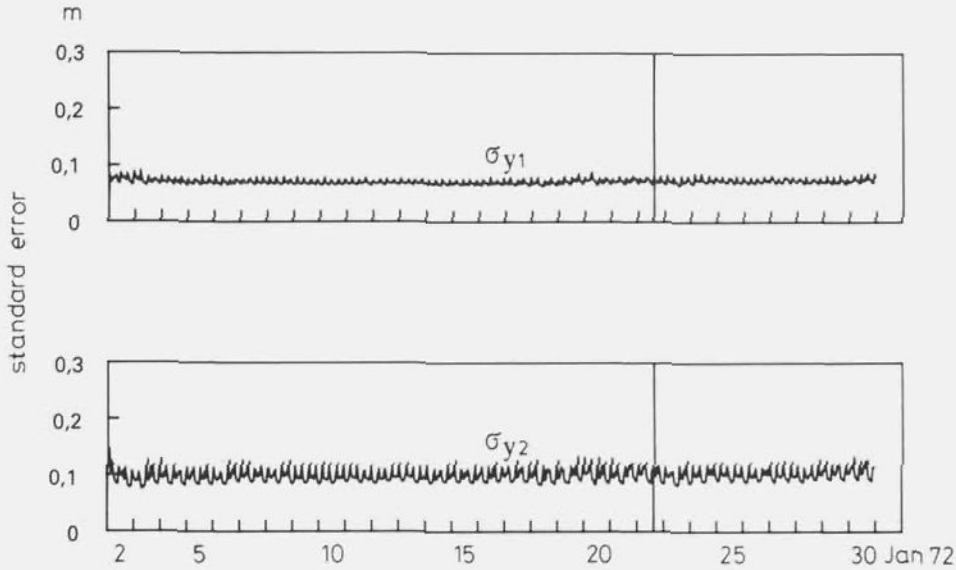


Fig. 8-16 Evolution of water levels noise (Western Scheldt example, case 2)

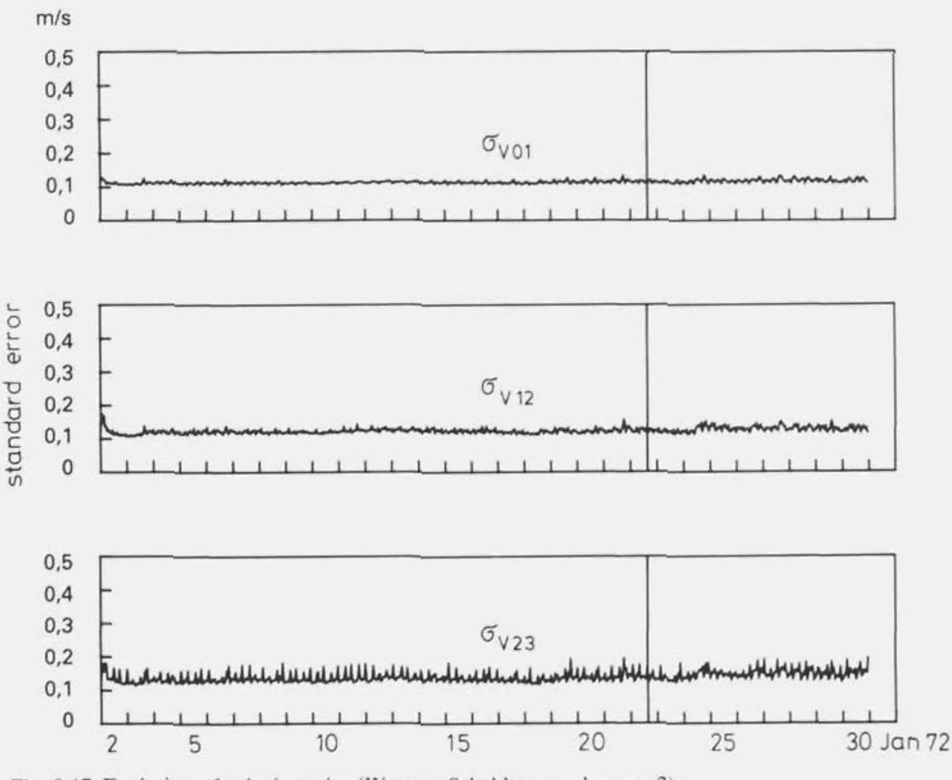


Fig. 8-17 Evolution of velocity noise (Western Scheldt example, case 2)

The standard errors of the calculated water levels and of the velocities at the intermediate site are shown in Figs. 8-16 and 8-17. They do not differ noticably from those of case 1.

Fig 8-18 shows the evolution of the standard error of C^* (= 10 000/C^2). It varies around a value of 1,5 s^2/m. This means that the roughness coefficient with an expected value of 40 m $^{1/2}$/s may vary between 32,9 m $^{1/2}$/s and 55,4 m $^{1/2}$/s at the 95% confidence level. Hero too the fortnighty cycle can be recognized.

This cycle is found once again in the evolution of the standard errors of the flow conveying and the storage widths (Fig. 8-19 and 8-20, respectively). The curves resemble the corresponding curves of case 1 although this time the curve for the standard error of $W_{f(12)}$ is more stable.

8.10.2. Calculations without Kalman filter, using revised parameters

After the experiments of the preceding section, calculations were carried out without using the Kalman filter, i.e. just like those of Section 7.11, but this time using

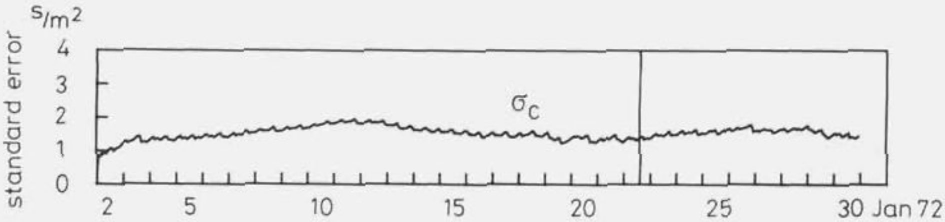


Fig. 8-18 Evolution of roughness coefficient noise (Western Scheldt example, case 2)

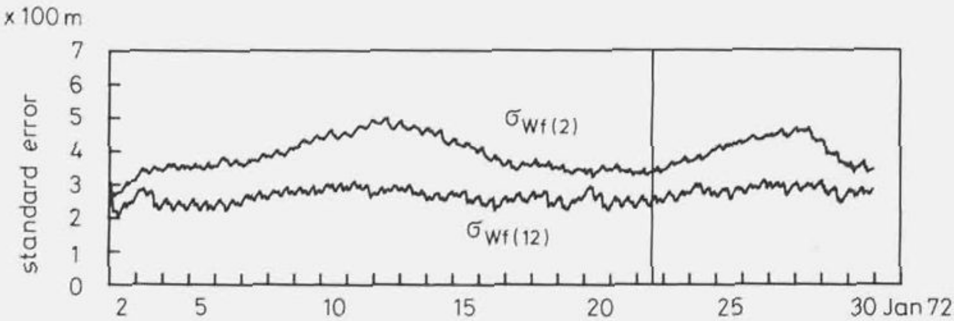


Fig. 8-19 Evolution of noise of flow conveying widths (Western Scheldt example, case 2)

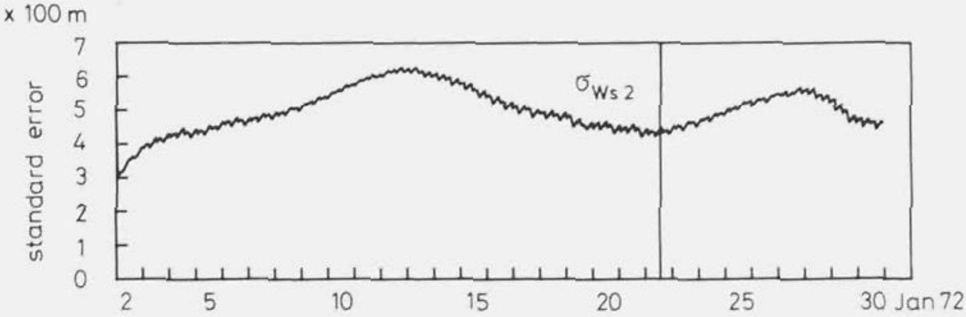


Fig. 8-20 Evolution of noise of storage widths (Western Scheldt example, case 2)

the parameter values, assessed in Section 8.10.1. These values are summarized in Table 8-3.

parameter	case 1	case 2
C (m $^{1/2}$/s)	60	40
$W_{f(01)}$ (m)	5000	5000
$W_{f(12)}$ (m)	4000	3300
$W_{f(23)}$ (m)	3000	2000
W_{s1} (m)	5500	5500
W_{s2} (m)	5500	5000

Table 8-3 Parameter values in two alternative cases

The calculations related to the same period as the example of Section 7.11, i.e. *20h*-22 January 1972 to 9h-25 January 1972. The results are shown in the following figures:

Fig. 8-21: water levels and velocities; case 1
Fig. 8-22: water levels and velocities; case 2
Fig. 8-23: deviations Δy; case 1
Fig. 8-24: deviations Δy; case 2

statistics	case 1	case 2	case *S.7.11*	case *S.7.12*
mean values (cm):				
Terneuzen $\bar{\Delta}\bar{y}_1$	4,405	5,470	6,326	0,731
Hansweert $\bar{\Delta}\bar{y}_2$	*-0,720*	*0,452*	*1,643*	*1,508*
standard errors (cm):				
Terneuzen $\sigma\Delta y_1$	6,890	6,577	10,099	6,104
Hansweert $\sigma\Delta y_2$	8,031	9,039	11,842	7,537
Correlation coeff				
$\varrho\Delta y_1\,\Delta y_2$	0,580	0,589	0,781	0,568

Table 8-4 Statistics of the deviations found in cases 1 and 2, compared with the statistics, found in Chapter 7.

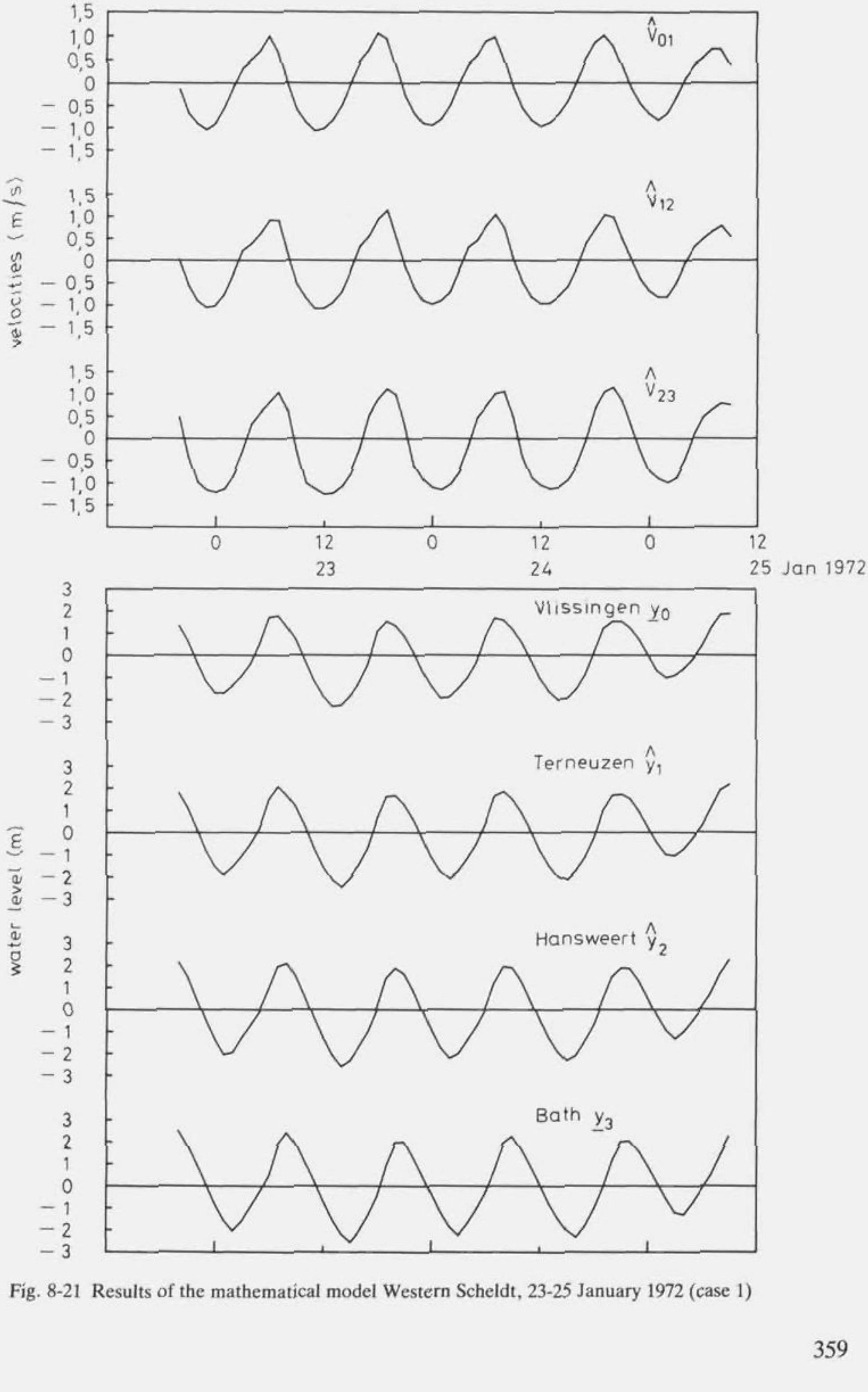


Fig. 8-21 Results of the mathematical model Western Scheldt, 23-25 January 1972 (case 1)

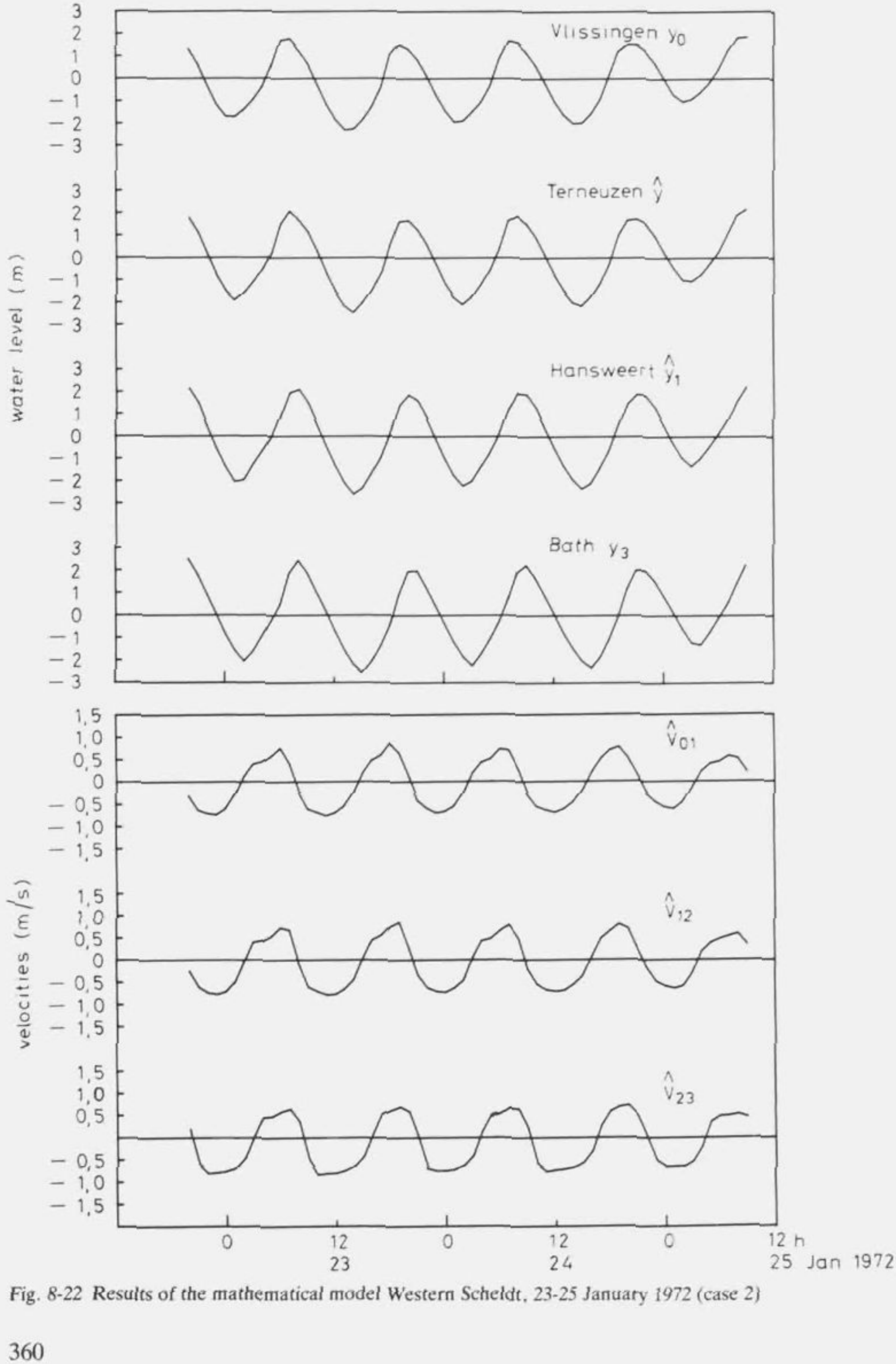


Fig. 8-22 Results of the mathematical model Western Scheldt, 23-25 January 1972 (case 2)

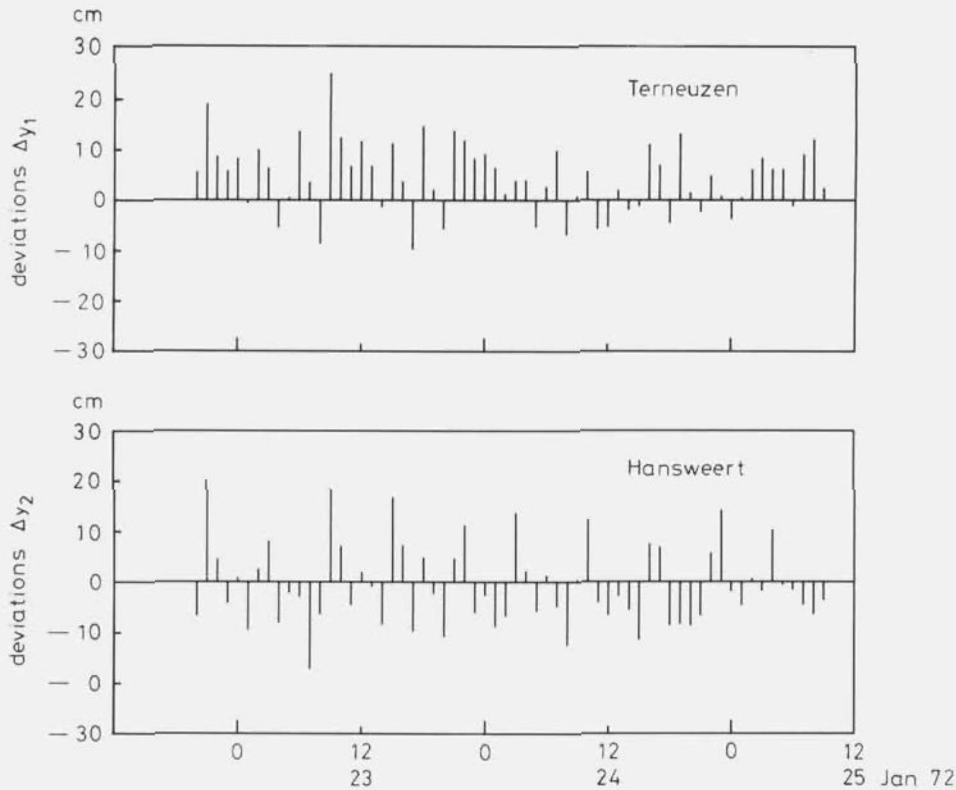


Fig. 8-23 Deviations between measured and calculated water levels (mathematical model; case 1)

The statistics of the deviations are presented in Table 8-4, together with those of Section 7-11 (model, with original parameters) and of Section 7-12 (lineair regression) for comparison.

Compared with the mathematical model of Section 7.11 the results are much better. Apparently the parameters as adjusted by the Kalman filter procedure fit the actual conditions for the model better than those originally used in Section 7.11. Compared with the results of the lineair regression method described in Section 7.12 the same order of magnitude was found at least as concerns the standard errors. It is difficult to decide which of the cases 1 or 2 is to be preferred, although as a whole those of case 1 seem to be somewhat better.

8.10.3 Determination of water levels at non gauged sites

This time only measurements at one of the intermediate stations are used in the calculations. The levels at the other intermediate site are calculated by the model and subsequently adjusted, using the measurement results at the measured station. The non measured stations get an element 'zero' in the measurement matrix H.

In this application of the Kalman filter, not too many parameters should be made adjustable, (i.e. be affected by noise) since they will have the tendency to focus on the measured water levels, without taking into account the behaviour of the levels at the non measured station. Therefore only two alternatives will be considered,

a) with noise affecting water levels and velocities only;
b) like a), but with additional noise affecting the roughness coefficient, C.

The flow conveying widths and the storage widths will be treated as fixed values, corresponding to the values used in the above cases 1 and 2 previously. All these experiments were carried out for the cases
- Terneuzen (y_1) measured, Hansweert (y_2) calculated
- Hansweert (y_2) measured, Terneuzen (y_1) calculated

Table 8-5 Summarizes the cases considered

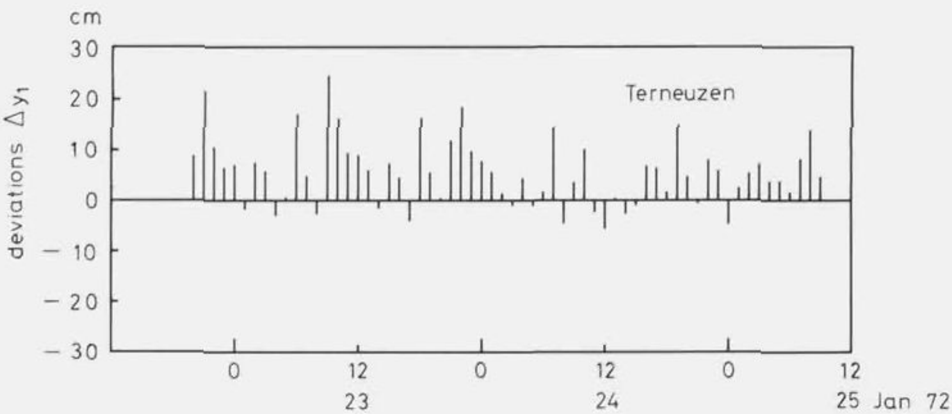
case	parameters from case	adjustable parameter	measurements at station
1 aT	1	-	Terneuzen (y_1)
1 bT	1	C	Terneuzen (y_1)
2 aT	2	-	Terneuzen (y_1)
2 bT	2	C	Terneuzen (y_1)
1 aH	1	-	Hansweert (y_2)
1 bH	1	C	Hansweert (y_2)
2 aH	2	-	Hansweert (y_2)
2 bH	2	C	Hansweert (y_2)

Tabel 8-5 Cases considered for measurements at one intermediate station

The statistics of the results are given in Table 8-6

station under examination	case							
	1 aT	1 bT	2 aT	2 bT	1 aH	1 bH	2 aH	2 bH
mean values (cm):								
Terneuzeny $\bar{\Delta}\bar{y}_1$	-	-	-	-	1,686	1,448	2,135	2,057
Hansweert $\bar{\Delta}\bar{y}_2$	-1,703	-2,067	-1,597	-1,998	-	-	-	-
standard errors (cm):								
Terneuzen $\sigma\Delta y_1$	-	-	-	-	5,231	5,242	5,306	5,344
Hansweert $\sigma\Delta y_2$	6,186	6,767	7,257	8,061	-	-	-	-

Table 8-6 Statistics of the results of Kalman filter calculations with measurements at one intermediate station.



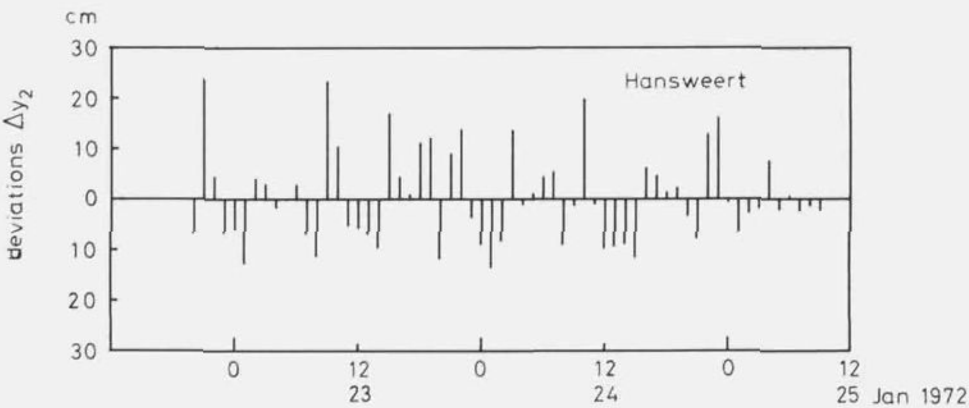


Fig. 8-24 Deviations between measured and calculated water levels (mathematical model, case 2)

Before discussing the results a comparison will be made with the results of a multiple regression calculation. Here also, the following cases were considered.

- Values for the Hansweert station were calculated using data from the Vlissingen, Terneuzen and Bath stations;
- Values for the Terneuzen station were calculated, using data from the Vlissingen, Hansweert and Bath stations

Here also the period 20h-22 January 1972, to 9h-25 January 1972 was used. The regression coefficients were based on the whole year 1971. When calculating the values for t_0 + 1h at the station under examination, data for the three other stations, at t_0-1h, t_0 and t_0 + 1h were used as input. In order to make the results comparable with those of the mathematical model no data measured after t_0+1h were used, although this is possible in the multiple linear regression method.

The results are given in table 8-7

station under examination	measurements at	
	Terneuzen	Hansweert
mean values (cm):		
Terneuzen $\bar{\Delta}\bar{y}_1$	-	-0,121
Hansweert $\bar{\Delta}\bar{y}_2$	1,087	-
standard errors (cm):		
Terneuzen $\sigma\Delta y_1$	-	4,796
Hansweert $\sigma\Delta y_2$	5,955	-

Table 8-7 Statistics of the results of the multiple linear regression method with measurements at one intermediate station

For this example the standard errors $\sigma\Delta y$, produced by multiple linear regression (Table 8-7) were somewhat better than the lowest in Table 8-6, although the differences are not important.

If the mathematical model is used (Table 8-6) the best results might be obtained for the cases 1aT and 1aH, i.e. the cases with fixed roughness coefficient and using the dimensions of case 1 (Table 8-3). The values of $\sigma\Delta y_1$ for Terneuzen, if measurements are carried out in Hansweert (i.e. for the cases, indicated by H) are of the same magnitude in both Table 8-6 and Table 8-7. However for the cases with measurements in Terneuzen (i.e. the cases indicated by T) there are remarkable differences in the $\sigma\Delta y_2$-values for Hansweert. Apparantly the model is more sensitive to the hydraulic conditions at Hansweert than to those at Terneuzen.

If one had to decide which of the two intermediate sites, Hansweert or Terneuzen, is most suitable for establishing a gauging station, then, on the basis of the above results one would surely choose for a station at Hansweert.

The deviations Δy for some of the cases which were examined are shown in the following figures:

Fig. 8-25: Δy_1 for Terneuzen (case 1aH and multiple linear regression)
Fig. 8-26: Δy_2 for Hansweert (case 1aT and multiple linear regression)

To help relate the results to the tidal cycle the measured water levels at the stations concerned are also shown in these figures. The deviations produced by the mathematical model show differences with those from the multiple linear regression method. These results will not be discussed in detail since they concern just an arbitrary example.

When deciding which method is most suitable for application in an actual case t following points should be taken into consideration.

The mathematical model
- requires knowledge of model parameters (dimensions and roughnes coefficient);
- produces water levels, velocities and, when desired, discharges (see Annex IV);
- requires a complicated and time consuming computer programme;
- allows to study possible future changes in the streamflow sections.

The multiple linear regression method
- requires long time series for determining the regression coefficients;
- needs a simple and short computer programme (for interpolation even a pocket calculator can be used);
- for calculating earlier water levels (hindcasting) measurements made later on can be used.

The combined use of a mathematical model and the Kalman filter
- requires only approximative knowledge of the model parameters.
- requires a much more complicated and time and capacity consuming computer programme than the mathematical model alone.

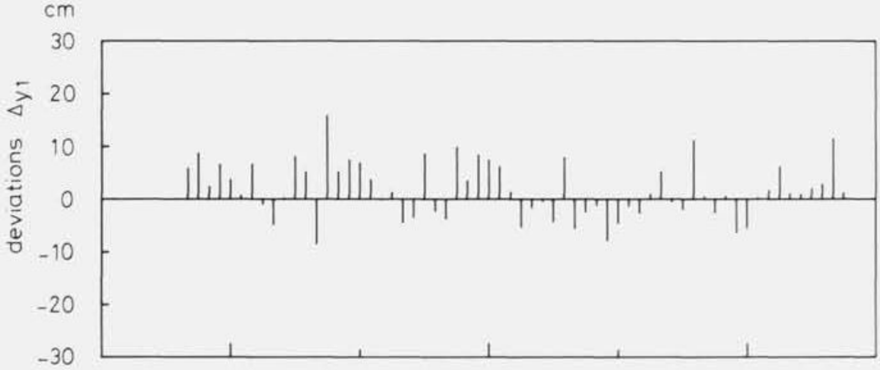
8.11 Use in network design

In principle a physically based, mathematical model, like the one described, can be used in testing an existing network in the same way as was done using the multiple regression method, as described in Chapter 5.

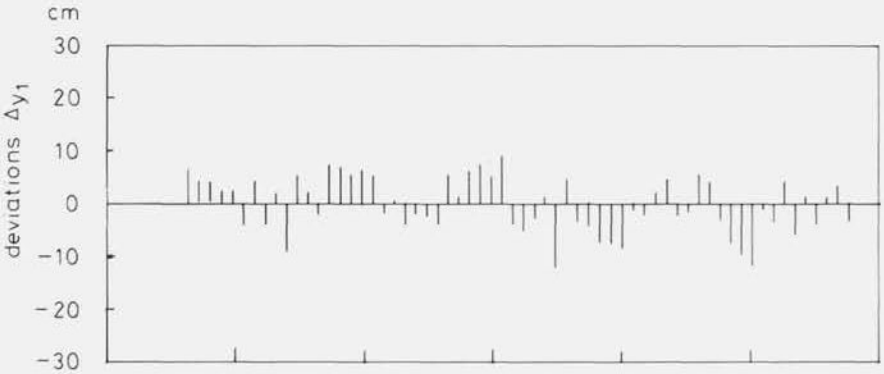
Water level data could be calculated using data of various combinations of gauging stations, acting as a network. For each case the results could be compared with actual measurements, not used in the calculations. The set of gauging stations, which can provide the best results, might be selected to be the future network. For the Western Scheldt example, for instance, this was demonstrated by Table 8-6.

For a river reach with a great number of existing gauging stations a number of examinations could be carried out for various station distances. The results could be used to draw a graph like Fig. 2-1, from which the required station distance could be easily derived.

For intermediate, i.e. non gauged sites a mathematical model can reproduce water level data as well. If combined with a Kalman filter it can also give an impression of



Deviations mathematical model (case 1aH)



Deviations multiple linear regression method

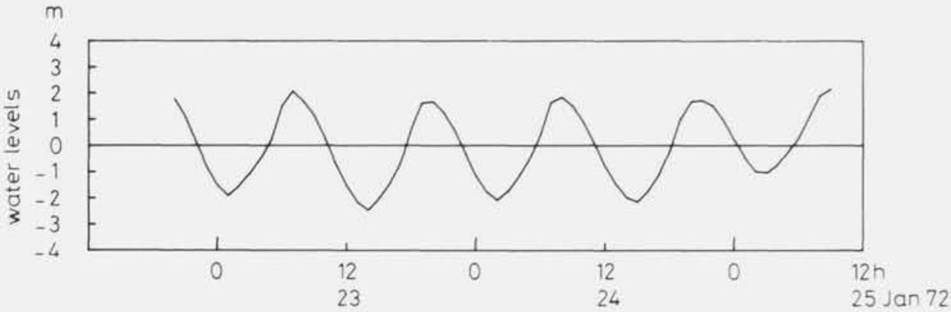
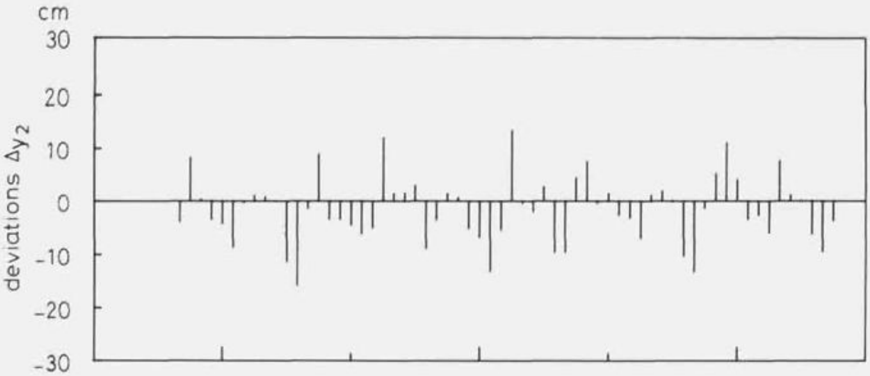
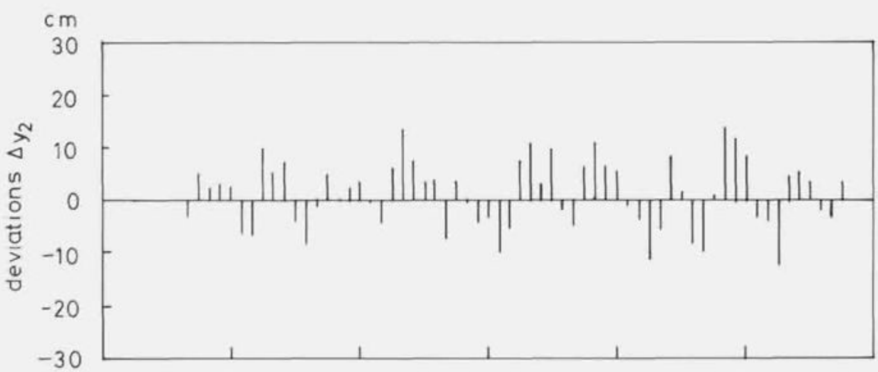


Fig. 8-25 Deviations Δy_1 and water levels (Terneuzen)

It is not the intention of this study to express a preference to one of the methods discussed. It is possible that the results of both methods may be improved, the



Deviations mathematical model (case 1aT)



Deviations multiple linear regression method

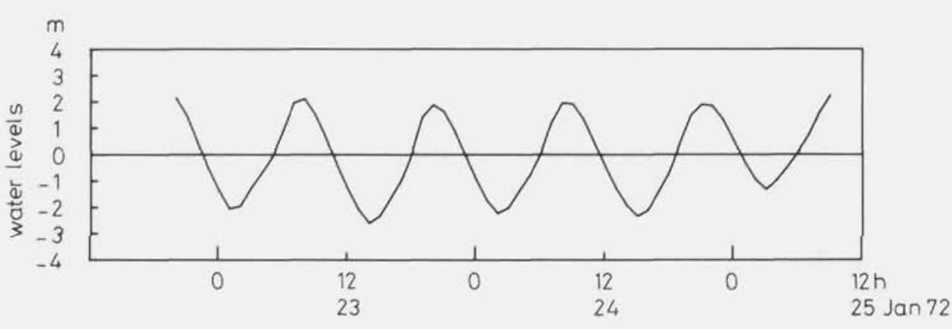


Fig. 8-26 Deviations Δy_2, and water levels (Hansweert)

mathematical model method by a more refined schematization, and the regression method by taking into account measurements, made after the time of examination.

the accuracy. The latter requires some trial-and-error approach, since the model error variance matrix must be specified , before starting the calculations.

The advantages and disadvantages of both the multiple regression method and the mathematical model, eventually combined with a Kalman filter, have been discussed at several places in this study. It is recommended that both methods be considered in practical applications and a final selection made on the basis of the quality of the results obtained and of the practical facilities of their execution.

Annex I

Some properties of Fourier transforms

In relation to the techniques used for network design, as described in Chapter 3, the Fourier transform technique can be a useful aid for examining the behaviour of the correlation coefficient. In this Annex the background of this technique will be described in simple terms with particular emphasis given to its application to network design problems.

Background of the Fourier transform technique

Consider a single harmonic function:

$$f(t) = A \cos \omega t \qquad \text{(I-1)}$$

where:

A = amplitude
ω = angular frequency
t = time

This function can be presented as a function of time, t, in Fig. I-1. This corresponds to the function's representation in the time domain. It can also be depicted in the frequency domain, where it is regarded as a harmonic function with frequency ω_1 and amplitude A. The corresponding graph is shown in Fig. I-2.

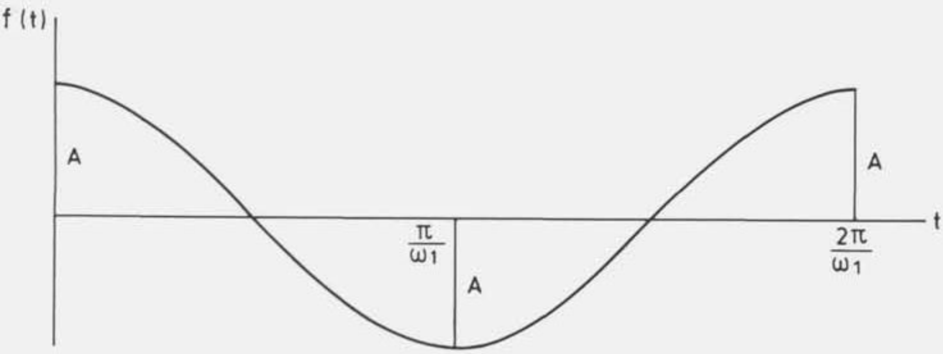


Fig. I-1. Single harmonic function (time domain).

This function has a value A for $\omega = \omega_1$ and is zero for all other values of ω. It can be expressed as a formula as follows:

$$G(\omega) = A \quad \text{for} \quad \omega = \omega_1$$
$$G(\omega) = 0 \quad \text{for} \quad \omega \neq \omega_1 \qquad \text{(I-2)}$$

Instead of a discrete frequency ω_1 the value A could be considered to be made up of contributions from all frequencies between ω_1 and $\omega_1 + \Delta\omega$. This is schematized in Fig. I-3, where the single line of Fig. I-2 has been replaced by a block. The area of this block is equal to the amplitude A and its height is $A/\Delta\frac{\omega}{2\pi}$ so that the vertical dimension is [It]. The units along the ordinate can thus be given, for instance, in ms.

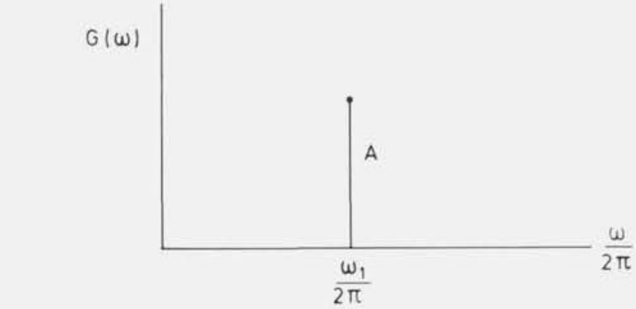


Fig. I-2 Single harmonic function (frequency domain).

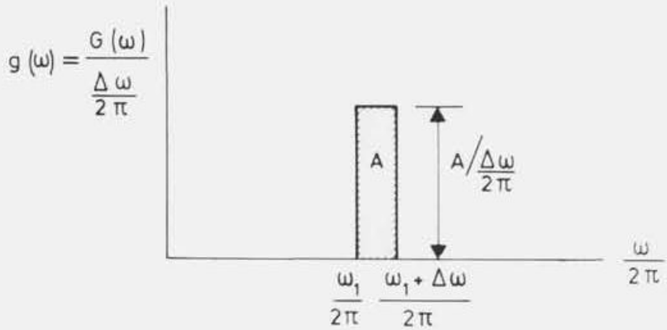


Fig. I-3 Single harmonic function in block representation (frequency domain).

Now let

$$g(\omega) = G(\omega)/\Delta\frac{\omega}{2\pi} \text{ for } \omega_1 < \omega < \omega_1 + \Delta\omega \qquad \text{(I-3)}$$

Further let $G(\omega)$ have a fixed value $G(\omega_1)$ over that range, so that

$$G(\omega_1) = g(\omega_1)\Delta\frac{\omega}{2\pi} \qquad \text{(I-4)}$$

or, more precisely

For

$$G(\omega_1) = \frac{1}{2\pi} \int_{\omega_1}^{\omega_1+\Delta\omega} g(\omega)d\omega \tag{I-5}$$

$$G(\omega_1) = A \tag{I-6}$$

and

$$g(\omega) = A/\Delta\frac{\omega}{2\pi} \tag{I-7}$$

one may write

$$G(\omega_1) = \frac{1}{2\pi} \int_{\omega_1}^{\omega_1+\Delta\omega} (A/\Delta\frac{\omega}{2\pi})d\omega = A \int_{\omega_1}^{\omega_1+\Delta\omega} (1/\Delta\omega)d\omega. \tag{I-8}$$

In the discrete case of Fig. I-2, then $\Delta\omega \rightarrow 0$. This case can be expressed by using the delta or Dirac function as follows:

$$g(\omega) = A\,\delta(\omega - \omega_1) \tag{I-9}$$

and

$$G(\omega) = A \int_{-\infty}^{\infty} \delta(\omega - \omega_1)\,d\omega \tag{I-10}$$

For this limit case

$$\delta(\omega - \omega_1) = \lim_{\substack{\Delta\omega \rightarrow 0 \\ \omega = \omega_1}} (1/\Delta\omega). \tag{I-11}$$

For discrete cases, figures like Fig. I-3 are in fact impossible, and a presentation like Fig. I-2 should be given.

Now consider a harmonic function made up of two components:

$$f(t) = A_1 \cos \omega_1 t + A_2 \cos \omega_2 t \tag{I-12}$$

For $\omega_2 = 2\omega_1$ and $A_2 = A_1$, this function, plotted in the time domain, is shown in Fig. I-4.

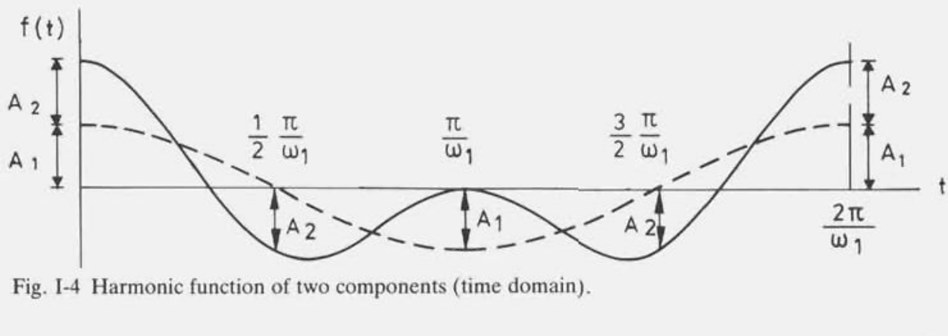


Fig. I-4 Harmonic function of two components (time domain).

In the frequency domain the function is

$$g(\omega)=A_1\delta(\omega-\omega_1)+A_2\delta(\omega-\omega_2) \qquad \text{(I-13)}$$

and corresponding graph is given in Fig. I-5.

Fig. I-5 Harmonic function of two components (frequency domain).

For more components the functions in the time domain and in the frequency domain become respectively:

$$f(t) = \sum_{i=1}^{n} a_i \cos \omega_i t \qquad \text{(I-14)}$$

and

$$g(\omega)= \sum_{i=1}^{n} A_i\delta(\omega-\omega_i). \qquad \text{(I-15)}$$

The function $g(\omega)$ is called the (amplitude)-spectrum of the time function $f(t)$. A graph of $G(\omega)$ for many components can for instance look like Fig. I-6.

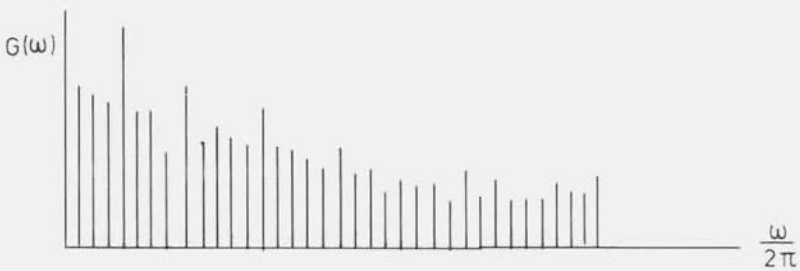


Fig. I-6 Amplitude spectrum of a time function of many harmonic components

If there are so many components in the spectrum that this is in fact a continuous function, then use should be made of a presentation which corresponds to Fig. I-3. This leads to a graph like Fig. I-7.

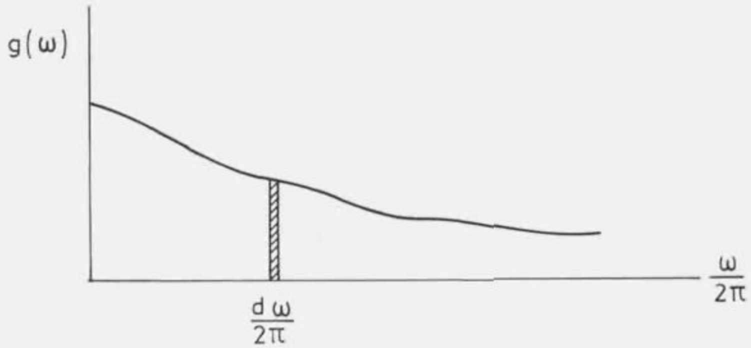


Fig. I-7 Continuous amplitude spectrum of a time function.

In order to derive the function in the time domain, i.e. f(t), A_i in eq (I-14) has to be replaced by g(ω)dω, and the sommation sign Σ transforms into an integral with boundaries from 0 to ∞. Thus:

$$f(t) = \sum_{i=1}^{n} A_i \cos\omega_i t \rightarrow \int_0^\infty g(\omega)\cos\omega t d\omega \tag{I-16}$$

This is the Fourier transform of even functions. These are functions, which take the same value for positive and for negative values of equal magnitude, thus f(t) = f(-t). For other than even functions the Fourier transform includes complex variables. This case will not be discussed here.

The time function will consist of an infinite number of small harmonic functions, but for the whole the harmonic character is not visible.

Relation between the Fourier transform and the correlation structure of the time functions

Consider a time series y(t). The autocorrelation coefficient over a timelag τ for this series is (compare eq (3-11):

$$\varrho(\tau) = \frac{\text{Cov } y(t).y(t-\tau)}{\text{Var } y} . \tag{I-17}$$

The numerator expresses the autocovariance between y(t) and y(t-τ), and will be denoted by Cov y(τ). For n measurements, and if the mean value of y is zero, then

$$\mathrm{Cov}\ y\ (\tau) = \frac{\sum_{i=1}^{n} y(t_i).y(t_i-\tau)}{n} \tag{I-18}$$

For continuous measurements made during an infinitely long period T this transforms into the integral:

$$\mathrm{Cov}\ y\ (\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T y(t).y(t-\tau)dt. \tag{I-19}$$

Now let y(t) consist of a number of consine functions, as expressed by eq (I-14). First consider a single cosine function, like eq. (I-1), so

$$y(t) = A\cos\omega_1 t \tag{I-20}$$

The covariance for a time τ is according to eq (I-19)

$$\mathrm{Cov}\ y(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T A\cos\omega_1 t.A\cos\omega_1(t-\tau)dt$$

$$= \lim_{T\to\infty} \frac{A^2}{T} \int_0^T \cos\omega_1 t.\cos\omega_1(t-\tau)dt. \tag{I-21}$$

Further elaboration of the integral yields:

$$\int_0^T \cos\omega_1 t.\cos\omega_1(t-\tau)dt =$$

$$\int_0^T \cos\omega_1 t(\cos\omega_1 t\cos\omega_1\tau - \sin\omega t\sin\omega_1\tau)dt =$$

$$\cos\omega_1\tau \int_0^T \cos^2\omega_1 t\,dt - \sin\omega_1\tau \int_0^T \sin\omega_1 t \cos\omega_1 t\,dt =$$

$$\frac{1}{2} \cos\omega_1\tau \int_0^T (\cos 2\omega_1 t + 1)dt - \frac{1}{2} \sin\omega_1\tau \int_0^T \sin 2\omega_1 t\,dt =$$

$$\frac{\cos\omega_1\tau}{?.2\omega_1} \sin 2\omega_1 T + \frac{\cos\omega_1\tau t}{2} T + \frac{\sin\omega_1\tau}{2.2\omega_1} \cos 2\omega_1 T - \frac{\sin\omega_1\tau}{2.2\omega_1}. \qquad \text{(I-22)}$$

From eq (I-21) then

$$\text{Cov } y(\tau) = \frac{A_2}{4\omega_1} \lim_{T\to\infty} \frac{1}{T} (\cos\omega_1\tau \sin 2\omega_1 T + \sin\omega_1\tau \cos 2\omega_1 T - \sin\omega_1\tau)$$

$$+ \frac{1}{2} A^2 \cos\omega_1\tau, \qquad \text{(I-23)}$$

or

$$\text{Cov } y(\tau) = \frac{1}{2} A^2 \cos\omega_1\tau. \qquad \text{(I-24)}$$

In fact, this is a time function like eq (I-20), but this time expressed in terms of τ. Here too, as in the case of eq (I-1) a transform can be given, which is similar to eq (I-9):

$$g_p(\omega) = \frac{1}{2} A^2 \delta(\omega - \omega_1). \qquad \text{(I-25)}$$

Since the potential energy of the time function is proportional to A.1/2 A = $1/2A^2$, eq (I-25) is generally called the energy spectrum or power spectrum. Apparently the power spectrum, eq (I-25), is the Fourier transform of the covariance function, eq (I-24). The index p in g_p stands for power.

The covariance transforms into the variance if the time interval τ becomes zero, thus, eq (I-24) gives

$$\text{Var } y = \frac{1}{2} A^2 \qquad \text{(I-26)}$$

According to eq (I-17) the autocorrelation coefficient will be

$$\varrho(\tau) = \cos \omega_1 \tau \qquad \text{(I-27)}$$

This is also a time function of τ, and it is similar to the original time function eq (I-20), and to the autocovariance function eq (I-24). However this time the amplitude amounts to 1.

Now consider the case of a time function of two cosine functions, like eq (I-12):

$$y(t) = A_1 \cos \omega_1 t + A_2 \cos \omega_2 t. \qquad \text{(I-28)}$$

For the covariance function at lagtime τ can be written, after substituting eq (I-28) into eq (I-19), as

$$\text{Cov } y(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T (A_1 \cos\omega_1 t + A_2 \cos\omega_2 t).\{A_1 \cos\omega_1 (t-\tau) + A_2 \cos\omega_2 (t-\tau)\} dt. \qquad \text{(I-29)}$$

When elaborating the integrand there appear four terms:

a) $\dfrac{A_1^2}{T} \displaystyle\int_0^T \cos\omega_1 t \cos\omega_1 (t-\tau) dt,$

b) $\dfrac{A_2^2}{T} \displaystyle\int_0^T \cos\omega_2 t \cos\omega_2 (t-\tau) dt,$

c) $\dfrac{A_1 A_2}{T} \displaystyle\int_0^T \cos\omega_1 t \cos\omega_2 (t-\tau) dt,$

d) $\dfrac{A_1 A_2}{T} \displaystyle\int_0^T \cos\omega_2 t \cos\omega_1 (t-\tau) dt.$

The terms a) and b) concern respectively the frequencies ω_1 and ω_2 only. Their integrals will, regarding eq (I-24) become for $T \to \infty$

a) $\frac{1}{2} A_1^2 \cos \omega_1\tau$

b) $\frac{1}{2} A_1^2 \cos \omega_2\tau$

The terms c) and d) contain different frequencies ω_1 and ω_2. Now consider the term c), so

$$\frac{A_1A_2}{T}\int_0^T \cos\omega_1 t \cos\omega_2(t-\tau)dt =$$

$$\frac{A_1A_2}{T}\int_0^T \cos\omega_1 t(\cos\omega_2 t\cos\omega_2\tau - \sin\omega_2 t\sin\omega_2\tau)dt =$$

$$\frac{A_1A_2}{T}\left\{\cos\omega_2\tau\int_0^T \cos\omega_1 t\cos\omega_2 t\,dt - \sin\omega_2\tau\int_0^T \cos\omega_1 t\sin\omega_2 t\,dt\right\}. \quad \text{(I-30)}$$

Considering the first integral of eq (I-30), if

$$p = \omega_1/\omega_2 \quad \text{(I-31)}$$

then this integral gives:

$$\int_0^T \cos\omega_1 t \cos p\,\omega_1 t\,dt. \quad \text{(I-32)}$$

Partial integration yields:

$$\frac{1}{\omega_1}\int_0^T \cos p\omega_1 t\, d\sin\omega_1 t =$$

$$\frac{1}{\omega_1}\left\{\cos p\omega_1 t\sin\omega_1 t\,\Big|_0^T - \int_0^T \sin\omega_1 t\, d\cos p\omega_1 t\right\} =$$

$$\frac{1}{\omega_1}\left\{\cos p\omega_1 T \sin\omega_1 T - p \sin p\omega_1 t \cos \omega_1 t \Big|_0^T + p \int_0^T \cos\omega_1 t \, d \sin p\omega_1 t\right\} =$$

$$\frac{1}{\omega_1}\left\{\cos p\omega_1 T \sin\omega_1 T - p \sin p\omega_1 T \cos \omega_1 T + p^2\omega_1 \int_0^T \cos\omega_1 t \cos p\omega_1 t \, dt\right\}. \quad \text{(I-33)}$$

The last integral is equal to eq (I-32). From eqs (I-33) and (I-32) it can be derived that

$$\int_0^T \cos\omega_1 t \cos p\omega_1 t \, dt = \frac{1}{\omega_1(1-p^2)} \{\cos p\omega_1 T \sin\omega_1 T - p \sin p\omega_1 T \cos\omega_1 T\} \quad \text{(I-34)}$$

For $p \neq 1$ this function has a finite value, but in the limit case for $T \rightarrow \infty$ its contribution to eq (I-30) becomes zero.

In a similar way it can be shown that the second term of eq (I-30):

$$\frac{A_1 A_2}{T} \sin\omega_2 T \int_0^T \cos\omega_1 t \sin\omega_2 t \, dt$$

becomes zero for $T \rightarrow \infty$. This implies that the entire expression of eq (I-30) becomes zero and with it the term c) that followed from eq (I-29). For the term d) the same proof can be given, so that finally only the terms a) and b) of eq (I-29) remain. This leads to the following value of the autocovariance:

$$\text{Cov } y(\tau) = \frac{1}{2} A_1^2 \cos\omega_1 t + \frac{1}{2} A_2^2 \cos\omega_2 t. \quad \text{(I-35)}$$

If there are n terms in the original time function, then

$$y(t) = \sum_{i=1}^{n} A_i \cos\omega_i t, \quad \text{(I-36)}$$

and it can be shown in a similar way as above that

$$\text{Cov } y(\tau) = \frac{1}{2} \sum_{i=1}^{n} A_i^2 \cos\omega_i t \quad \text{(I-37)}$$

The corresponding power spectrum is

$$g_p(\omega) = \frac{1}{2} \sum_{i=1}^{n} A_i^2 \delta(\omega - \omega_i). \qquad \text{(I-38)}$$

What was already shown earlier for a single harmonic function has been proven now for multiple harmonic functions, and, as a consequence, for the limit case of an infinite number of harmonic functions, and also for continuous functions. In other words the power spectrum has been shown to be the Fourier transform of the covariance function.

Thus, if the power spectrum is known, the covariance function can be derived, and with it the autocorrelation function. From eq (I-37) for $\tau = 0$ it follows that

$$\text{Var } y = \sum_{i=1}^{n} A_i^2, \qquad \text{(I-39)}$$

and thus, according to eq (I-17):

$$\varrho(\tau) = \frac{\sum_{i=1}^{n} A_i^2 \cos\omega_i\tau}{\sum_{i=1}^{n} A_i^2}. \qquad \text{(I-40)}$$

In the case of a continuous power spectrum the covariance function of eq (I-37) transforms into the integral:

$$\text{Cov } y(\tau) = \int_0^\infty g_p(\omega)\cos\omega\tau d\omega \qquad \text{(I-41)}$$

Here $g_p(\omega)$ is the power density spectrum. The dimension of $g_p(\omega)$ is $[l^2t]$.

Influence of the power spectrum on the shape of the covariance and correlation functions

Various cases of power spectra and the corresponding covariance and correlation functions will be considered now. In particular the influence of high frequent variations on the covariance and correlation functions for small time intervals will be examined. In the following graphs, the power spectra are presented as well as the covariance functions. The correlation functions are not given, but these can easily be imagined by standardizing the covariance functions in such a way that for $\tau=0$ a value $\varrho(\tau)=1$ is obtained. As was mentioned earlier the power spectra are the Fourier transforms of the covariance functions.

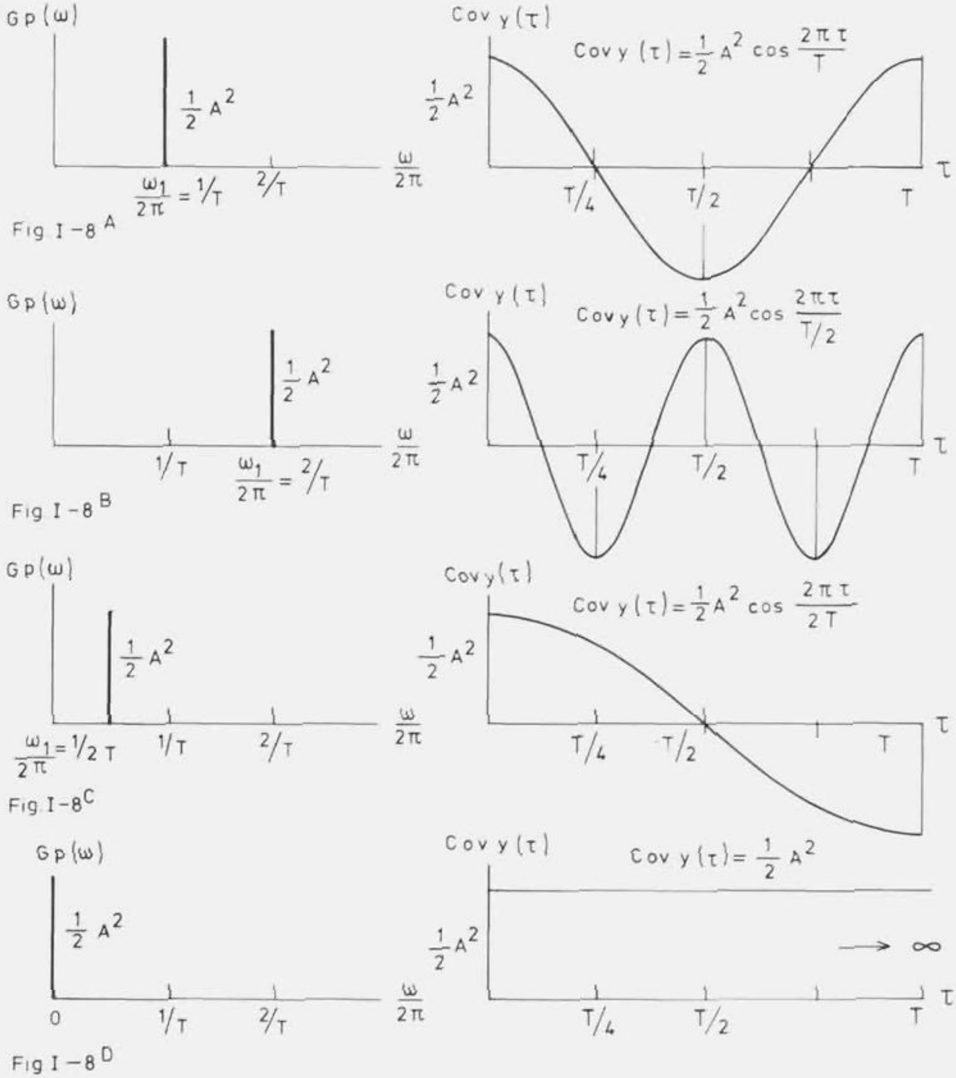


Fig. I-8. Power spectra and covariance functions of single harmonic functions.

If the variations only have one frequency ω_1, then the covariance function consists of a cosine function with a period of $T_1=2\pi/\omega_1$ (Fig. I-8A). A frequency $\omega_2=2\omega_1$ yields a covariance function with a period of $T_2=2\pi/\omega_2=2\pi/2\omega_1$ (Fig. I-8B). Thus a higher frequency implies a sharper shape of the covariance function. On the other hand a lower frequency will lead to a widening of the covariance function, i.e. with a slower decay of the covariance with increasing time lag (Fig. I-8C). If, in the limit case, the frequency becomes zero, i.e. no motions at all, the covariance has a constant value (Fig. I-8D). In that case the correlation coefficient remains 1 for all values of τ.

Since the Fourier transform is a linear operation the function to be transformed can be split up in partial functions which can be transformed separately. Finally their transforms can be added to obtain the resulting transform.

If the energy $1/2\ A^2$ is divided over two frequencies, one being zero and the other $\omega=2.2\pi/T$ (Fig. I-9A), then a combination can be formed of the cases of Figs. I-8A and I-8D. The transform is a cosine function around a value $1/4\ A^2$ (Fig. I-9A, right hand figure). In Fig. I-9B the energy is partitioned over three frequencies 0, $2\pi/T$ and $4\pi/T$. The covariance function is a combination of two cosine functions and a constant value.

Distributing the energy over more frequencies, for instance over 5 (Fig. I-9C) or 9 frequencies (Fig. I-9D) yields corresponding covariance functions. Increasing this number of frequencies will finally lead to a continuous power spectrum. In Fig. I-9E this has a constant value $1/4\ A^2T$, ranging from $\omega=0$ to $\omega=2.2\pi/T$. The covariance function can be found by replacing the function of Fig. I-9D by the expression

$$\text{Cov y}(\tau) = \frac{1}{2} A^2 \frac{\int_0^{2\frac{2\pi}{T}} \cos\omega\tau d\omega}{\int_0^{2\frac{2\pi}{T}} d\omega} = \frac{1}{2} A^2 \frac{\sin 2(2\pi/T)\tau}{2(2\pi/T)\tau}. \quad \text{(I-42)}$$

The fraction on the right hand side is the Dirichlet function of $2(2\pi/T)\tau$, expressed as

$$\text{dif } 2(2\pi/T)\tau = \frac{\sin 2(2\pi/T)\tau}{2(2\pi/T)\tau}, \quad \text{(I-43)}$$

so that

$$\text{Cov y}(\tau) = \frac{1}{2} A^2 \text{ dif } 2(2\pi/T)\tau \quad \text{(I-44)}$$

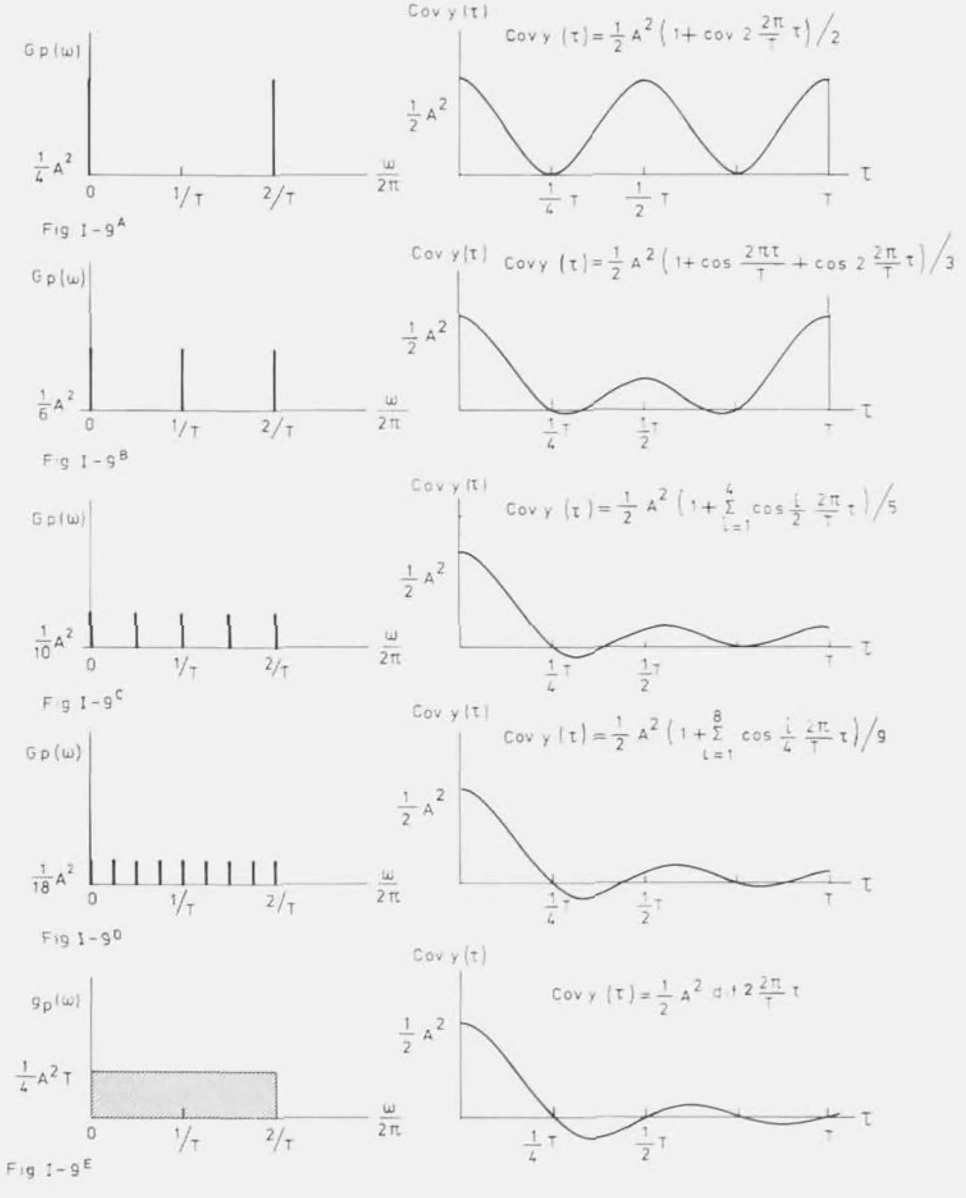


Fig. I-9. Power spectra and covariance functions of discrete harmonic functions, growing into a block.

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Fig. I-10. Power spectra and covariance functions of continuous functions, with increasing energy in high frequencies.

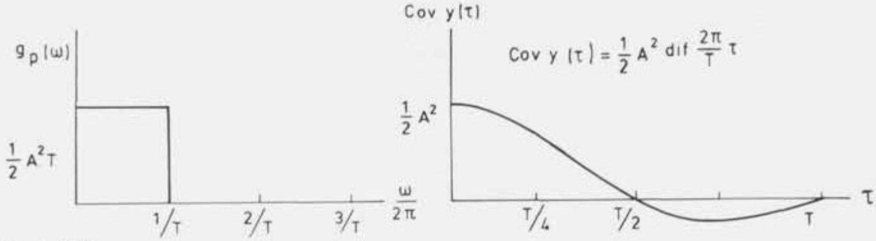

Fig I-10A

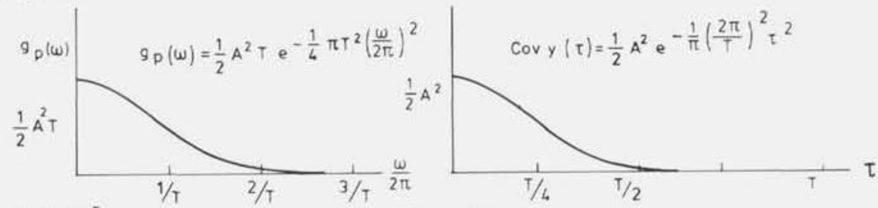

Fig I-10B

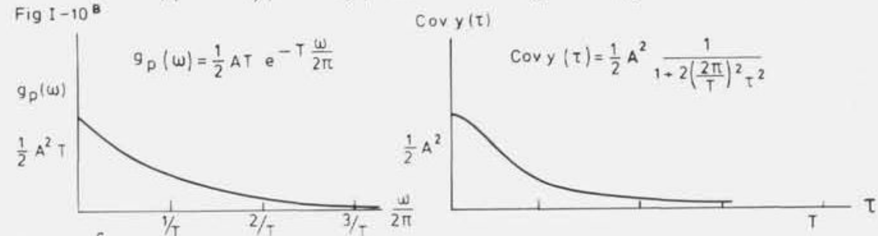

Fig I-10C

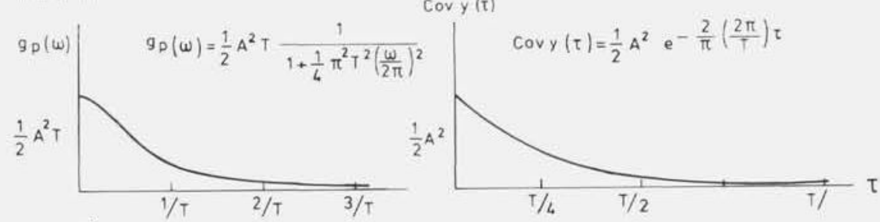

Fig. I-10D

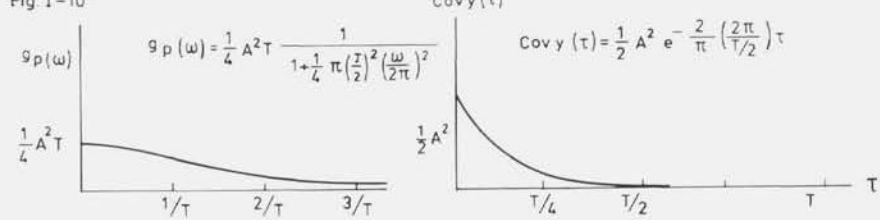

Fig. I-10E

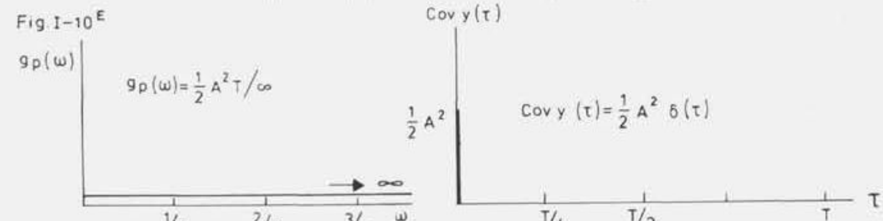

Fig. I-10F

This function shows a maximum covariance for $\tau=0$, decreasing and oscillating to zero with increasing τ. The frequencies included are cut off at $\omega=2.2\pi/T$. The shape of the covariance function near $\tau=0$ is rounded.

Reduction of the upper limit of the frequencies to $\omega=2\pi/T$ (Fig. I-10A) leads to a slower recession of the covariance function. Replacing the power spectrum by a gaussian function (Fig. I-10B) with an area of $\frac{1}{2}A^2$ and an initial value of $\frac{1}{2}A^2T$ carries a part of the low frequent energy to higher frequencies, asymptotically decreasing to zero at infinitely high frequencies. The covariance function, which for a gaussian power spectrum is also a gaussian function, retains it rounded shape near $\tau=0$.

In the above case the approach of the energy to zero for high frequencies was of a square exponential order. If the approach is somewhat weaker, following a linear exponential order (Fig. I-10C), which implies more energy in the higher frequencies, then the covariance function will be a Cauchy function, which still shows a rounded shape near $\tau=0$.

If the approach of the energy to zero is following a square power function, (i.e. the power spectrum itself follows a Cauchy function, Fig. I-10D) then there is so much energy in the high frequencies that the recession of the covariance function from $\tau=0$ is no longer rounded, but commences immediately to follow a certain slope.

The covariance function will recede stronger if more energy is carried over to high frequencies, as in Fig. I-10E. If finally all energy is uniformly distributed over all frequencies from 0 to ∞ then the covariance is described by a Dirac function $\delta(\tau)$, which means that it has a value of $\frac{1}{2}A^2$ for $\tau=0$ only, but becomes zero for all other time lags (Fig. I-10F). The correlation coefficient is 1 for $\tau=0$ and drops down to zero immediately. (Fig. I-10F).

Consequences of measurement errors to the covariances function

An important point that follows from the above discussion is the fact that absence of energy in the highest frequencies implies a horizontal initial course of the correlogram at $\tau=0$. This is of importance for the examination of the correlation coefficient in the near vicinity of a gauging station. Since infinitely high frequencies do not exist in the recorded data, a horizontal initial course is most likely in practical cases. This is the reason that, in extrapolating the correlograms to $\tau=0$, in several cases in Chapter 3, a horizontal initial course was observed.

A horizontal initial course of the correlogram is not a special feature of the above examples, but it is a general phenomenon if there are no components in the frequencies, greater than a certain finite value. This can be explained when considering the number of zero crossings of the variable under consideration. Papoulis (1965) shows that the probability $p_1(\tau)$ of one zero crossing in a time interval $(t, t+\tau)$ amounts to

$$p_1(\tau) = \frac{1}{\pi} \sqrt{- \frac{2\varrho'(0).\ \tau}{\varrho(0)}} \tag{I-45}$$

If $p_1(\tau) = 0$, but $\tau \neq 0$, then it follows that

$$\varrho'(0) = 0 \tag{I-46}$$

As was described in Chapter 3 the measured water level data were the result of a complicated filtering and elaboration process. Whatever the result of this process may be, variations below a certain limit are excluded. On the other hand it seems reasonable to assume that measurement errors will occur in all remaining frequencies. Since there is no reason that certain frequencies might be preferred, the assumption can be made that the measurement errors will be equally distributed over the frequencies concerned.

For the measurement errors, a spectrum as given in Fig. I-10A, showing a block function seems acceptable. The variance amounts to $\frac{1}{2}A^2=\varepsilon^2$, whereas the covariance function starts with a value of ε^2 for $\tau=0$. If the upper limit of the frequencies is at $\omega=2\pi/T$ the autocorrelation coefficient is zero for $\tau=T/2$, and shows only small values for longer time lags. If for instance $T=10$ min. the noticable influence of the measurement errors on the covariance ranges no further than up to a time lag of 5 min.

Now consider a power spectrum consisting of equally partitioned low power measurement errors and gaussion partitioned high power variations, ranging over relatively long time lags, thus over relatively low frequencies.

As an example, let the following values be given to the power density spectra of their variations:

- measurement errors:
  - variance $\frac{1}{2}A_1^2 = \varepsilon^2 = 6{,}25\ cm^2$
  - maximum measured frequency*) $\omega_1 = 2\pi/T_1 = 2\pi/10\ min^{-1}$

*) This is half of the frequency of measurement.

- high power variations:
  - variance $\frac{1}{2} A_2^2 = \text{Var } y = 25 \text{ cm}^2$
  - frequency parameter $\omega_2 = 2\pi/T_2 = 2\pi/360$ min (see Fig. I-10B)

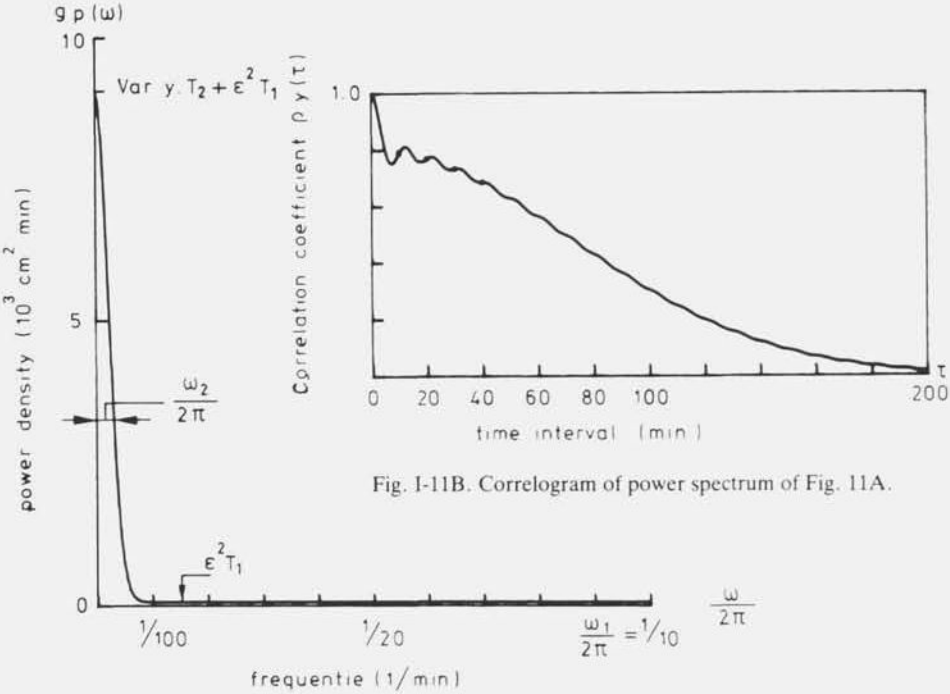

Fig. I-11B. Correlogram of power spectrum of Fig. 11A.

Fig. I-11A. Power density spectrum.

The power density spectrum of both variations together is shown in Fig. I-11A, and it is defined as

$$g_p(\omega) = \varepsilon^2 T_1 \beta(0,\omega,2\pi/T_1) + \text{Var } y . T_2 \exp\{-\frac{1}{4}\pi T_2^2(\omega/2\pi)^2\} \qquad \text{(I-47)}$$

The two right hand terms can be recognized by comparison with Figures I-10A and I-10B. The covariance function can be derived from the same figures. It follows:

$$\text{Cov } y(\tau) = \varepsilon^2 \text{dif}(2\pi/T_1)\tau + \text{Var } y.\exp\left\{ -\frac{1}{\pi}\left(\frac{2\pi}{T_2}\right)^2 \tau^2 \right\} \qquad \text{(I-48)}$$

The autocorrelation coefficient can be derived by standardizing this function at a value 1 for $\tau=0$, i.e. by dividing by

$$\text{Cov } y(0)= \varepsilon^2+\text{Var } y \ (=\text{Var } \underset{\sim}{y}). \qquad \text{(I-49)}$$

Thus

$$\varrho(\tau) = \frac{1}{\varepsilon^2+\text{Var } y}\left[\varepsilon^2 \text{dif}(2\pi/T_1) + \text{Var } y.\exp\left\{-\frac{1}{\pi}\left(\frac{2\pi}{T_2}\right)^2\tau^2\right\}\right] \qquad \text{(I-50)}$$

This function, called the correlogram, is shown in Fig. I-11B. Starting at $\varrho=1$ for $\tau=0$ it rapidly drops down to a much lower value where, after some small oscillations it takes the gaussian shape for increasing time intervals.

When calculating the correlogram from a series of measurements, carried out with 10/2=5 min. intervals, these oscillations appear as points, as seen in the plot of Fig. I-11B. Fitting a curve through these points and extrapolating to $\tau=0$ yields a value of $\varrho(0)<1$. This is the value $\tilde{\varrho}(0)$ of Fig. 3-6. The numerical value follows from eq (I-48), disregarding the first term in brackets. Thus

$$\tilde{\varrho}(0) = \frac{\text{Var } y}{\varepsilon^2+\text{Var } y}. \qquad \text{(I-51)}$$

This coefficient can give an indication of the standard error of measurement ε, as was discussed in Section 3.3.3.

In the second example one harmonic function with a period of T_3 = 180 min is added to the power spectrum. Let this have an amplitude of H = 7 cm. Then the variance amounts to

$$\tfrac{1}{2} H^2 = 25 \text{ cm}^2$$

In Fig. I-12A the corresponding power spectrum is shown. This figure is of a mixed character, since the discrete part of the power spectrum is expressed in variance $[l^2]$, but the continuous part in variance density (or power density) $[l^2t]$. Thus two different vertical scales are used.

Because of the great variations in the values of the power function $g_p(\omega)$ a log scale is often preferred, thus reducing the high values and increasing the low values. A log scale is used in Fig. 1-12B.

In the correlation function the harmonic component has a contribution of (see also Fig. I-8A)

$$\tfrac{1}{2}H^2\cos(2\pi/T_3)\omega. \qquad \text{(I-52)}$$

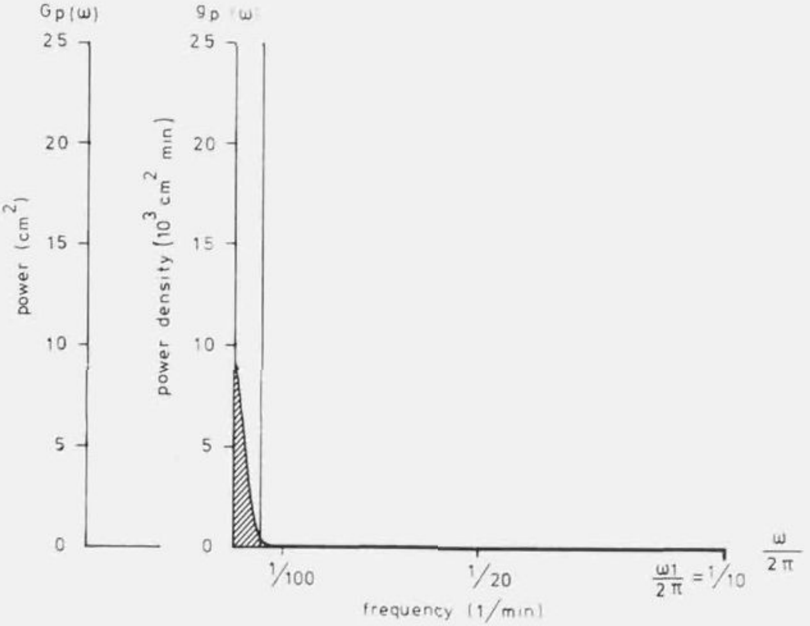

Fig. I-12A. Power spectrum of Fig. I-11A, with an additional harmonic component.

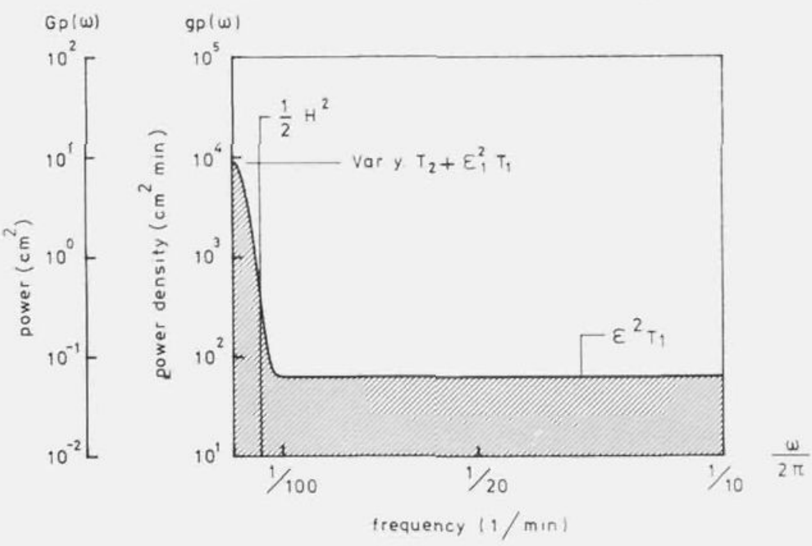

Fig. I-12B. Power spectrum of fig. I-12A displayed using a log scale.

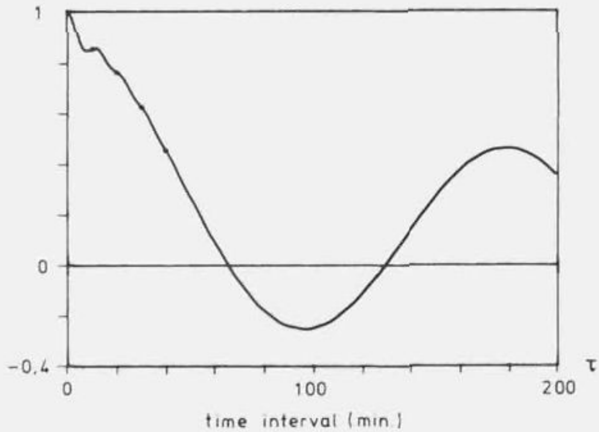

Fig. I-12C. Correlogram of power spectrum of Fig. I-12A.

This leads to the following equation for the covariance function (compare eq (I-46)):

$$\text{Cov } y(\tau) = \varepsilon^2 \text{dif}(2\pi/T_1)\tau + \text{Var } y.\exp\left\{-\frac{1}{\pi}\left(\frac{2\pi}{T_2}\right)^2\tau^2\right\} + \frac{1}{2}H^2\cos(2\pi/T_3)\tau \tag{I-53}$$

Dividing by:

$$\text{Cov } y(0) = \varepsilon^2 + \text{Var } y + \frac{1}{2}H^2 \tag{I-54}$$

then gives the equation for the correlation coefficient as

$$\varrho(\tau) = \frac{1}{\varepsilon^2 + \text{Var } y + {}^1\!/\!_2 H^2}\left[\varepsilon^2 \text{dif}(2\pi/T_1)\tau + \text{Var } y.\exp\left\{-\frac{1}{\pi}\left(\frac{2\pi}{T^2}\right)^2\tau^2\right\} + \frac{1}{2}H^2\cos(2\pi/T_3)\tau\right]. \tag{I-55}$$

This function is shown in Fig. I-12C, where the influence of the harmonic function is clearly shown. Correlograms like this are typical of tidal waters. In this connection reference is made to Fig. 3-3. The sampling frequency will hide the oscillations near $\tau = 0$ from view.*)

*) Extension of the sampling frequency too will not show these oscillations, because in that case their frequency will be increased to the same extent.

As a rule in the tidal waters a number of harmonic components will be present in the power spectrum. This produces will cause a more complicated correlogram, but no specific new problems are introduced. The components with the highest frequencies considered in practice always should have lower frequencies than the measurement frequency.

Annex II

Correlation over time and distance

Coherence between water levels exists not only over time intervals, as was decribed in Annex I, but also over distance intervals. In principle the same discussion for time series is relevant for functions of distance. In the distance case frequency is expressed in times per unit length instead of in times per time unit. In particular the discussion about the correlation decay from the origin (this time at $z = 0$) is similar for time and for distance.

Correlation over both time and distance can also be considered. If, for instance, a wave phenomenon moves unchanged with a propagation velocity c, then a level y, occurring at t_0 in $z = 0$ will be in $z = z_1$ at a time $t_0 + \tau_1$, where $\tau_1 = z_1/c$; then

$$y(z=0;\ \tau=0) = y(z=z_1;\ \tau=\tau_1=z_1/c) \qquad \text{(II-1)}$$

For a site, $z=z_1$, the behaviour of the waterlevel y can be described by the equation, holding for $z=0$, but with τ replaced by $\tau=z/c$ (Fig. II-1).

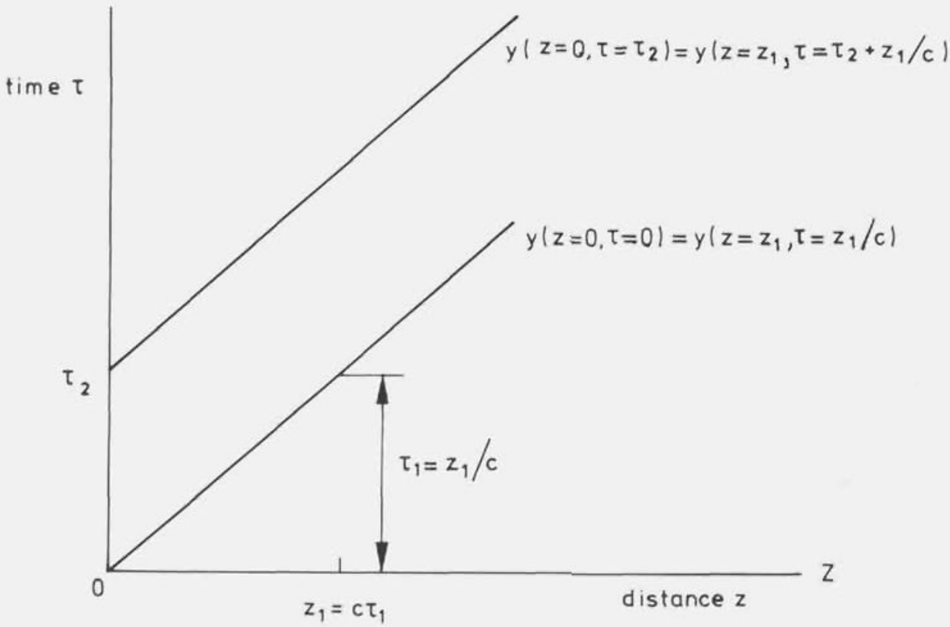

Fig. II-1. Time distance diagram showing the relation between waterlevels at various sites.

In this example the individual waves remain unchanged when moving ahead, but each wave can have a different shape.

The correlation between waterlevels at different sites and at different times can be expressed by replacing the time interval τ by a time-distance interval $\tau = z/c$. If the correlation coefficient between levels at different times at one station is given by eq (I-50), namely

$$\varrho(\tau) = \frac{1}{\varepsilon^2 + \text{Var } y}\left[\varepsilon^2 \text{dif}(2\pi/T_1)\tau + \text{Var } y.\exp\left\{-\frac{1}{\pi}\left(\frac{2\pi}{T_2}\right)^2 \tau^2\right\}\right]$$

then the correlation coefficient between levels at different times and at two stations located a distance z apart, is

$$\varrho(z,\tau) = \frac{1}{\varepsilon^2 + \text{Var } y}\left[\varepsilon^2 \text{dif}(2\pi/T_1)(\tau - z/c) + \text{Var } y.\exp\left\{-\frac{1}{\pi}\left(\frac{2\pi}{T_2}\right)^2 (t - z/c)^2\right\}\right] \tag{II-2}$$

This is called the cross correlation function. Its shape is similar to Fig. I-11B when shifted horizontally over an amount z/c.

When $\tau = 0$ eq (II-2) transforms into the following expression for spatial correlation:

$$\varrho(z,0) = \frac{1}{\varepsilon^2 + \text{Var } y}\left[\varepsilon^2 \text{dif}(2\pi/T_1)(z/c) + \text{Var } y.\exp\left\{-\frac{1}{\pi}\left(\frac{2\pi}{T_2}\right)^2 (z/c)^2\right\}\right] \tag{II-3}$$

If moving along a line $z = c\tau$ then the correlation coefficient remains 1, so

$$\varrho(z, z/c) = 1 \tag{II-4}$$

The correlation coefficient $\varrho(z,z/c)$ can also be influenced by factors, affecting distance only, e.g. by local disturbances. Suppose the spatial correlation has a similar structure to that of the autocorrelation. Then the correlation coefficient no longer remains constant at 1 when moving along in line $z = c\tau$ but will, for instance, have a behaviour described by

$$\varrho(z, z/c) = \frac{1}{\varepsilon^2 + \text{Var } y}\left[\varepsilon^2 \text{dif}(2\pi/Z_1)z + \text{Var } y.\exp\left\{-\frac{1}{\pi}\left(\frac{2\pi}{Z_2}\right)^2 z^2\right\}\right] \tag{II-5}$$

The correlation coefficient for the general case can be expressed as

$$\varrho(z,\tau) =$$

$$\frac{1}{\varepsilon^2+\text{Var } y}\left[\varepsilon^2\text{dif}(2\pi/T_1)(\tau\text{-}z/c).\text{dif}(2\pi/Z_1)z+\right.$$

$$\left.\text{Var } y.\exp\left[-\frac{1}{\pi}\left\{\left(\frac{2\pi}{T_2}\right)^2(\tau\text{-}z/c)^2+\left(\frac{2\pi}{Z_2}\right)^2 z^2\right\}\right]\right]. \qquad \text{(II-6)}$$

The dif-functions have only values beyond the points and the times where measurements are carried out. So these functions will never be found by measurement. Measurement can only produce the relation

$$\varrho(z,\tau)= \frac{1}{\varepsilon^2 + \text{Var } y}\text{Var } y.\ \exp\left[-\frac{1}{\pi}\left\{\left(\frac{2\pi}{T_2}\right)^2(\tau\text{-}z/c)^2+\left(\frac{2\pi}{Z_2}\right)^2 z^2\right\}\right] \qquad \text{(II-7)}$$

Extrapolation to z=0 and τ=0 yields:

$$\tilde{\varrho}(0,0) = \frac{\text{Var } y}{\varepsilon^2+\text{Var } y}, \qquad \text{(II-8)}$$

which is equal to eq (I-49) and again gives the correlation coefficient for distance zero when influenced by measurement errors.

In the case where harmonic components are present in the water movement the correlation function of eq (II-7) can be transformed in such a way that these components are included. If there is one harmonic component the correlation coefficient might be as follows:

$$\varrho(z,\tau) = \frac{1}{\varepsilon^2+\text{Var } y+{}^1\!/\!_2 H^2}\left[\text{Var } y.\exp\left[-\frac{1}{\pi}\left\{\left(\frac{2\pi}{T_2}\right)^2(\tau\text{-}z/c)^2+\left(\frac{2\pi}{Z_2}\right)^2 z^2\right\}\right]\right.$$

$$\left.+{}^1\!/\!_2 H^2\cos\left(\frac{2\pi}{T_3}\right)(\tau\text{-}z/c).\right] \qquad \text{(II-9)}$$

If there are n tidal components, then

$$\varrho(z,\tau)=$$

$$\frac{1}{\varepsilon^2+\mathrm{Var}\ y+{}^1\!/\!_2\sum_{i=1}^{n} H_i^2}\left[\mathrm{Var}\ y.\exp\left[-\frac{1}{\pi}\left\{\left(\frac{2\pi}{T_2}\right)^2(\tau\text{-}z/c)^2+\left(\frac{2\pi}{Z_2}\right)^2 z^2\right\}\right]\right.$$

$$\left.+\frac{1}{2}\sum_{i=1}^{n} H_i^2\cos\left(\frac{2\pi}{T_3}\right)(\tau\text{-}z/c).\right] \qquad \text{(II-10)}$$

It is assumed here that the tidal components are not subject to recession but that an approximately fixed relation over time and distance will remain.

Annex III

The variance of the errors of estimate

In this Annex an expression is derived for the variance of the errors of estimate, i.e. Var $\Delta\hat{y}$.

From eq *(5-30)*:

$$\text{Var}\ \Delta\hat{y} = \text{Var}\ \Delta\hat{y} - \varepsilon_y^2. \tag{III-1}$$

Further, according to eq (5-32):

$$\text{Var}\ \Delta y = \frac{\sum_{j=1}^{n} \Delta y_j^2}{n}, \tag{III-2}$$

and, according to eq (5-34):

$$\sum_{j=1}^{n} \Delta y_j^2 = \underline{Y}^T\underline{Y} - (\underline{X}^T\underline{Y})^T \{(\underline{X}^T\underline{X})^{-1}\}^T(\underline{X}^T\underline{Y}). \tag{III-3}$$

Now consider the matrix $(\underline{X}^T\underline{X})^{-1}$. According to Cramer's rule, this inversion may be written

$$(\underline{X}^T\underline{X})^{-1} = \frac{\{(\underline{X}^T\underline{X})^*\}^T}{|\underline{X}^T\underline{X}|} \tag{III-4}$$

where:

$(\underline{X}^T\underline{X})^*$ = the matrix of the minors of $\underline{X}^T\underline{X}$

$|\underline{X}^T\underline{X}|$ = the value of the determinant of $\underline{X}^T\underline{X}$

The second term of (III-3) can be written as

$$(\underline{X}^T\underline{Y})^T \{(\underline{X}^T\underline{X})^{-1}\}^T(\underline{X}^T\underline{Y}) = \frac{1}{|\underline{X}^T\underline{X}|} (\underline{X}^T\underline{Y})^T \{(\underline{X}^T\underline{X})^*\} (\underline{X}^T\underline{Y}) \tag{III-5}$$

where the expression $(\underline{X}^T\underline{Y})^T (\underline{X}^T\underline{X})^* (\underline{X}^T\underline{Y})$ is a scalar. It can be expressed as the determinant of a matrix as given below:

$$(\underline{X}^T\underline{Y})^T \{(\underline{X}^T\underline{X})^*\} (\underline{X}^T\underline{Y}) = - \begin{vmatrix} 0 & (\underline{X}^T\underline{Y})^T \\ (\underline{X}^T\underline{Y}) & (\underline{X}^T\underline{X}) \end{vmatrix} \tag{III-6}$$

This can be readily seen by the fact that each term of the determinant development consists of a product of one element of the first row, namely $(\underline{X}^T\underline{Y})^T$, one element of the first column, namely $(\underline{X}^T\underline{Y})$ and the minor determinant of the corresponding element in the matrix $(\underline{X}^T\underline{X})$. Reference should be made to Fig. III-1A.

The minor determinants of $|\underline{X}^T\underline{X}|$ change sign when taken up into the determinant of eq (III-6), since a reverse sequence is introduced by the elements of the first row and column. This can best be demonstrated when considering the first element of each, leading to a negative sign. See Fig. III-1B.

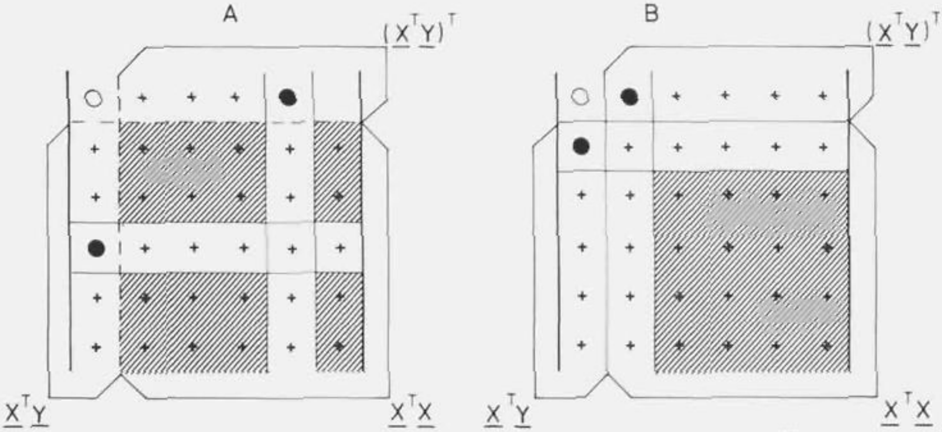

Fig. III-1. Elements contributing to the value of the determinant of eq (III-6).

Now substitution of eq (III-6) into eq (III-3) yields:

$$\sum_{j=1}^{n} \Delta y_j^2 = \frac{(\underline{Y}^T\underline{Y})|\underline{X}^T\underline{X}| - (\underline{X}^T\underline{Y})^T \{(\underline{X}^T\underline{X})^*\} (\underline{X}^T\underline{Y})}{|\underline{X}^T\underline{X}|} \tag{III-7}$$

$$= \frac{\underline{Y}^T\underline{Y}|\underline{X}^T\underline{X}| + \begin{vmatrix} 0 & (\underline{X}^T\underline{Y})^T \\ (\underline{X}^T\underline{Y}) & (\underline{X}^T\underline{X}) \end{vmatrix}}{|\underline{X}^T\underline{X}|}$$

$$= \frac{\begin{vmatrix} \underline{Y}^T\underline{Y} & (\underline{X}^T\underline{Y})^T \\ (\underline{X}^T\underline{Y}) & (\underline{X}^T\underline{X}) \end{vmatrix}}{|\underline{X}^T\underline{X}|} \qquad \text{(III-8)}$$

The numerator, to be called N, follows when elaborating eq (5-5)*)

$$N = \begin{vmatrix} \Sigma\underline{y}^2 & \Sigma\underline{y} & \Sigma\underline{x}_1\underline{y} & \cdots & \Sigma\underline{x}_m\underline{y} \\ \Sigma\underline{y} & \Sigma n & \Sigma\underline{x}_1 & & \Sigma\underline{x}_m \\ \Sigma\underline{x}_1\underline{y} & \Sigma\underline{x}_1 & \Sigma\underline{x}_1^{\,2} & & \Sigma\underline{x}_m\underline{x}_1 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \Sigma\underline{x}_m\underline{y} & \Sigma\underline{x}_m & \Sigma\underline{x}_1\underline{x}_m & \cdots & \Sigma\underline{x}_m^{\,2} \end{vmatrix} \qquad \text{(III-9)}$$

Exchanging the first and second rows and also the first and second columns, and dividing all elements by n, gives

$$n^{-(m+1)}.N = \begin{vmatrix} 1 & \bar{\underline{y}}_1 & \bar{\underline{x}}_1 & \cdots & \bar{\underline{x}}_m \\ \bar{\underline{y}} & \frac{\Sigma\bar{\underline{y}}^2}{n} & \frac{\Sigma\underline{x}_1\underline{y}}{n} & & \frac{\Sigma\underline{x}_m\underline{y}}{n} \\ \bar{\underline{x}}_1 & \frac{\Sigma\underline{x}_1\underline{y}}{n} & \frac{\Sigma\underline{x}_1^{\,2}}{n} & & \frac{\Sigma\underline{x}_m\underline{x}_1}{n} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \bar{\underline{x}}_m & \frac{\Sigma\underline{x}_m\underline{y}}{n} & \frac{\Sigma\underline{x}_1\underline{x}_m}{n} & \cdots & \frac{\Sigma\underline{x}_m^{\,2}}{n} \end{vmatrix} \qquad \text{(III-10)}$$

*) Here Σ stands for $\sum_{j=1}^{m}$

Subtracting from each i^{th} column except the first, the values of the first column, multiplied by the first element of the i^{th} column leaves the value of the determinant unchanged, and gives

$$n^{-(m+1)}.N = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ \bar{y} & \frac{\Sigma y^2}{n} - \bar{y}^2 & \frac{\Sigma x^1 y}{n} - \bar{x}_1\bar{y} & & \frac{\Sigma x_m y}{n} - \bar{x}_m\bar{y} \\ \bar{x}_1 & \frac{\Sigma x_1 y}{n} - \bar{x}_1\bar{y} & \frac{\Sigma x_1^2}{n} - \bar{x}_1^2 & & \frac{\Sigma x_m x_1}{n} - \bar{x}_m\bar{x}_1 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \bar{x}_m & \frac{\Sigma x_m y}{n} - \bar{x}_m\bar{y} & \frac{\Sigma x_1 x_m}{n} - \bar{x}_1\bar{x}_m & \cdots & \frac{\Sigma x^2_m}{n} - \bar{x}^2_m \end{vmatrix} \quad \text{(III-11)}$$

The elements of this matrix apart from the first rows and columns, are variances and covariances. Without changing the value of the matrix one can write:

$$n^{-(m+1)}.N = \begin{vmatrix} \text{Var } y & \text{Cov } x_1 y & \cdots & \text{Cov } x_m y \\ \text{Cov } x_1 y & \text{Var } x_1 & & \text{Cov } x_1 x_m \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \text{Cov } x_m y & \text{Cov } x_1 x_m & \cdots & \text{Var } x_m \end{vmatrix} \quad \text{(III-12)}$$

Now dividing the first row by $\sqrt{\text{Var } y}$, the second row by $\sqrt{\text{Var } x_1}$, etc., until the last row by $\sqrt{\text{Var } x_m}$ and repeating this operation for all columns yields the result that the value of the determinant is divided by

$$\text{Var } y \prod_{i=1}^{m} \text{Var } x_i, \quad \text{(III-13)}$$

whereas the matrix now contains the correlation coefficients $\varrho(\underline{y}\ \underline{x}_1)$ and $\varrho(\underline{x}_i\underline{x}_k)$ (i=1...m, k=1...m). Thus:

$$n^{-(m+1)}.N = \text{Var}\ \underline{y} \prod_{i=1}^{m} \text{Var}\ \underline{x}_i |R(\underline{XY})|,$$

or

$$N = n^{m+1}\ \text{Var}\ \underline{y} \prod_{i=1}^{m} \text{Var}\ \underline{x}_i R(\underline{XY})| \qquad \text{(III-14)}$$

where

$$|R(\underline{XY})| = \begin{vmatrix} 1 & \varrho(\underline{x}_1\underline{y}) & \cdots & \varrho(\underline{x}_m\underline{y}) \\ \varrho(\underline{x}_1\underline{y}) & 1 & & \varrho(\underline{x}_1\underline{x}_m) \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \varrho(\underline{x}_m\underline{y}) & \varrho(\underline{x}_1\underline{x}_m) & \cdots & 1 \end{vmatrix} \qquad \text{(III-15)}$$

The denominator of eq (III-7) can be developed along the same lines to give

$$|\underline{X}^T\underline{X}| = n^m \prod_{i=1}^{m} \text{Var}\ \underline{x}_i |R(\underline{XX})|, \qquad \text{(III-16)}$$

where $|R(\underline{X}\ \underline{X})|$is the covariance matrix of the $\underline{x}_i$-values only.

Now, substituting the numerator N, as expressed by eq (III-14), and the denominator $|\underline{X}^T\underline{X}|$, as expressed by eq (III-16) into eq (III-7), gives

$$\Sigma\Delta y_j^2 = \frac{n^{m+1}\ \text{Var}\ \underline{y} \prod_{i=1}^{m} \text{Var}\ \underline{x}_i |R(\underline{XY})|}{n^m \sum_{i=1}^{m} \text{Var}\ \underline{x}_i |R(\underline{XX})|} = n\ \frac{|R(\underline{XY})|}{|R(\underline{XX})|}\ \text{Var}\ \underline{y}. \qquad \text{(III-17)}$$

With regard to eq (III-2) it follows that

$$\text{Var}\ \Delta y = \frac{|R(\underline{XY})|}{|R(\underline{XX})|}\ \text{Var}\ \underline{y} \qquad \text{(III-18)}$$

Note that this relates to samples variances but not to population variances. Now the following relation holds:

$$\text{Var } \underline{y} \text{ (population)} = \frac{n}{n-1} \text{ Var } \underline{y} \text{ (sample)} \qquad \text{(III-19)}$$

It follows for Var Δy, since there are (n-m-1) degrees of freedom, that

$$\text{Var } \Delta y \text{ (population)} = \frac{n}{n-m-1} \text{ Var } \Delta y \text{ (sample)} \qquad \text{(III-20)}$$

So, if eq (III-18) is converted to population variances using eqs (III-19) and (III-20) then

$$\text{Var } \Delta y = \frac{n-1}{n-m-1} \cdot \frac{|R(\underline{XY})|}{|R(\underline{XX})|} \text{ Var } \underline{y}. \qquad \text{(III-21)}$$

However, since as a rule in water level network problems the number of measurements n is large with respect to the number of measurement stations, m, the expression in eq (III-21) before the determinants quotient will be almost 1, i.e.

$$\frac{n-1}{n-m-1} \approx 1 \text{ if } n \gg m$$

So, for practical problems in most cases eq (III-18) can be used.

Finally it follows from eq (III-1) that the variance of estimate is

$$\text{Var } \Delta \hat{y} = \frac{|R(\underline{XY})|}{|R(\underline{XX})|} \text{ Var } \underline{y} - \varepsilon_y^2 \qquad \text{(III-22)}$$

Annex IV

Examination of the discharges in a tidal river reach

From the model, described in Chapters 7 and 8, it follows that it is possible to calculate, besides the water levels, also the flow velocities and discharges. Although it is not the primary aim of this study to examine the discharges, it may be interesting to find out to what extent the calculated discharges correspond to possible measured values.

The model estimate the velocities, averaged over the flow conveying cross sections. As was stated in Section 8.10, the calculated water levels and velocities depend on the ratio between the flow conveying widths and the storage widths, but not on the values of the widths themselves. However the discharges are proportional to the flow conveying widths. If the width ratios are correct, but the width values are not, then the calculated discharges will be only proportional to their actual values, but will numerically be different from them. Therefore in the context for which this model was developed not too much weight should be given to the numerical values of the calculated discharges. However the shape of the calculated discharge hydrograph can give some indication for the behaviour of the model, when compared with measured discharges.

Discharge measurements were carried out in the Western Scheldt estuary at 11 May 1982, 8 June 1982 and 11 June 1982 at sites indicated in Fig. IV-1. The sites as well as the time of measurement differ from those for which the calculations were carried out. For this reason a comparison between both results should be focussed on general features but not on numerical details.

Fig. IV-1 shows the results of these measurements, each related to one tidal cycle, Fig. IV-2 the calculated discharges for case 1 of Chapter 8, and Fig. IV-3 those for case 2. Case 1 is seen to produce too high discharges, whereas for case 2 better fit to the measured data of Fig. IV-1 is achieved. Note particularly the irregularity in the tidal curves, shortly before the maximum inflow. This phenomenon can be recognized in some of the calculated curves, and in particular in those of case 2 (Fig. IV-3).

Finally, Fig. IV-4 shows the results of case 1, but this time with all widths reduced to 75% of their original values. This did not influence the calculated water levels and velocities, but the discharges showed greater correspondence to the measured ones than previously (see Fig. IV-2).

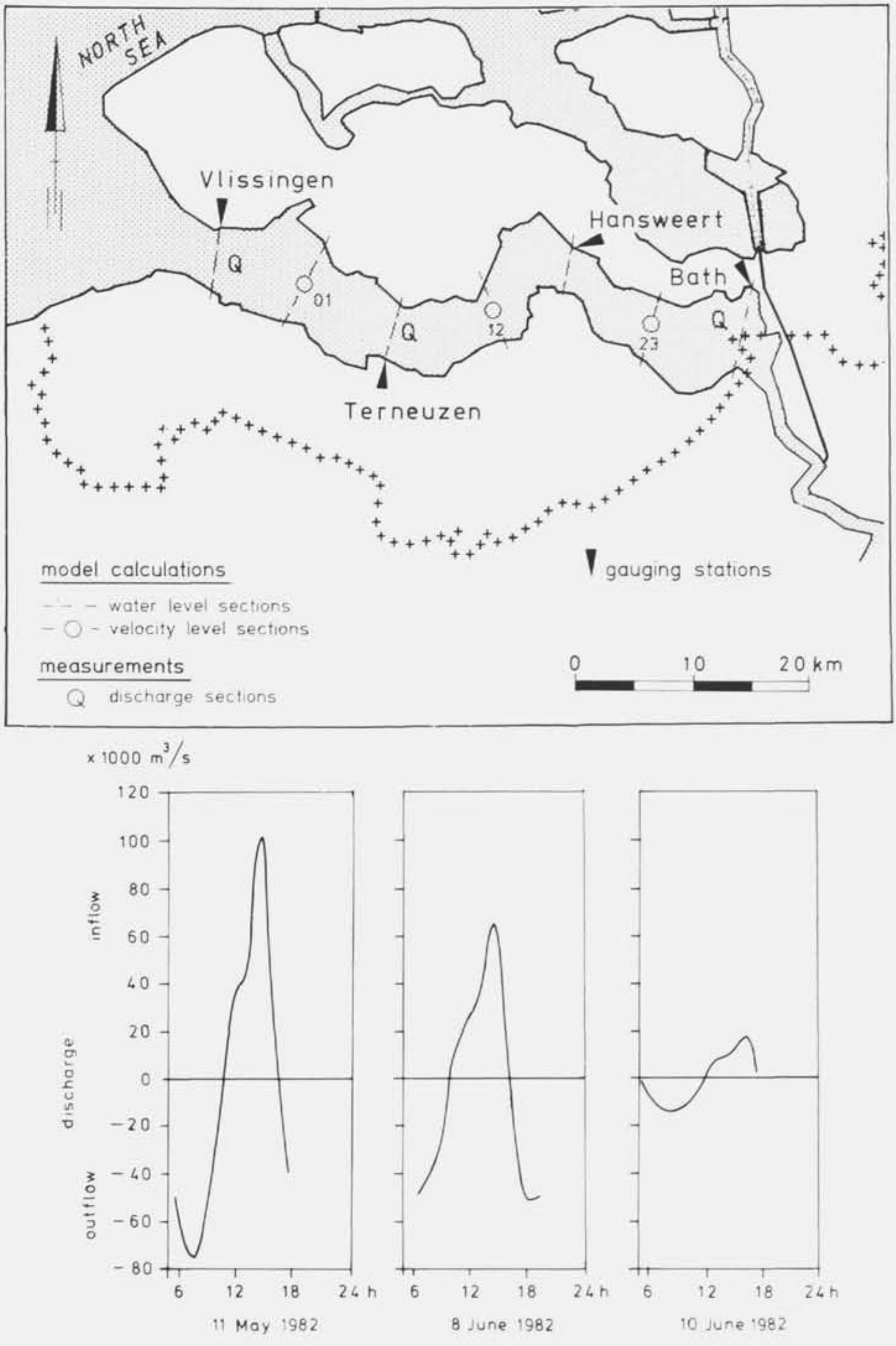

Fig. IV-1 Discharge measurements in the Western Scheldt estuary.

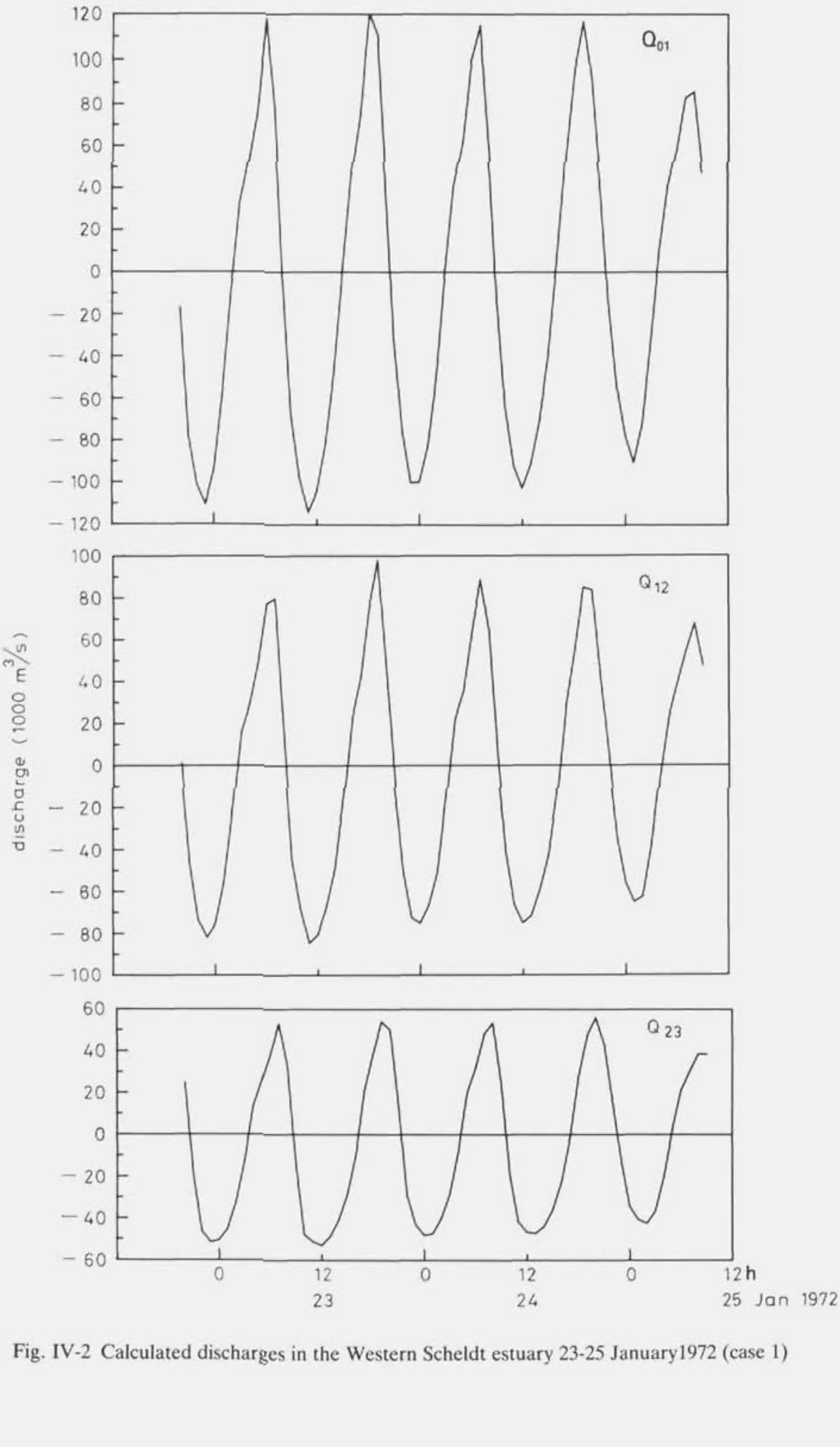

Fig. IV-2 Calculated discharges in the Western Scheldt estuary 23-25 January1972 (case 1)

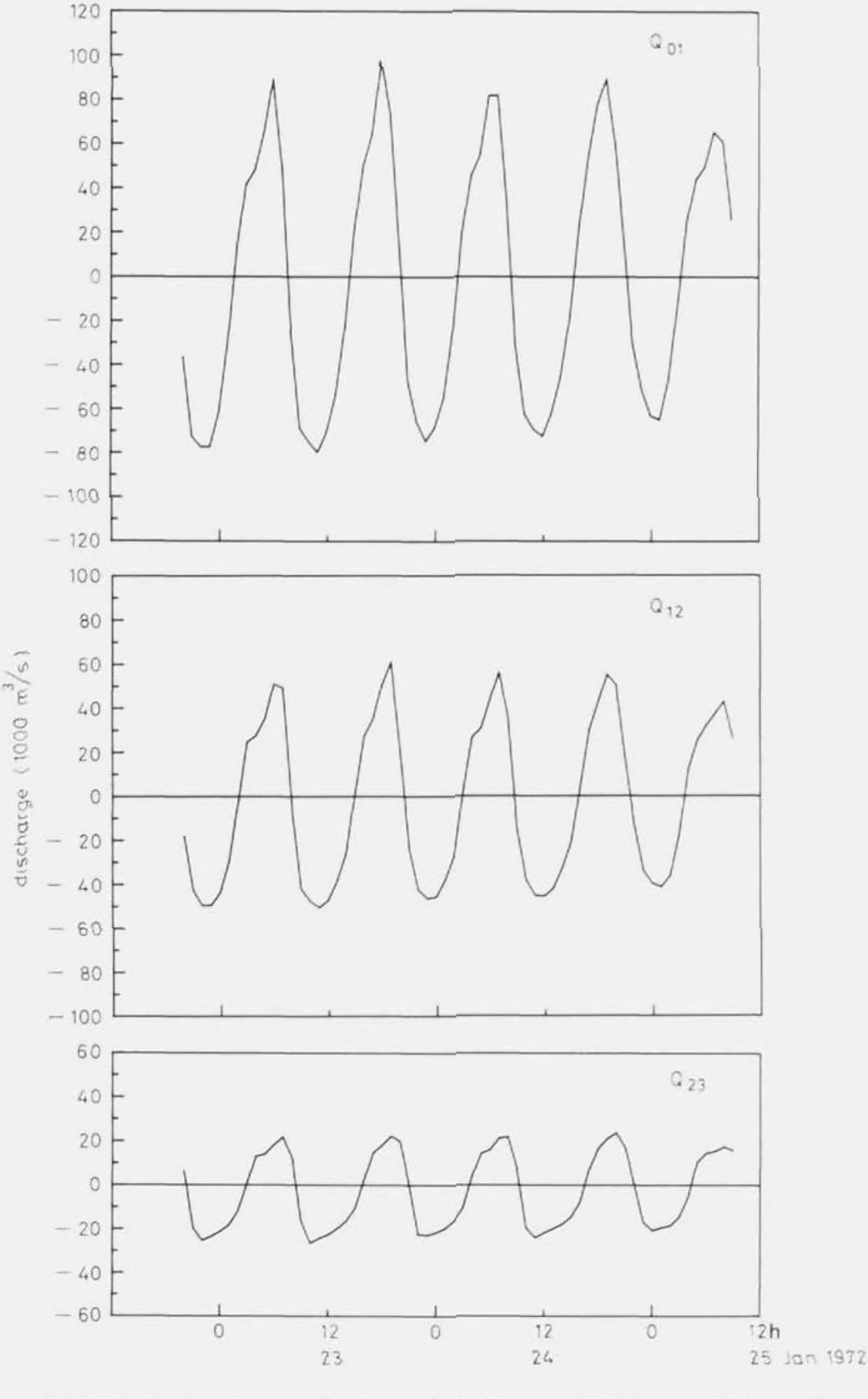

Fig. IV-3 Calculated discharges in the Western Scheldt estuary 23-25 January 1972 (case 2)

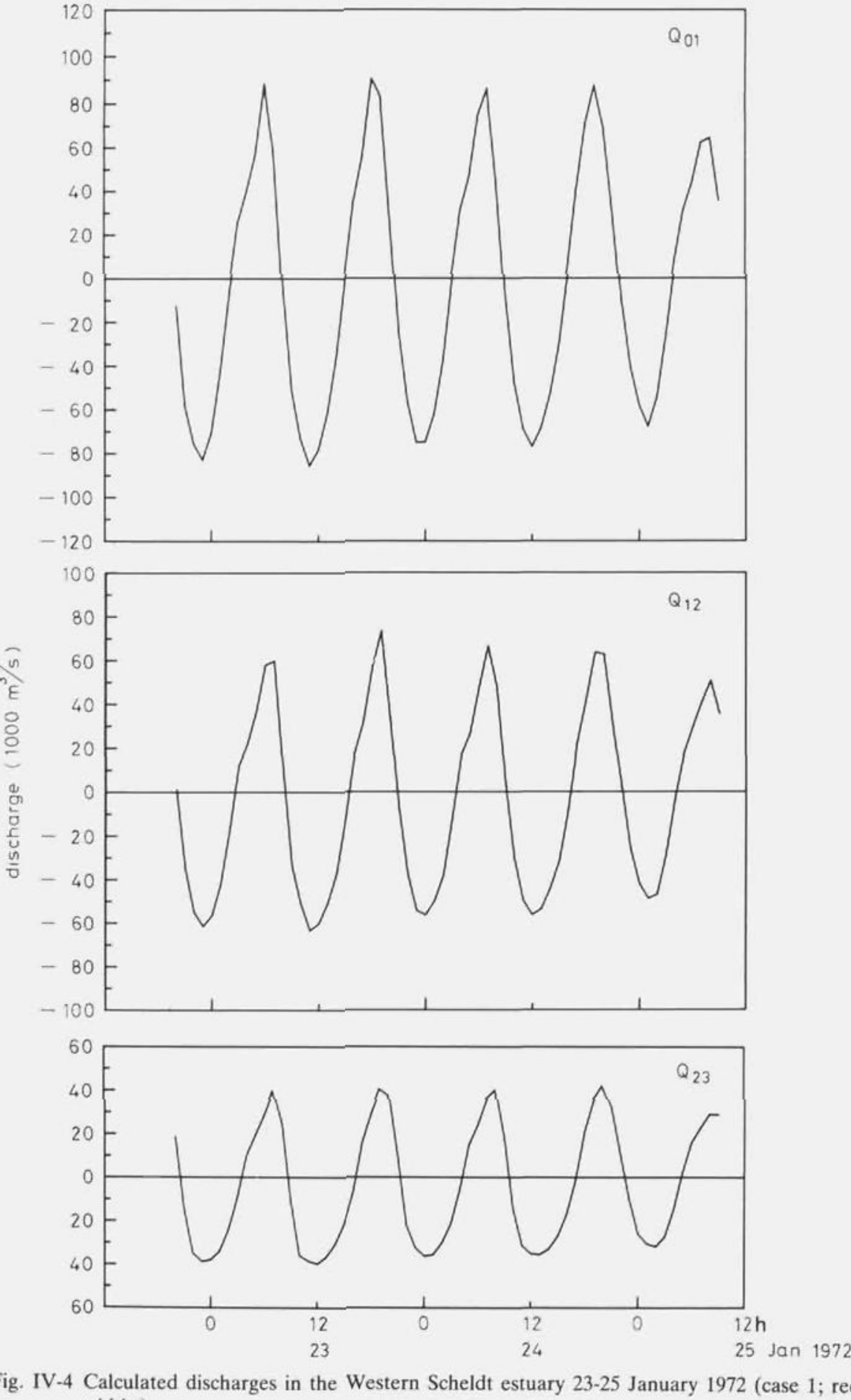

Fig. IV-4 Calculated discharges in the Western Scheldt estuary 23-25 January 1972 (case 1; reduced widths)

It should be remarked that for a good adjustment to actual discharge a number of measurements should be carried out during the period of examination. The results of these measurements could be used in the Kalman filter procedure. For exercises like this a long series of discharge data is required.

Annex V

Examination of the stability of the mathematical model

In this Annex the problem of model stability, introduced in Section 7.7 will be discussed in greater detail. Stability is assured if disturbance in the water level and velocity data is not amplified in the course of successive calculation steps. In order to examine the propagation of disturbances first consider the basic equations (7-10/11) and (7-12). The following simplifications will be introduced:

1. The flow conveying width is equal to the storage width, and they have also the same value along the whole river reach under consideration, so

   $$W_s = W_f = W. \qquad \text{(V-1)}$$

2. The streamflow profile, W(y-b), is also constant along the river reach. This means a fixed depth, H, where

   $$H = \bar{y}\text{-}\bar{b}. \qquad \text{(V-2)}$$

3. The river sections are of equal length, where

   $$d = 2\,\Delta z. \qquad \text{(V-3)}$$

4. In the first case to be considered there is no bottom friction, so

   $$1/C^2 = 0. \qquad \text{(V-4)}$$

   Later on a case with bottom friction will be examined.

5. The longitudinal changes of the velocity, expressed in the Bernouilli term, are neglected.

1. Case without bottom friction.

Because of these simplifications the basic equations become

Equation of continuity

$$\frac{\Delta y_i}{\Delta t} = -H\,\frac{v_{i,i+1} - v_{i-1,i}}{2\Delta z} \qquad \text{(V-5)}$$

Equation of motion

$$\frac{\Delta v_{i,i+1}}{\Delta t} = - g \frac{y_{i+1} - y_i}{2\Delta z} . \qquad \text{(V-6)}$$

The values of the water levels and velocities at time t + Δt , given the corresponding data at time t, are determined using the calculation procedure described in Section 7.6. Finite difference equations will be derived now, following the calculation scheme of Fig. V-1.

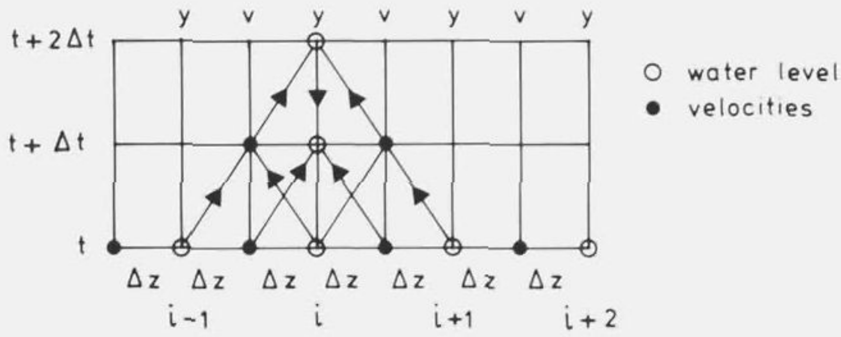

Fig. V-1 Calculation scheme

Remember that first provisional data are derived for time t + Δt and that these in turn are used to derive provisional data for time t + 2 Δt. The latter are used to find the final values for time t + Δt (see also to Figure 7-7).

First consider the water levels. The provisional value (indicated by*) for time t + Δt follows from eq (V-5)

$$\frac{y_i (t + \Delta t)^* - y_i(t)}{\Delta t} = - H \frac{v_{i,i+1} (t) - v_{i-1,i}(t)}{2\Delta z} , \qquad \text{(V-7)}$$

so that

$$y_i(t + \Delta t)^* = y_i(t) - H \frac{v_{i,i+1}(t) - v_{i-1,i} (t)}{2\Delta z}\Delta t. \qquad \text{(V-8)}$$

From eq (V-6) it follows in the same manner, that

$$v_{i,i+1}(t + \Delta t)^* = v_{i,i+1}(t) - g \frac{y_{i+1}(t)-y_i(t)}{2\Delta z} \Delta t, \tag{V-9}$$

and also

$$v_{i-1,i} (t + \Delta t)^* = v_{i-1,i}(t) - g \frac{y_i(t)-y_{i-1}(t)}{2\Delta z} \Delta t. \tag{V-10}$$

Now applying eq (V-7) one time step later, and using the above provisional data as input, it follows that

$$\frac{y_i(t + 2\Delta t)^* - y_i(t + \Delta t)^*}{\Delta t} = -H \frac{v_{i,i+1}(t+\Delta t)^* - v_{i-1,i}(t + \Delta t)^*}{2\Delta z}, \tag{V-11}$$

Substituting eqs (V-9) and (V-10) into eq (V-11), after some algebra, leads to

$$\frac{y_i(t + 2\Delta t)^* - y_i(t+\Delta t)^*}{\Delta t} = - H \frac{v_{i,i+1}(t) - v_{i-1,i}(t)}{2\Delta z} + $$

$$+ gH \frac{y_{i+1}(t) - 2 y_i(t) + y_{i-1}(t)}{4\Delta z^2} \Delta t. \tag{V-12}$$

The final value of $y_i(t + \Delta t)$ is derived using the weighted means of derivates of eqs (V-7) and (V-12)

$$y_i(t + \Delta t) = y_i(t)+\theta \frac{y_i(t + 2\Delta t)^* - y_i(t+\Delta t)^*}{\Delta t} + (1-\theta) \frac{y_i(t+\Delta t)^* - y_i(t)}{\Delta t} \tag{V-13}$$

so that,

$$y_i(t + \Delta t) = y_i(t) - H \frac{v_{i,i+1}(t) - v_{i-1,i}(t)}{2\Delta z} \Delta t$$

$$+ \theta gH \frac{y_{i+1}(t) - 2 y_i(t) + y_{i-1}(t)}{4\Delta z^2} \Delta t^2, \tag{V-14}$$

This differs from eq (V-8) in the third right hand term. A similar expression for the velocities can be derived, along the same lines giving

$$v_{i,i+1}(t + \Delta t) = v_{i,i+1}(t) - g \frac{y_{i+1}(t) - y_i(t)}{2\Delta z} \Delta t + \theta g H \frac{v_{i+1,i+2}(t) - 2\, v_{i,i+1}(t) + v_{i-1,i}(t)}{4\Delta z^2} \Delta t^2. \qquad \text{(V-15)}$$

Now assume disturbance Δv and Δy to occur, and that these may be amplified to the same extent as the variables themselves. This means that for the disturbances the same transformations can be used as for the levels and velocities.

The stability test of Neumann makes use of Fourier transforms of the data. One may write

$$y_i(t) = \sum_{\omega=-\infty}^{\infty} Y_\omega(t).e^{-j\omega z}, \qquad \text{(V-16a)}$$

$$v_{i,i+1}(t) = \sum_{\omega=-\infty}^{\infty} V_\omega(t).e^{-j\omega(z+\Delta z)}, \qquad \text{(V-16b)}$$

$$y_{i+1}(t) = \sum_{\omega=-\infty}^{\infty} Y_\omega(t).e^{-j\omega(z+2\Delta z)} \quad \text{etc}, \qquad \text{(V-16c)}$$

where $j = \sqrt{-1}$. (V-17)

Substitution of eqs (V-16a,b,c), and similar expressions for the other values required into eqs (V-14) and (V-16) yields, for each of the separate Fourier components, after some elaboration

$$Y_\omega(t + \Delta t) = Y_\omega(t) - H \frac{\Delta t}{\Delta z} V_\omega(t) \frac{e^{-j\omega\Delta z} - e^{j\omega\Delta z}}{2} + \theta g H \left(\frac{\Delta t}{\Delta z}\right)^2 Y_\omega(t) \frac{e^{-2j\omega\Delta z} - 2 + e^{2j\omega\Delta z}}{4}, \qquad \text{(V-18)}$$

which is equal to

$$Y_{\omega}(t + \Delta t) = Y_{\omega}(t) + H \frac{\Delta t}{\Delta z} V_{\omega}(t).j\sin\omega\Delta z - \theta gH \left(\frac{\Delta t}{\Delta z}\right)^2 Y_{\omega}(t)\sin^2\omega\Delta z. \tag{V-19}$$

It holds also that

$$V_{\omega}(t + \Delta t) = V_{\omega}(t) + g \frac{\Delta t}{\Delta z} Y_{\omega}(t).j\sin\omega\Delta z - \theta gH \left(\frac{\Delta t}{\Delta z}\right)^2 V_{\omega}(t)\sin^2\omega\Delta z. \tag{V-20}$$

Equations (V-19) and (V-20) can be expressed in matrix notation as

$$\begin{bmatrix} Y_{\omega}(t + \Delta t \\ V_{\omega}(t + \Delta t) \end{bmatrix} = \begin{bmatrix} 1 - \theta gH \left(\frac{\Delta t}{\Delta z}\right)^2 \sin^2\omega\Delta z & H \frac{\Delta t}{\Delta z}.\ j\sin\omega\Delta z \\ g \frac{\Delta t}{\Delta z} j\sin\omega\Delta z & 1 - \theta gH \left(\frac{\Delta t}{\Delta z}\right)^2 \sin^2\omega\Delta z \end{bmatrix} \begin{bmatrix} Y_{\omega}(t) \\ V_{\omega}(t) \end{bmatrix} \tag{V-21}$$

The necessary condition for stability is that the moduli of the eigenvalues of the transformation matrix in eq (V-21), to be called Φ (also called the amplification matrix), will both be less than or equal to unity

The eigenvalues, λ, are calculated by setting the determinant of the matrix $\lambda I - \Phi$ to zero, so

$$\begin{vmatrix} \lambda - 1 + \theta gH \left(\frac{\Delta t}{\Delta z}\right)^2 \sin^2\omega\Delta z & - H \frac{\Delta t}{\Delta z} j\sin\omega\Delta z \\ - g \frac{\Delta t}{\Delta z} j\sin\omega\Delta z & \lambda - 1 + \theta gH \left(\frac{\Delta t}{\Delta z}\right)^2 \sin^2\omega\Delta z \end{vmatrix} = 0 \tag{V-22}$$

or

$$\left\{\lambda - 1 + \theta gH \left(\frac{\Delta t}{\Delta z}\right)^2 \sin^2\omega\Delta z\right\}^2 + gH \left(\frac{\Delta t}{\Delta z}\right)^2 \sin^2\omega\Delta z = 0, \tag{V-23}$$

so that

$$\lambda = 1 - \theta gH \left(\frac{\Delta t}{\Delta z}\right)^2 \sin^2 \omega\Delta z \pm j \sqrt{gH}\frac{\Delta t}{\Delta z} \sin\omega\Delta z. \qquad (v\text{-}24)$$

The modulus of both λ - values is given by

$$|\lambda| = \sqrt{\left\{1 - \theta gh\left(\frac{\Delta t}{\Delta z}\right)^2 \sin^2 v\Delta z\right\}^2 + gH\left(\frac{\Delta t}{\Delta z}\right)^2 \sin^2\omega\Delta z}. \qquad (V\text{-}25)$$

This relation makes sense only for $0 \leqslant \sin^2 \omega\Delta z \leqslant 1$. Between these values it has no *maximum, because of the positive sign of the first term under the square root sign,* and which is 2nd-degree. Thus values of $\sin^2 \omega\Delta z$ between the boundaries 0 and 1 can never produce values of $|\lambda|$ which exceed the greatest of the boundary values. So it is sufficient to examine these boundary values only. For $\sin^2 \omega\Delta z = 0$, *then:*

$$|\lambda| = 1,$$

which just fulfils the requirement mentioned above.

For $\sin^2 \omega\Delta z = 1$, then

$$|\lambda| = \sqrt{\left\{1 - \theta gH \left(\frac{\Delta t}{\Delta z}\right)^2\right\}^2 + gH \left(\frac{\Delta t}{\Delta z}\right)^2}. \qquad (V\text{-}26)$$

For this expression it follows that

$$|\lambda| \leqslant 1 \text{ if } \theta \geqslant \frac{1 - \sqrt{1 - gH \left(\frac{\Delta t}{\Delta z}\right)^2}}{gh \left(\frac{\Delta t}{\Delta z}\right)^2} \qquad (V\text{-}27)$$

This equation shows that a solution can be found only if

$$gH \left(\frac{\Delta t}{\Delta z}\right)^2 \leqslant 1. \qquad (V\text{-}28)$$

If

$$gh \left(\frac{\Delta t}{\Delta z} \right)^2 = 1,$$

then $\theta = 1$, which is a very unfavourable condition for the accuracy (see eq (7-61)).

For small time intervals the lower limit of θ can be calculated as

$$\lim_{t \to 0} \theta = 0{,}5 \qquad \text{(V-29)}$$

so that

$$0{,}5 \leq \theta \leq 1 \qquad \text{(V-30)}$$

This means that the conditions for stability and accuracy *) cannot be fulfilled at the same time. Since stability must be assured some loss in accuracy has to be conceeded. As a rule θ can be chosen such that

$$0{,}5 < \theta < 0{,}6$$

2. Case with bottom friction

According to eq(7-12) a term, which includes the value $v_{i,i+1}|v_{i,i+1}|$ should be added to the right hand side of eq (V-6). However, the method described above can only be applied if linear relations are used. This is approximated by the term $-av_{i,i+1}$, in which

$$a = g \frac{\lceil v \rceil}{C^2 H} \qquad \text{(V-31)}$$

where $\lceil v \rceil$ denotes the mean of the absolute values of the velocities.

Thus eq (V-6) is replaced by

$$\frac{\Delta v_{i,i+1}}{\Delta t} = - g \frac{y_{i+1} \text{-} y_i}{2\Delta z} - av_{i,i+1} \qquad \text{(V-32)}$$

*) Eq (7-61), one of the accuracy conditions, reads: $\theta = 0{,}5$.

Repeating the above development, but this time using eq (V-32), gives the following relations:

$$y_i(t+\Delta t) = y_i(t) - H(1-\theta a\Delta t)\ \frac{v_{i,i+1}(t)-v_{i-1,i}(t)}{2\Delta z}\ \Delta t$$

$$+\ \theta gH\ \frac{y_{i+1}(t)-2y_i(t)+y_{i-1}(t)}{4\Delta z^2}\ \Delta t^2 \qquad \text{(V-33)}$$

and

$$v_{i,i+1}(t+\Delta t) = (1-a\Delta t + \theta a^2\Delta t^2)v_{i,i+1}(t) - g(1-\theta a\Delta t)\frac{y_{i+1}(t)-y_i(t)}{2\Delta z}\ \Delta t$$

$$+\theta gH\ \frac{v_{i+1,\ i+2}(t)-2v_{i,i+1}(t)+v_{i-1,i}\ (t)}{4\Delta z^2} \qquad \text{(V-34)}$$

which replace eqs (V-14) and (V-15).

Continuing the development as before gives a transformation matrix

$$\Phi = \begin{bmatrix} 1-\theta gH\left(\dfrac{\Delta t}{\Delta z}\right)^2 \sin^2\omega\Delta z & (1-\theta a\Delta t)H\ \dfrac{\Delta t}{\Delta z}\ j\sin\omega\Delta z \\ (1-\theta a\Delta t)g\ \dfrac{\Delta t}{\Delta z}\ j\sin\omega\Delta z & 1-a\Delta t+\theta a^2\Delta t^2-\theta gH\left(\dfrac{\Delta t}{\Delta z}\right)^2 \sin^2\omega\Delta z \end{bmatrix}$$

(V-35)

and the eigenvalues follow from

$$\left\{\lambda-1+\theta g\ H\left(\frac{\Delta t}{\Delta z}\right)^2 \sin^2\omega\Delta z\right\}\left\{\lambda-1+a\Delta t-\theta a^2\Delta t^2+\theta gH\left(\frac{\Delta t}{\Delta z}\right)^2 \sin^2\omega\Delta z\right\}$$

$$+\ (1-\theta a\Delta t)^2\ gH\left(\frac{\Delta t}{\Delta z}\right)^2 \sin^2\omega\Delta z = 0 \qquad \text{(V-36)}$$

Solving this quadratic equation in λ gives

$$\lambda = 1\text{-}\theta gH \left(\frac{\Delta t}{\Delta z}\right)^2 \sin^2\omega\Delta z - \frac{a\Delta t-\theta a^2\Delta t^2}{2}$$

$$\pm\sqrt{\left(\frac{a\Delta t-\theta a^2\Delta t^2}{2}\right)^2 - (1-\theta a\Delta t)^2 gH \left(\frac{\Delta t}{\Delta z}\right)^2 \sin^2\omega\Delta z} \qquad \text{(V-37)}$$

If $\sin \omega\Delta z = 0$ then

$$\lambda = 1 - \frac{a\Delta t-\theta a^2\Delta t^2}{2} \pm \frac{a\Delta t-\theta a^2\Delta t^2}{2} \qquad \text{(V-38)}$$

so that the two roots are

$$\lambda_1 = 1 \qquad \text{(V-39)}$$
$$\lambda_2 = 1\text{-}a\Delta t(1-\theta a\Delta t). \qquad \text{(V-40)}$$

As a rule $a\Delta t$ will be small. In the examples of Chapter 7 (see Table V-1) it has values in the order of 0,02 to 0,04. Further $0 < \theta < 1$. Taking this into account then it follows from eq (V-40) that $0<\lambda_2<1$.

For a certain value of $\sin^2 \omega\Delta z$ the square root of eq (V-37) becomes zero. In that case

$$gH \left(\frac{\Delta t}{\Delta z}\right)^2 \sin^2 \omega\Delta z = \left(\frac{a\Delta t}{2}\right)^2 \qquad \text{(V-41)}$$

and

$$\lambda = 1 - \frac{a\Delta t}{2}\left(1\text{-}\theta\frac{a\Delta t}{2}\right), \qquad \text{(V-42)}$$

for which, because of similar reasons as for eq (V-40), it holds also that

$o<\lambda<1$.

For $\sin^2 \omega\Delta z$, when moving from zero to the value following from eq (V-41), the two λ-values develop monotonously from those, given by eqs (V-39) and (V-40) to the single value of eq (V-42). This is depicted in Fig. V-2.

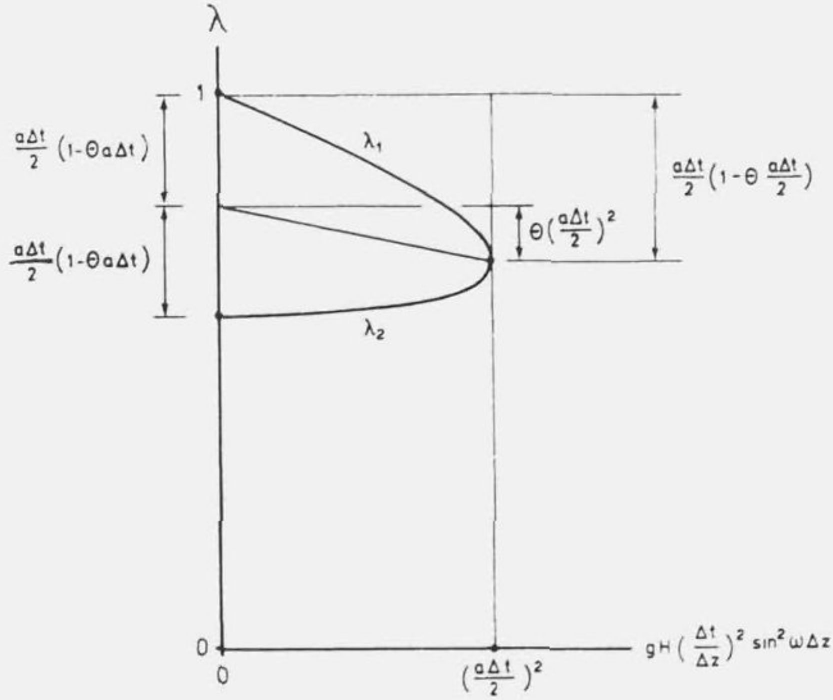

Fig. V-2 Course of λ vs. $gH \left(\frac{\Delta t}{\Delta t}\right)^2 \sin^2\omega\Delta z$

For values of $\sin^2 \omega\Delta z$ greater than the value corresponding to eq (V-41) the square root form of eq (V-37) is negative. For those cases it can be written that

$$\lambda = 1\text{-}\theta gH \left(\frac{\Delta t}{\Delta z}\right)^2 \sin^2\omega\Delta z\text{-}a\Delta t \left(\frac{1\text{-}\theta a\Delta t}{2}\right) \pm$$

$$\pm\, j \sqrt{(1\text{-}\theta a\Delta t)^2\, gH \left(\frac{\Delta t}{\Delta z}\right)^2 \sin^2\omega\Delta z\text{-}a^2\Delta t^2 \left(\frac{1\text{-}\theta a\Delta t}{2}\right)^2} \qquad \text{(V-43)}$$

The condition for stability is that $|\lambda| \leqslant 1$, or

$$|\lambda|^2 = \left\{ 1\text{-}\theta gH \left(\frac{\Delta t}{\Delta z}\right)^2 \sin^2\omega\Delta z\text{-}a\Delta t \left(\frac{1\text{-}\theta a\Delta t}{2}\right)^2 \right\} +$$

$$(1\text{-}\theta a\Delta t)^2\, gH \left(\frac{\Delta t}{\Delta z}\right)^2 \sin^2\omega\Delta z\text{-}a^2\Delta t^2\left(\frac{1\text{-}\theta a\Delta t}{2}\right)^2 \leqslant 1 \qquad \text{(V-44)}$$

The greatest values of this expression are found either for the greatest possible value of $\sin^2 \omega\Delta z$, i.e. for $\sin^2 \omega\Delta z = 1$ or for the value corresponding to eq (V-41). So it is sufficient to examine the condition

$$|\lambda|^2 = \left\{1\text{-}\theta gH\left(\frac{\Delta t}{\Delta z}\right)^2 \text{-}a\Delta t\ \frac{1\text{-}\theta a\Delta t}{2}\right\}^2 + (1\text{-}\theta a\Delta t)^2 \left\{gH\left(\frac{\Delta t}{\Delta z}\right)^2 \text{-} \left(\frac{a\Delta t}{2}\right)^2\right\} \leqslant 1$$

(V-45)

The variables acting in this condition are θ, $a\Delta t$ and $gH\left(\frac{\Delta t}{\Delta z}\right)^2$

Because of its complexity this equation will not be reduced to an explicit equation in one of these variables. For actual applications it can be used directly through substitution of the relevant values.

A summary of applications of the stability theory to the case studies dealt with in Chapter 7 is given in Table V-1

Case		Δt	$a\Delta t$	$gh\left(\frac{\Delta t}{\Delta z}\right)^2$	θ	$\lvert\lambda\rvert$
Hypothetical case						
without friction	(Section 7.9)	180 s	0	0,0900	0.6	0,992
		360 s	0	0,3600	0,6	0,987
		720 s	0	1,4400	0,6	1,208
with friction	(Section 7.10)	360 s	0,0353	0,3600	0,5	0,996
Western Scheldt (Section 7.11)						
combination of mean values		360 s	0,0317	0,0221	0,5	0,988
unfavourable combination		360 s	0,0245	0,0308	0,5	0,997

Table V-1. Examples Chapter 7: Results of examination to stability

Apparently all applications are stable except the third one, which already appeared to be unstable (see Tables 7-1 and 7-2). The other $|\lambda|$-values are smaller than 1, but very close to it. As such this is an indication for a small numerical damping, i.e. for reliable results.

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List of symbols

This list of symbols is divided into five parts on the basis of the following types of symbols used:

1. Latin capital
2. Latin lower case
3. Greek capital
4. Greek lower case
5. Miscellaneous

Since some of the symbols are used with different meanings, the list includes the equation, table or figure where the particular symbol is used for the first time.

1. Latin Capital

A	– unit economical value of water level data (per unit standard error and per unit river length) in eq (1-3) – matrix of eq (4-7) – matrix of A-coefficients in eq (5-6) – coefficient in eq (7-15) – coefficient of linear relation in eq (8-7) – amplitude in eq (I-1)
A_o, A_i ($i = 1 \dots m$)	– coefficients of linear relation in eq (2-19)
A_i	– elements of matrix A of eq (4-7)
$A_o \dots A_m$	– regression coefficients in eq (5-1)
A_{1E}	– regression coefficient between E (x_t) and y_t in eq (5-65)
A_E	– matrix A in eq (5-90) with elements of expected values
A_r	– matrix of regression coefficients in eq (5-164)
A_f	– area of streamflow profile in eq (7-8)

B	– coefficient in eq (7-15) – regression coefficient for time shifted measurements in eq (5-169) – coefficient in linear relation of eq (8-7)
B_i	– function of eq (4-51)
C	– total macro economic costs in eq (1-5) – column matrix in eq (4-3) – regression coefficient for time shifted measurements in eq (5-169) – friction or roughness coefficient of Chézy in eq (7-6) – coefficient in eq (7-15) – coefficient of linear relation in eq (8-7)
C_i	– total value of information loss in eq (1-3) – function of eq (4-51)
C_s	– total costs of gauging stations along a river reach in eq (1-4)
$C_0 \ldots C_n$	– coefficients of polynomial in eq (4-1)
D	– coefficient in eq (1-1) – increase of draught of a ship in eq (2-1) – station distance for given variance of estimate in Fig. 2-1 – matrix of eq (4-74) – transformation matrix of measured values into expected values in eq (5-92) – regression coefficient for time shifted measurements in eq (5-169) – length of correlation scale in eq (6-14) – coefficient of linear relation in eq (8-7)
D'	– station distance for corrected variance of estimate in Fig. 2-1
D_1, D_2	– lengths of correlation scales along the main directions in eq (6-36)
E	– minimum value of standard error SE in eq (1-1) – design criterion for standard error of estimate in eq (2-2) – regression coefficient for time shifted measurements in eq (5-169)

E ( )	- expected value
E_x, E_y	- measurement error variance matrices in eq (8-60)
E	- measurement error variance matrix in Fig. 8-2
F	- matrix of eq (5-100) - combined forces on a unit mass of water in eq (7-2)
F (χ^2)	- (Cumulative) χ^2 - distribution function in eq (2-35)
G_i (z)	- function of eq (4-30)
G(ω)	- function of frequency (amplitude spectrum) of eq (I-2)
H	- measurement matrix in eq (8-50) - mean water depth in eq (V-2)
H_i	- amplitude of i^{th} tidal component in eq (6-29)
I I^*	- Unit matrix in eq (5-115) and eq (8-59) - matrix of eq (4-71)
I_o	- matrix of eq (4-85)
I_t	- transverse slope due to streamflow curvature in eq (3-2)
II	- matrix of eq (4-74)
K	- matrix of eq (4-98) - Kalman gain matrix in eq (8-34)
K^J	- *Kalman gain matrix, to be compared with K in eq (8-67)*
L	- length of correlation scale in eq (1-2) - wavelength in footnote Section 3.4.3 and in eq (7-71) - linear coefficient of u in eq (8-73)
M	- matrix of eq (4-48) - number of examined stations in eq (5-163) - *row matrix with one zero-value in eq (8-66)* - linear coefficient of u in eq (8-73)

M_1	–	matrix of eq (4-72)
M_2	–	matrix of eq (4-83)
$M\downarrow$	–	lower m rows of M in eq (4-90)
M^{-1} (1/2)	–	left hand half of M^{-1} in eq (4-91)
$\|M\|$	–	minor of $\|R\ (\underset{\sim}{X}\ \underset{\sim}{Y})\|$ in eq (5-157)
N	–	number of tidal components in eq (6-29)
	–	number of measurement intervals in Fig. 7-8
N^*	–	number of tidal components in one group in eq (6-30)
P	–	general weight factor in eq (4-87)
P (-)	–	covariance matrix of $\Delta\hat{Y}$-values before correction in eq (8-62)
P (+)	–	covariance matrix in $\Delta\hat{Y}$-values after correction in eq (8-60)
Q	–	discharge in eq (7-1)
R	–	unit value of information loss in eq (1-3)
	–	k × k serial correlation matrix between Δy-values in eq (2-36)
	–	model error covariance matrix in eq (8-60)
R_Δ	–	correlation coefficient $\varrho_\Delta(\tau)$ for $\tau = 0$ in eq (3-31)
$R_{\Delta Y}$	–	autocorrelation matrix of Δy-values in eq (5-106)
R^+ $(\underset{\sim}{X}\underset{\sim}{X})$	–	extended correlation matrix in eq (6-5)
R^+ $(\underset{\sim}{X}Y)$	–	column correlation matrix in eq (6-8)
R $(\underset{\sim}{X}\underset{\sim}{X})$, R $(\underset{\sim}{X}\underset{\sim}{Y})$	–	correlation matrices in eq (5-35)
S	–	costs of a gauging station in eq (1-3)

$S^+(X)$	–	extended standard error diagonal matrix of x in eq (6-6)
$S^+(Y)$	–	extended standard error diagonal matrix of y in eq (6-9)
S_b	–	bottom slope in eq (7-3)
S_w	–	slope of the water pressure in eq (7-4)
S_r	–	slope of the energy loss due to friction in eq (7-5)
SE	–	standard error of estimate in eq (1-1)
SD	–	natural standard deviation of levels in eq (1-2)
T	–	wave period in footnote of Section 3.4.3 and in eq (7-71) – long time period, going to infinity in eq (I-19) – constant time parameter in variogram of Fig. I-8
T_o	–	time correlation scale in eq (6-29)
T_i	–	period of i^{th} tidal component in eq (6-29)
U	–	column matrix of u-values in eq (8-74)
V	–	matrix of eq (4-74)
$V_{\underline{Y}\|\underline{X}}$	–	autocovariance matrix of $\underline{y}$ for given $\underline{X}$ in eq (5-102)
V_A	–	covariance matrix of eq (5-102)
V	–	covariance matrix of $\Delta\underline{u}$ in eq (8-76)
V_ω	–	*amplitude of Fourier component of velocity disturbance* with frequency ω in eq (V-16b)
Var	–	variance
Var Δy_k	–	variance of Δy in group of k Δy-values in eq (2-32)
Var Δy^*	–	corrected value of Var Δy in eq (2-37)
$\bar{\text{Var}}$ $\Delta y(\tau)$	–	extrapolated value of Var $\Delta y(\tau)$ for $\tau = 0$ in eq (3-28)

W – general indication for width in eq (7-67)
– mean width in eq (V-1)

W_f – flow conveying width in eq (7-9)

W_s – storage width in eq (7-1)

X – column matrix of x-values at m stations in eq (4-23)

– column matrix of x-values in eq (8-55)

$\underline{X}$ – column matrix of measured x-values in eq (4-3)
– right hand matrix of eq (4-48)
– matrix of measured x-values in eq (5-6)
– model parameter in eq (8-35)

$\hat{X}$ – matrix of calculated x and x′-values in eq (4-48)

$\underline{X}^+$ – matrix of eq (4-98)

$\bar{\underline{X}}$ – row matrix of means of measured values in eq (6-7)

Y – column matrix of y-values in eq (8-53)

$\hat{Y}$ – column matrix of calculated $\hat{y}$-values in eq (5-6)

$\underline{Y}$ – column matrix of measured $\underline{y}$-values in eq (5-7)

$\underline{Y}r$ – matrix of $\underline{y}$-values in eq (5-163)

Y_ω – amplitude of Fourier component of water level disturbance with frequency ω in eq (V-16a)

Z – column matrix of z-values in eq (4-3)
– distance parameter in spatial variogram in eq (II-5)

[ZX] – right hand matrix of eq (4-19)

2. Latin lower case

a – linear resistance coefficient in eq (V-31)

b	–	bottom level with respect to horizontal reference level in eq (7-3)
c	–	celerity, footnote in Section 3.4.3
	–	celerity of tidal wave in eq (6-29)
	–	celerity in eq (7-73)
	–	celerity in eq (II-1)
d	–	variable station distance in Fig. 2.1
	–	distance between 2 stations in eq (6-20)
	–	distance between 2 stations in eq (V-3)
	–	distance between points for which water levels are considered in Fig. 7-1
dif	–	Dirichlet function in Fig. I-10
e_{ij}	–	elements of matrix $E(X_t)$ in eq (5-90)
f	–	function of x_1 and x_2 in eq (8-38)
$f(\chi^2)$	–	χ^2-distribution density in eq (2-34)
f(t)	–	function of time in eq (I-1)
f_t	–	acceleration owing to transversal forces in eq (7-77)
f_{cor}	–	acceleration of Coriolis in eq (7-78)
f_{cent}	–	centrifugal acceleration in eq (7-79)
g	–	acceleration due to gravity in eq (3-1) and in eq (7-3)
	–	function of x_1 and x_2 in eq (8-38)
$g(\omega)$	–	amplitude density spectrum in eq (I-3)
$g_p(\omega)$	–	power density spectrum in eq (I-25)
h	–	mean water depth in footnote of Section 3.4.3
	–	number of unit times in timeshift of Section 5.7
i	–	general serial number
	–	indication of points for which water levels are considered in eq (7-10)
j	–	general serial number
	–	station at which changes are assumed in Fig. 4-32 ($j = k + p$)
	–	number of measurements in eq (5-4)
	–	$\sqrt{-1}$ in eq (V-17)

k	– number of Δy-values in groups for which a χ^2-distribution is examined in eq (2-31) – number of measured non-transition points in eq (4-92) – station with fixed slope ($i = k$) in Fig. 4-32) – serial number of an estimation in eq (5-110) – serial order of a time step Δt in eq (7-20) – correction coefficient in eq (8-4)
m	– number of network stations in eq (2-3) – number of stations in eq (4-16) – number of network stations in eq (5-1) – number of measured interval variables in Section 8.7
$\underline{m}$	– error, owing to propagated measurement errors in eq (2-13)
n	– general indication for number – number of measurements in eq (3-3) – degree of the function in eq (4-1) – number of measurement sets in eq (5-4) – number of time steps in a measurement interval in eq (7-20) – number of internal variables in Section 8.7 – number of harmonic components in eq (I-14)
n_e	– number of external variables in Section 8.7
p	– number of stations after k in eq (4-105) – ratio between two frequencies ω_2 and ω_1 in eq (I-31)
p_i	– weight factor for x_i in eq (4-63)
q	– number of stations between k and p in eq (4-105)
r	– models error in eq (2-12) – radius of the curve in eq (3-2) – distance between two sites in polar coordinates in eq (6-36) – radius of the flow line in eq (7-79) – column matrix of r_y-values (model errors) in eq (8-58)
r_e	– radius of correlation ellipse (function of φ) in eq (6-37)
r_{yl}	– model error in eq (8-9)
s.o.t.	– second order terms in eq (7-44)

t	–	general indication for time
t_o	–	time of examination in Section 5.7
	–	starting time of calculations in Section 7.2.3
u	–	external influence on $\hat{y}_1$ (-) and $\hat{y}_2$ (-) in eq (8-73)
v	–	velocity in eq (3-1), eq (7-2)
x	–	general indication for independent variable (water level at network station) in eq (2-3)
	–	a variable in footnote Section 3.2.6
x_t	–	true water level
$\underline{x}$, $\underline{x}_i$	–	measured water level (water levels) at n+1 stations in eq (4-2)
$\hat{x}$	–	calculated watert level
$\underline{x}_k$	–	measured water level at a non-transition point in eq (4-92)
$\vec{\underline{x}}$	–	row matrix of measured x-values in eq (5-13)
$\bar{\underline{x}}$	–	mean value of $\underline{x}_i$ in eq (5-42)
y	–	general indication for dependent variable (water level at site or station under examination)
	–	water level with respect to horizontal reference level in eq (7-1)
y_t	–	true water level
$\underline{y}$	–	measured water level
$\hat{y}$	–	calculated water level, eq (4-1), eq (5-1)
y_0	–	constant value of y in eq (3-3)
$\bar{\underline{y}}$	–	mean value of $\underline{y}_i$ in eq (3-3), eq (5-24)
y_i	–	continuous value of y between z_i and z_{i+1} in eq (4-50)
$y_0 - \bar{y}$	–	wave amplitude in eq (7-71)
$y_t \vert \hat{y}$	–	true value of y, if measured value is $\hat{y}$ in eq (5-79)
$\hat{y}_1$ (-), $\hat{y}_2$ (-)	–	calculated values before Kalman filter correction in eq (8-2)
$\hat{y}_1$ (+), $\hat{y}_2$ (+)	–	calculated values after Kalman filter correction in eq (8-2)
y_d	–	difference of water levels in transversal direction in eq (7-77)
z	–	horizontal distance, eq (7-1), eq (4-1), eq (II-1)
	–	station distance in eq (1-1)
	–	horizontal coordinate in eq (4-1)
	–	distance between station and site under examination in eq (6-21)
z_m	–	economic optimum of station distance in eq (1-6).

3. Greek capital

Δ	– general indication for finite difference
$\Delta \underline{x}$, $\Delta \underline{y}$	– error of measurement in eq (2-4)
$\Delta \hat{x}$, $\Delta \hat{y}$	– error of estimate in eq (2-5)
Δx, Δy	– difference between measured and calculated water level in eq (2-6)
ΔY	– $1 \times k$ row matrix of Δy-values in eq (2-36)
	– column matrix of Δy-values in eq (5-8)
Δh	– velocity head in eq (3-1)
ΔT	– time interval between subsequent measurements in eq (3-37)
	– measurement interval in Fig. 7-3
$\bar{\Delta}\bar{y}$	– mean value of Δy in Table 3-1
Δz_i	– horizontal distance in eq (4-31)
Δy_s	– systematic error of measurement in eq (5-71)
Δt	– time shift in Section 5.7
	– time step in the calculations in eq (7-10)
$\Delta \hat{Y}, \Delta \underline{Y}$	– column matrices of $\Delta \hat{y}$, $\Delta \underline{y}$ in eq (8-56)
$\bar{\Delta}\bar{Y}$	– column matrix of $\bar{\Delta}\bar{y}$ in eq (8-53)
$\Delta^* Y$	– column matrix of $\Delta^* y$ in eq (8-89)
$\Delta^* y$	– increase of $\hat{y}(+)$, used to calculate Φ in eq (8-88)
$\Delta y(\tau)$	– difference between $\underline{y}(t)$ and $\underline{y}(t-\tau)$ in Fig. 3-1
θ	– weight coefficient in eq (7-21)
Λ	– external transformation matrix in eq (8-74)
Π	– continuous product in eq (8-86)
Σ	– general indication for sum
	– matrix of eq (4-19)

Φ	–	internal transformation matrix of eq (8-74)
	–	amplification matrix in eq (V-21)
Φ^*	–	transformation matrix over time step Δt in eq (8-86)

4. Greek lower case

α	–	standard normally distributed stochastic variable in eq (5-138)
	–	angle of direction of greatest correlation in eq (6-36)
β	–	standard normally distributed stochastic variable in eq (5-138)
	–	block function in eq (I-47)
γ^2	–	correction coefficient of Var Δy in eq (2-39)
δ	–	difference between logs of calculated and actual values of correlation coefficients in eq (6-42)
	–	2^{nd}-degree approxination of ∂ in eq (7-41)
	–	Dirac function in eq (I-10)
ε	–	standard error of measurement ($= \sigma\Delta y$)
ε_s	–	systematic error of measurement in eq (3-35)
λ	–	eigen values of matrix Φ in eq (V-22)
λ^2	–	correction coefficient of Var r in eq (2-40)
ϱ	–	general indication of correlation coefficient
$\varrho(\tau)$	–	correlation coefficient between $\underline{y}(t)$ and $\underline{y}(t-\tau)$ in eq (3-9)
$\varrho\Delta(\tau)$	–	correlation coefficient between $\Delta\underline{y}(t)$ and $\Delta\underline{y}(t-\tau)$ in eq (3-17)
$\varrho_t(\tau)$	–	correlation coefficient between $\underline{y}_t(t)$ and $\underline{y}_t(t-\tau)$ in eq (3-18)
$\tilde{\varrho}(0)$	–	extrapolated value of $\varrho(\tau)$ in for $\tau=0$ in eq (3-27)
ϱ_Δ	–	correlation coefficient between $\Delta\underline{y}$ and $\Delta\underline{x}$ in eq (3-53)
$\varrho\Delta y_1\Delta y_2$	–	correlation coefficient between Δy_1 and Δy_2 in eq (8-1)
$\varrho(\tau)$	–	correlation coefficient between y(t) and y(t-τ) in eq (I-17)
$\tilde{\varrho}(0)$	–	extrapolated value of $\varrho(\tau)$ for $\tau=0$ in eq (I-51)

$\tilde{\varrho}(0, 0)$	–	extrapolated value of $\varrho(z,\tau)$ for $z=0$ and $\tau=0$ in eq (II-8)
$\varrho(z,\tau)$	–	cross correlation coefficient between $y(0,t)$ and $y(-z, t-\tau)$ in eq (II-6)
σ	–	general indication for standard error
	–	standard error of estimate in eq (2-2)
σ_1, σ_2	–	standard error of estimate in eq (2-1)
$\tilde{\sigma}\Delta y(0)$	–	extrapolated value of $\sigma\Delta y\ (\tau)$ for $\tau=0$ in eq (3-28)
σ_m	–	standard error of propagated measurement errors in Fig. 6-3
τ	–	time between measurements (subsequent or not) in eq (3-6)
	–	time lag in Fig. 5-2
	–	time interval after t in eq (I-17)
φ	–	correction coefficient in eq (2-2)
	–	*angle of direction in eq (6-36)*
	–	local altitude in eq (7-78)
$\varphi_{i,j}$	–	elements of matrix Φ in eq (8-89)
χ^2	–	elements of chi-square distribution in eq (2-31)
ω	–	earth rotation velocity in eq (7-78)
	–	angular frequency in eq (I-1)
	–	freqiemcu pf Fourier components in eq (V-16a, b, c)

5. Miscellaneous

*	–	indication of provisional values in eqs (7-23) and (7-24)
~	–	indication of extrapolation to τ=o
∂	–	partial derivative

Samenvatting

Een meetnet is een stelsel van samenhangende meetstations, meet- en/of bemonsteringspunten (TNO-1986). Het doel van een meetnet is te voorzien in een dichtheid en een verdeling van meetstations over een gebied, zodanig dat door interpolatie tussen de gegevens van verschillende stations het mogelijk is om, met een voor praktische doeleinden voldoende nauwkeurigheid, de grootte van de beschouwde grootheid voor ieder punt binnen het bestreken gebied af te leiden (WMO-1981).

Deze studie heeft betrekking op waterstandsmetingen. De eis van voldoende nauwkeurigheid werd zodanig geformuleerd dat een afgeleid gegeven voor ieder punt binnen het door het meetnet bestreken gebied ten hoogste een standaardafwijking mag hebben, gelijk aan de standaardafwijking in de metingen. De standaardafwijking van de geïnterpoleerde waarden is afhankelijk van de afstand tussen de stations. Wanneer dit verband bekend is heeft men de mogelijkheid de afstanden zodanig te kiezen dat de vereiste standaardafwijking inderdaad niet wordt overschreden.

Voor de schatting van de standaardmeetafwijking kunnen drie methoden worden gebruikt:

1. Vergelijking tussen metingen aan het onderzochte station op verschillende tijdstippen.
2. Vergelijking tussen metingen aan het onderzochte station en aan een nabij gelegen station.
3. Vergelijking tussen metingen aan het onderzochte station en aan een reeks stations, gelegen op verschillende afstanden ervan.

Hoewel geen van deze methoden op zichzelf tot een oplossing zal leiden, zal men, indien mogelijk, door toepassing van twee ervan of van alle drie het vraagstuk kunnen insluiten en zodoende tot een uitspraak kunnen komen. Voor de Nederlandse omstandigheden kan de standaard-meetafwijking worden vastgesteld op 2,5 cm.

Interpolatiemethoden die voor het ontwerpen van een meetnet, en later voor het operationele gebruik kunnen worden toegepast, zijn:

1. methoden, gebaseerd op mathematisch gedefinieerde interpolatiekrommen (machtsfuncties en deelpolynomen);
2. statistische methoden, gebaseerd op de correlatie tussen de standen aan de

3. methods based on mathematical models (based on the mass and energy conservation laws);
4. methods composed of a combination of a mathematical model and a statistical model (e.g. a Kalman filter).

This study in particular is devoted to methods mentioned under 2, 3 and 4. Methods discussed under 1 are applied to interpolation of derived statistics such as means, standard errors and correlation coefficients, and are subsequently available for use in the statistical methods.

For the statistical methods long data series should be available, and for the mathematical models a good knowledge of the dimensions and roughness of the streamflow sections is required. The mathematical models give a better understanding of the water motion as a whole: as well as water levels these models also estimate velocities and discharges. However they require a laborious and time consuming calculation procedure. An advantage of the statistical methods, if applied for hindcasting, is that data measured after the time of examination can be included in the calculations as well.

The Kalman filter can be used for two purposes:
1. to determine the parameters of the mathematical model, and
2. to determine water levels at ungauged sites.

Both applications are described in the present study.

It is recommended that both statistical methods and mathematical models are used for designing a network, and subsequently a decision made as to which of these methods is most suitable for use in the operational phase.

RÉSUMÉ

Un réseau de mesure est un système des stations de jaugeage cohérentes, de points de mesure ou d'échantillonage (TNO-1986). Le but d'un réseau de mesure est de pourvoir à une densité et une distribution des stations de jaugeage sur la région couverte par le réseau, de telle façon que, par interpolation entre les données des stations différentes, il soit possible de dériver la valeur de l'élement considéré pour chaque lieu dans la région couverte, avec une accuratesse, suffisante aux fins practiques (OMM, 1981).

Cette étude se rapporte à des mesures de niveaux d'eau. L'exigence de l'accuratesse suffisante fut formulée tellement, qu'une donnée derivée pourra avoir un écart type, partout dans la région couverte par le réseau, au maximum égal à l'écart type des mesures elles-mêmes. L'écart type des quantités interpolées dépend de la distance entre les stations. Dès que cette relation serra connue on a la possibilité de choisir les distances tellement que l'écart type exigé ne sera vraiment dépassé.

Pour estimer l'écart type des mesures on pourrait utiliser les trois méthodes suivantes:
1. Comparaison des mesures à la station examinée à de différents de moments.
2. Comparaison des mesures à la station examinée et à une station voisine.
3. Comparaison des mesures à la station examinéee et à une série des stations, situées à des distances différentes de celle-là.

Quoiqu'aucune de ces méthodes en elles-mêmes puisse résulter à une solution, on pourra, si possible, enclure le problème, en appliquant deux de ces méthodes ou tous les trois, et arriver ainsi à une décision. Pour les conditions néerlandaises on peut déterminer l'écart type des mesures à 2,5 cm.

Les méthodes d'interpolation qui pourraient être appliquées pour dessiner un réseau, et après pour l'utilisation opérationelle de celui-ci, sont les suivantes:
1. des méthodes, basées sur des courbes définiées mathématiquement (des fonctions de pouvoir, des polynomes partiels);
2. des méthodes statistiques, basées sur la corrélation entre les niveaux aux stations de jaugeage (par exemple la régression linéaire multiple, menant à l'interpolation optimale);
3. des méthodes, basées sur des modèles mathématiques (basées sur les lois de conservation de la masse et de l'énergie);

4. des méthodes, composées d'une combinaison d'un modèle mathématique et d'un modèle statistique (par exemple le filtre Kalman).

L'étude est consacrée en particulier aux méthodes nommées sous 2, 3 et 4. Les méthodes, nommées sous 1 furent appliquées à l'interpolation des quantités charactéristiques, comme des moyennes, des écarts type et des coefficients de corrélation, au besoin des méthodes statistiques.

Pour les méthodes statistiques il faut disposer de longues séries de données; pour les modèles mathématiques une bonne connaissance des dimensions et des rugosités des profiles de courant est nécessaire. Les modèles mathématiques donnent plus de connaissance sur le mouvement des eaux en général: outre des niveaux d'eau elles produisent également des vitesses d'eau et des débits. Mais la procedure de calculation demande beaucoup de travail et de temps. Un avantage des méthodes statistiques, si elles sont utilisées pour des calculations après, est qu'on peut également inclure dans les calculations des données, mesurées après le moment examiné.

On peut appliquer le filtre Kalman à deux buts:
1. pour la détermination des paramètres du modèle mathématique.
2. pour la détermination des niveaux d'eau non mesurés.

Ces deux applications sont décrites dans l'étude présente.

Il est recommandable d'utiliser pour le dessin d'un réseau de mesure des méthodes statistiques ainsi que des modèles mathématiques et en suite de choisir laquelle de ces méthodes doit être prise en considération pour l'application dans la phase opérationelle.

Curriculum vitae

Johannes Willem van der Made werd op 15 september 1928 geboren te Rotterdam. In 1946 behaalde hij het einddiploma van de 1e Gemeentelijke Hogere Burgerschool aldaar. Daarna liet hij zich inschrijven aan de Technische Hogeschool te Delft, waar hij in 1951 slaagde voor Civiel Ingenieur. Na zijn militaire dienst trad hij in 1953 toe tot de Rijkswaterstaat, Arrondissement 'Het Noordzeekanaal'. In 1955 werd hij tewerkgesteld bij de Studiedienst IJmuiden en in 1959 bij het Arrondissement 's-Hertogenbosch.

In 1962 werd hij overgeplaatst naar de Afdeling Hydrometrie van de Directie Waterhuishouding en Waterbeweging. Daar was hij werkzaam op het gebied van afvoerhydrologie, gegevensbewerking, statistiek, getijbeweging en stormvloedvoorspelling. In 1973 werd de afdeling omgevormd tot de Operationele Afdeling van de Hoofdafdeling Waterhuishouding. Van zijn studies zijn een groot aantal publicaties en bijdragen aan symposia verschenen.

In internationaal verband vervult hij functies in het kader van de hydrologische programma's van de Unesco en de Meteorologische Wereldorganisatie (WMO). Tevens was hij van 1970 tot 1987 algemeen secretaris van de Internationale Commissie voor de Hydrologie van het Rijngebied. Momenteel is hij werkzaam bij de Hoofdafdeling Advies en Ondersteuning van de Dienst Getijdewateren.